

# Hadronic Vacuum Polarization for the Muon $g - 2$ from Lattice QCD: Long-Distance and Full Light-Quark Connected Contribution

Alexei Bazavov<sup>1</sup>, Claude W. Bernard<sup>2</sup>, David A. Clarke<sup>3</sup>, Christine Davies<sup>4</sup>, Carleton DeTar<sup>3</sup>, Aida X. El-Khadra<sup>5,6</sup>, Elvira Gámiz<sup>7</sup>, Steven Gottlieb<sup>8</sup>, Anthony V. Grebe<sup>9</sup>, Leon Hostetler<sup>8</sup>, William I. Jay<sup>10</sup>, Hwancheol Jeong<sup>8</sup>, Andreas S. Kronfeld<sup>9</sup>, Shaun Lahert<sup>3,\*</sup>, Jack Laiho<sup>11</sup>, G. Peter Lepage<sup>12</sup>, Michael Lynch<sup>5,6,†</sup>, Andrew T. Lytle<sup>5,6</sup>, Craig McNeile<sup>13</sup>, Ethan T. Neil<sup>14</sup>, Curtis T. Peterson<sup>15</sup>, James N. Simone<sup>9</sup>, Jacob W. Sitison<sup>14,‡</sup>, Ruth S. Van de Water<sup>9</sup>, and Alejandro Vaquero<sup>3,15,16</sup>

(Fermilab Lattice, HPQCD, and MILC Collaborations)

<sup>1</sup>*Department of Computational Mathematics, Science and Engineering, and Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824, USA*

<sup>2</sup>*Department of Physics, Washington University, St. Louis, Missouri 63130, USA*

<sup>3</sup>*Department of Physics and Astronomy, University of Utah, Salt Lake City, Utah 84112, USA*

<sup>4</sup>*SUPA, School of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, United Kingdom*

<sup>5</sup>*Department of Physics, University of Illinois Urbana-Champaign, Urbana, Illinois 61801, USA*

<sup>6</sup>*Illinois Center for Advanced Studies of the Universe, University of Illinois Urbana-Champaign, Urbana, Illinois 61801, USA*

<sup>7</sup>*CAFPE and Departamento de Física Teórica y del Cosmos, Universidad de Granada, E-18071 Granada, Spain*

<sup>8</sup>*Department of Physics, Indiana University, Bloomington, Indiana 47405, USA*

<sup>9</sup>*Theory Division, Fermi National Accelerator Laboratory, Batavia, Illinois 60510, USA*

<sup>10</sup>*Department of Physics, Colorado State University, Fort Collins, Colorado 80523, USA*

<sup>11</sup>*Department of Physics, Syracuse University, Syracuse, New York 13244, USA*

<sup>12</sup>*Laboratory for Elementary-Particle Physics, Cornell University, Ithaca, New York 14853, USA*

<sup>13</sup>*Centre for Mathematical Sciences, University of Plymouth, Plymouth PL4 8AA, United Kingdom*

<sup>14</sup>*Department of Physics, University of Colorado, Boulder, Colorado 80309, USA*

<sup>15</sup>*Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain*

<sup>16</sup>*Center for Astroparticles and High Energy Physics (CAPA), Calle Pedro Cerbuna 12, 50009 Zaragoza, Spain*



(Received 22 January 2025; revised 25 April 2025; accepted 20 May 2025; published 1 July 2025)

