Comment on "Evolution Operator Can Always Be Separated into the Product of Holonomy and Dynamic Operators"

In a recent Letter [1], Yu and Tong derived an expression for a generic time evolution operator acting on a subspace, claimed to constitute a general separation into a product of a holonomy operator and a dynamic operator. This result appears to be in conflict with prior work by Anandan [2], in which the time evolution was found not to separate into a holonomic and a dynamic part in general. In this Comment, we show that the claim in Ref. [1], that the time evolution operator always can be written as a product of a holonomy operator and a dynamic operator, is false, as it is based on a circular use of the time evolution operator.

The key finding of Ref. [1] is that, given a Hilbert space $\mathcal{H}$ of some quantum system and the ℓ dimensional ($\ell \leq \dim \mathcal{H}$) subspace spanned by $\{|\psi_j(t)\rangle\}_{j=1}^{\ell}$, where each $|\psi_j(t)\rangle$ is a solution of the Schrödinger equation with Hamiltonian $H(t)$, the time evolution operator $U(t,0) = |\psi_j(t)\rangle\langle\psi_j(0)|$ satisfies the differential equation

$$\dot{U}(t,0) = \dot{P}(t)U(t,0) + U(t,0)F(t,0), \tag{1}$$

where $P(t) \equiv |\psi_j(t)\rangle\langle\psi_j(t)|$ is the projector on the subspace, $F(t,0) \equiv F_{jk}(t)|\psi_j(0)\rangle\langle\psi_k(0)|$ with $F_{jk}(t) \equiv -i\langle\psi_j(t)|H(t)|\psi_k(t)\rangle$, we use Einstein's summation convention, i.e., repeated indices are implicitly summed, and we put $\hbar = 1$. Formally, Eq. (1) implies that

$$U(t,0) = \mathcal{P}e^{\int_0^t \dot{P}(\tau)d\tau} P(0) \bar{\mathcal{T}} e^{\int_0^t F(\tau,0)d\tau}, \tag{2}$$

which is the expression for $U(t,0)$ found in Ref. [1] that was claimed to imply a separation into a product of a holonomy operator and a dynamic operator, given by a path ordered exponential $\mathcal{P}e^{\int_0^t \dot{P}(\tau)d\tau}P(0)$ and a reverse time ordered exponential $P(0)\bar{\mathcal{T}}e^{\int_0^t F(\tau,0)d\tau}$, respectively.

To understand why Eq. (2) does not imply the claimed separation, one should note that $F(t,0)$, in fact, depends on the time evolution operator itself

$$\begin{aligned} F(t,0) &= -i|\psi_j(0)\rangle\langle\psi_j(t)|H(t)|\psi_k(t)\rangle\langle\psi_k(0)| \\ &= -iU^\dagger(t,0)H(t)U(t,0). \end{aligned} \tag{3}$$

This observation, which seems to be overlooked in Ref. [1], allows us to write Eq. (2) as

$$U(t,0) = \mathcal{P}e^{\int_0^t \dot{P}(\tau)d\tau} P(0) \bar{\mathcal{T}} e^{-i\int_0^t U^\dagger(\tau,0)H(\tau)U(\tau,0)d\tau}. \tag{4}$$

Evidently, the "dynamic" operator, the last factor on the right-hand side, contains contributions from the holonomy operator via $U(t,0)$ in a circular manner. In other words, if one insists on calling Eq. (2) a separation into a product of a holonomy operator and a dynamic operator, it follows, by Eq. (4), that the dynamic operator depends nontrivially on the holonomy operator. Thus, contrary to the claim in Ref. [1], Eq. (2) does not constitute a general separation of $U(t,0)$ into a holonomy operator and dynamic operator.

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Received 13 August 2025; accepted 17 December 2025; published 12 February 2026
DOI: 10.1103/d5rt-jpjk

Data availability—No data were created or analyzed in this study.

[1] X.-D. Yu and D. M. Tong, Evolution operator can always be separated into the product of holonomy and dynamic operators, Phys. Rev. Lett. **131**, 200202 (2023).
[2] J. Anandan, Non-adiabatic non-Abelian geometric phase, Phys. Lett. A **133**, 171 (1988).

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