We present results for the dominant light-quark connected contribution to the long-distance window of the hadronic vacuum polarization (HVP) contribution to the muon  $g - 2$  from lattice quantum chromodynamics. Specifically, with a new determination of the lattice scale on MILC's physical-mass HISQ ensembles, using the  $\Omega^-$  baryon mass, we obtain a result of  $a_\mu^{\text{H,L,D}}(\text{conn}) = 400.2(2.3)_{\text{stat}}(3.7)_{\text{syst}}[4.3]_{\text{total}} \times 10^{-10}$ . Summing this result with our recent determinations of the light-quark connected contributions to the short- and intermediate-distance windows, we obtain a subpercent precision determination of the light-quark-connected contribution to HVP of  $a_\mu^{\text{H}}(\text{conn}) = 655.2(2.3)_{\text{stat}}(3.9)_{\text{syst}}[4.5]_{\text{total}} \times 10^{-10}$ . Finally, as a consistency check, we verify that an independent analysis of the full contribution is in agreement with the sum of individual windows. We discuss our future plans for improvements of our HVP calculations to meet the target precision of the Fermilab  $g - 2$  experiment.

DOI: [10.1103/d583-yhfs](https://doi.org/10.1103/d583-yhfs)

\*Contact author: [shaun.lahert@gmail.com](mailto:shaun.lahert@gmail.com)

†Contact author: [ml11@illinois.edu](mailto:ml11@illinois.edu)

‡Contact author: [jacob.sitison@colorado.edu](mailto:jacob.sitison@colorado.edu)

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP<sup>3</sup>.

**Introduction**—The muon's anomalous magnetic moment or “muon ( $g - 2$ )” is one of the most sensitive probes of the standard model of particle physics, with both the experimental results [1,2] and the standard model prediction [3] reaching precisions well below one part per million. Tensions between theory and experiment in this quantity have long captured the attention of particle physicists since the results of the E821 experiment at Brookhaven [4]. With forthcoming experimental results from Fermilab [2] and J-PARC [5,6] expected to improve further the accuracy of

the muon ( $g - 2$ ) measurement, concurrent improvements in the standard model theoretical calculation are urgently needed.

Although they are a relatively small part of the total prediction, the contributions to the muon ( $g - 2$ ) from hadronic physics are among the most challenging to calculate due to strong coupling effects. The muon does not feel the strong force directly, so these contributions arise from corrections to either the photon two-point function (hadronic vacuum polarization, or HVP), or the photon four-point function (hadronic light-by-light scattering, or HLbL). As described in detail in Ref. [3], both HVP and HLbL can be estimated either from dispersive theory relying on experimental measurements of hadronic processes or calculated *ab initio* using lattice QCD.

Of these quantities, HLbL represents a smaller contribution to muon ( $g - 2$ ) but is also more difficult to calculate precisely. While previous summary reports from the Muon  $g - 2$  Theory Initiative [3,7] already found good agreement between dispersive evaluations [8–21] and lattice QCD calculations [22,23], more recent lattice and dispersive results [24–54] are rapidly pushing this quantity toward the precision goals needed for the final muon ( $g - 2$ ) experimental results.

For HVP, rapid progress on the lattice QCD side [55–67] has revealed tensions between lattice HVP results and data-driven dispersive estimates [7,68]. Efforts to understand these tensions, as well as to cross-check systematic effects in lattice calculations from different groups, have led to widespread study of HVP restricted to Euclidean windows [69], which isolate the contribution from a particular temporal range. Corresponding dispersive estimates of HVP restricted to the same windows [68,70–73] can be studied and compared with lattice results, an approach that can help shed light on these tensions.

In this Letter, we present results for the dominant long-distance (LD) window contribution to HVP, specifically the connected contributions from light (up and down) quarks in the isospin-symmetric limit, which comprise about 90% of the total HVP. LD window quantities suffer from large statistical noise and are sensitive to systematic effects from finite lattice volume and scale-setting [74] uncertainties. This work employs the high-statistics lattice datasets that were used to study the short- and intermediate-distance contributions in Ref. [67]. As in that work, we set the lattice scale at high precision using the mass of the  $\Omega^-$  baryon [75,76]. The combination of high statistics and improved scale setting allows us to achieve greatly improved precision for the LD HVP as well as for the full light-quark connected HVP by summing our LD window result with the other window observables calculated in Ref. [67], and updated here with the improved scale setting.

**Definitions**—The leading-order (LO) hadronic-vacuum-polarization contribution to the muon anomalous magnetic moment  $a_\mu \equiv (g - 2)_\mu/2$  is obtained via

$$a_\mu^{\text{HVP,LO}} = 4\alpha^2 \int_0^\infty dt C(t) \tilde{K}(t), \quad (1)$$

where  $\alpha$  is the fine-structure constant, the kernel  $\tilde{K}$  stems from quantum electrodynamics (QED) [77,78], and  $C(t) = \frac{1}{3} \sum_{k=1}^3 \sum_x \langle J^k(\mathbf{x}, t) J^k(0) \rangle$  is the Euclidean-time two-point correlation function of the electromagnetic current  $J^\mu(x) = \sum_f q_f \bar{\psi}_f(x) \gamma^\mu \psi_f(x)$ , summed over all quark flavors  $f \in \{u, d, s, c, b, t\}$  of electric charge  $q_f$ . Lattice-QCD calculations of  $a_\mu^{\text{HVP,LO}}$  separately compute the contributions from each quark flavor and from connected and disconnected Wick contractions. Additionally, these calculations are performed in the isospin-symmetric limit in pure QCD where the up- and down-quark masses are taken to be equal and QED effects are removed. The effects of isospin-breaking corrections are studied in Ref. [67,79]. In this Letter, we focus on the dominant isospin-symmetric, light-quark connected (LQC) contribution  $a_\mu^{\text{H}}(\text{conn})$ .

We also consider window observables [69] that restrict the Euclidean-time region over which  $C(t)$  is integrated,

$$a_\mu^{\text{win}} = 4\alpha^2 \int_0^\infty dt C(t) \tilde{K}(t) \mathcal{W}^{\text{win}}(t). \quad (2)$$

Here, we focus on the “long-distance” window defined by

$$\mathcal{W}^{\text{LD}}(t) = \frac{1}{2} \left[ 1 + \tanh\left(\frac{t - t_1}{\Delta}\right) \right], \quad (3)$$

with  $t_1 = 1$  fm, and  $\Delta = 0.15$  fm. This window is complemented by the “short” (SD) and “intermediate” (W) distance windows, defined in Eqs. (2.9)–(2.12) of Ref. [67] and studied there in detail, to cover the full integration region. The functions for the three complementary windows (SD, W, and LD) are shown in Fig. 1. The LD window covers the tail of the integrand, also shown in Fig. 1, where statistical errors increase significantly. Our strategies for alleviating this issue are discussed in detail below and in Supplemental Material [80].

**Physical inputs**—This calculation makes use of ensembles of four-flavor gauge-field configurations. Converting the lattice results into physical units requires the selection of a set of physical inputs that determine the quark masses and the lattice spacing (or scale). The masses of the  $u$ ,  $d$ ,  $s$ , and  $c$  quarks are determined by the masses of the  $\pi^+$ ,  $K^0$ ,  $K^+$ , and  $D_s^+$  mesons. Following our previous work [67], we define the physical point in the isospin-symmetric limit of pure QCD with  $M_{\pi^+} = 135.0$  MeV,  $M_K = 494.6$  MeV, and  $M_{D_s} = 1967$  MeV [85]. In this prescription, we work with the physical light-quark mass  $m_l \equiv (m_u + m_d)/2$ .

We set the lattice scale using the  $\Omega^-$  baryon mass [75,76]. To fix the scale to the pure-QCD world, we use the value of  $M_\Omega = 1.67126(32)$  GeV [75,76], obtained by subtracting the QED effects. The error includes statistical,

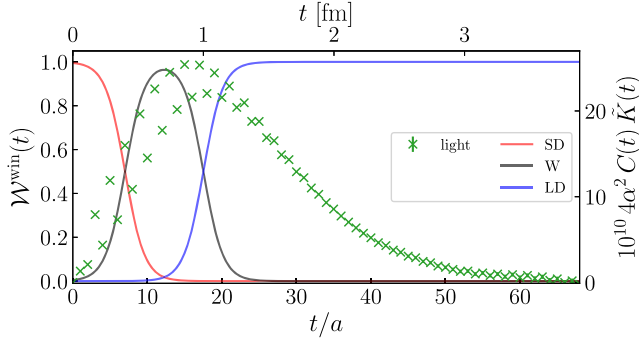


FIG. 1. The SD (red), W (black), and LD (blue) window functions as defined in the text, overlaid with raw lattice data for the LQC vector correlator multiplied by the integration kernel  $C(t)\tilde{K}(t)$  (green crosses) on our 0.06 fm ensemble. Note that the data points show the characteristic oscillating behavior of staggered fermions due to parity mixing at finite lattice spacing.

correlator-fit-systematic, and continuum-extrapolation uncertainties. Systematic errors from finite volume, strong isospin breaking, and charged sea-quark effects are sub-leading. We also consider a secondary scale setting based on the pion decay constant  $f_\pi$  [86–88] using the physical value  $f_\pi = 0.130\,50(13)$  GeV [85,89], allowing a consistency check between the two most commonly used scale-setting observables for  $a_\mu^{\text{ll}}(\text{conn})$ . Values for  $aM_\Omega$  and  $af_\pi$  on the ensembles used in this work are given in Table I.

*Lattice ensembles and correlation functions*—The lattice ensembles and correlation function datasets employed in this work are described in detail in Ref. [67]. Briefly, this work employs gauge-field ensembles generated by the MILC Collaboration [90–93] with light, strange, and charm quarks in the sea (all tuned to their physical values) spanning the range of lattice spacings of  $a \approx 0.15\text{--}0.06$  fm, as summarized in Table I [94]. Further details on the quark masses and corresponding meson masses are given in Table II of Ref. [67]. The vector-current two-point correlation function datasets are described in Table IV of Ref. [67] and the corresponding renormalization factors  $Z_V$  are taken from Refs. [96,97] and listed in Table III of Ref. [67]. We use the “local” and “one-link” current datasets, which correspond to two different discretizations of the vector current, on the

$a \approx 0.12, 0.09, 0.06$  fm ensembles, all of which were generated using low-mode averaging [98–100].

*Analysis strategy*—Our overall strategy for HVP observables is described in Ref. [67]. In particular, the lattice corrections, continuum fit functions and strategy for performing the continuum extrapolations, the Bayesian model averaging (BMA), and systematic error estimates follow the detailed description of Ref. [67]. As in Ref. [67], each individual HVP observable was multiplied with an unknown blinding factor, which was removed only after the analysis was finalized.

Long-distance HVP observables are affected by the well-known, rapid growth of statistical errors in the vector-current correlation functions [101]. While the exact low-modes used in the generation of the correlators help to reduce the statistical errors at large Euclidean times, these errors are still a significant source of uncertainty. We address this issue with a correlator-reconstruction strategy to replace  $C(t)$  at large times. In Ref. [102], a precise reconstruction of the vector-current correlator at large times, obtained from an exclusive channel study of the two-pion contributions, was used to test the fit method (described below). The fit was found to provide a reliable, albeit less precise, description of the vector-current correlator in the long time region. Hence, our results are obtained with this approach; we also employ the bounding method [69] as a cross-check (see Supplemental Material [80] and references).

*Fit method*—In this approach, the correlation function data are fit over the range  $t_{\min} \leq t \leq t_{\max}$  to a function matching the expected spectral decomposition. The noisy correlation function data is then replaced with the fit reconstruction at large Euclidean distances,  $t \geq t^*$ , where  $t^*$  is a hyperparameter that we optimize by minimizing the overall variance of  $a_\mu$ . Each fit makes use of the full sample covariance matrix over the fit range, which is well determined for all lattice spacings and currents.

The spectral decompositions of the staggered correlation functions employed in this analysis contain two towers of states, one of which has an oscillating phase, yielding the following fit function:

TABLE I. Ensemble parameters used in this Letter. The first column lists the approximate lattice spacings in fm using  $M_\Omega$  scale setting. The second column gives the spatial length  $L$  of the lattices in fm. The third column lists the volumes of the lattices in number of space-time points. The fourth column gives the sea-quark masses in lattice-spacing units. The fifth column lists the  $\Omega^-$  baryon mass in lattice units,  $aM_\Omega$  [75,76]. The sixth column lists the values of  $af_\pi$ .

| $\approx a/\text{fm}$ | $L_{M_\Omega}/\text{fm}$ | $N_s^3 \times N_t$ | $am_l^{\text{sea}}/am_s^{\text{sea}}/am_c^{\text{sea}}$ | $aM_\Omega$ | $af_\pi$      |
|-----------------------|--------------------------|--------------------|---------------------------------------------------------|-------------|---------------|
| 0.15                  | 5.00                     | $32^3 \times 48$   | 0.002 426/0.0673/0.8447                                 | 1.3246(26)  | 0.100 015(39) |
| 0.12                  | 5.95                     | $48^3 \times 64$   | 0.001 907/0.052 52/0.6382                               | 1.0494(17)  | 0.080 290(62) |
| 0.09                  | 5.70                     | $64^3 \times 96$   | 0.001 326/0.036 36/0.4313                               | 0.75372(97) | 0.058 145(63) |
| 0.06                  | 5.48                     | $96^3 \times 128$  | 0.0008/0.022/0.260                                      | 0.4834(11)  | 0.037 526(32) |

$$C_{\text{fit}}(t) = \text{const} + \sum_{n=0}^{N_{\text{states}}} [Z_n^2 (e^{-E_n t} + e^{-E_n (T-t)})] + (-1)^t \sum_{m=0}^{M_{\text{states}}} [Z_{m,\text{osc}}^2 (e^{-E_{m,\text{osc}} t} + e^{-E_{m,\text{osc}} (T-t)})], \quad (4)$$

where the energies ( $E_n, E_{m,\text{osc}}$ ), amplitudes ( $Z_n, Z_{m,\text{osc}}$ ), and constant term are the parameters obtained from the fit. Here,  $T = aN_t$  is the temporal extent of the lattice, and the term that is constant in time separation  $t$  incorporates the leading additional finite- $T$  effects [102,103]. The vector-current correlators are too noisy to resolve precisely the low-lying two-pion states that contribute to the correlator at large times. Hence, the low-lying spectrum obtained with the fit method is only approximate, representing a mix of the  $\rho$  meson and two-pion states. Tests of the fidelity of the fit in reproducing the correlator are given in Supplemental Material.

Rather than using a single fit ansatz, we carry out a model average over multiple values of  $t_{\min}$  using the Bayesian Akaike information criterion with the data-subset penalty [104,105], holding  $t_{\max}$  fixed at a value beyond which the noise-to-signal rises above 85%. The value of  $t_{\min}$  is allowed to vary over a restricted range. Log-normal priors are employed for the energies and amplitudes, following Ref. [106]; the prior for the constant term is set based on the energy and amplitude priors to match the leading constant contribution of  $Z_0^2 e^{-E_\pi T}$ . Choices of fit hyperparameters and results for ground-state energies are given in Supplemental Material.

Our final  $C(t)$ , which enters into Eqs. (1) and (2), is constructed by taking the raw data  $C(t)$  for  $t < t^*$  and  $C_{\text{recon}}(t)$  for  $t \geq t^*$ ; the latter corresponds to the prediction of the fit model Eq. (4) evaluated in the limit  $T \rightarrow \infty$ . Other variations of the analysis choices, such as changing the number of states, have been tested and have negligible impact on the results.

Following the correlator reconstruction, to calculate  $a_\mu$ , we extend the independent time range of our correlation function dataset from  $N_t/2 + 1$  to  $2N_t$  using the infinite- $T$  correlator reconstruction and then integrate Eqs. (1) and (2) using the trapezoidal rule. All error propagation is carried out using GVAR [107], cross-checked using jackknife resampling.

*Lattice corrections*—We perform explicit corrections for finite volume (FV), pion-mass ( $M_\pi$ ) mistuning, and (optionally) taste breaking (TB) in that order, using the effective field theory based correction schemes described in Ref. [67]; details are shown in Supplemental Material. Finite-time corrections, computed using next-to-leading order chiral perturbation theory (NLO  $\chi$ PT) [55], on the contribution from  $t < t^*$  are negligible at the current level of precision.

*Continuum extrapolations*—The continuum extrapolations of our  $a_\mu^{\text{LL,LD}}(\text{conn})$  and  $a_\mu^{\text{LL}}(\text{conn})$  data are based on

the form  $a_\mu(a, M_A) = a_\mu[1 + F^a(a) + F^M(M_A)]$ , where  $F^M(M_A)$  accounts for residual sea-quark mass-mistuning effects and is given in Eq. (3.7) of Ref. [67] and

$$F^a(a) = C_{a^2,n}(a\Lambda)^2 \alpha_s^n + \sum_{k=2}^m C_{a^{2k}}(a\Lambda)^{2k}, \quad (5)$$

where  $n = 1, 2$  ( $n = 0$ ) for the local (one-link) current, and  $m = 2, 3$ . As described in Ref. [67], the Gaussian prior  $C_{a^{2m}} = 0(2)$  is imposed on the highest-order terms. In joint fits to data obtained from the local and one-link currents, the parameters in  $F^M(M_A)$  and  $a_\mu$  are shared.

We first employ the empirical Bayes procedure discussed in Sec. III.C of Ref. [67] to obtain guidance for the choice of scale  $\Lambda$  and the relevant terms in the continuum fit function. The procedure is performed for both currents independently and jointly, and separately for data corrected and not corrected for TB effects. After varying the correction schemes and windows, as well as correlator-reconstruction methods, we find  $\Lambda \approx 0.5$  GeV for data corrected for TB effects and  $\Lambda \approx 1.0$  GeV for uncorrected data, reflecting the fact that discretization effects are larger in the uncorrected  $a_\mu^{\text{LL}}(\text{conn})$  data. The empirical Bayes analysis further reveals that the data corrected for TB is sensitive to discretization terms up to  $a^4$ , whereas the uncorrected data is sensitive to terms up to  $a^6$ . Hence, we consider variations of the functions  $F^a(a)$  where  $m = 2$  (quadratic) or  $m = 3$  (cubic). For the quadratic fits, we consider a variation where the 0.15 fm data point is dropped. For datasets not corrected for TB effects, we perform only joint fits and fix  $n = 2$  for the local current to account for the dominant TB effects [59].

*Continuum extrapolation BMA*—Systematic uncertainties are estimated using BMA [104,108], following the procedure and formulas detailed in Refs. [59,67]. In the BMA, we include variations for all FV,  $M_\pi$ -mistuning, and TB correction schemes discussed in Supplemental Material, namely next-to-next-to-leading order chiral perturbation theory, chiral model (CM), and Meyer-Lellouch-Lüscher-Gounaris-Sakurai. We always correct both currents using the same scheme. Extrapolations using Meyer-Lellouch-Lüscher-Gounaris-Sakurai exclude the coarsest data point (0.15 fm) as discussed in Ref. [67]. FV and  $M_\pi$ -mistuning corrections are always included, along with a 10% associated uncertainty to capture differences between effective field theory based and data-driven corrections [55,63,65]. Variations with and without TB corrections are considered.

We perform continuum extrapolations using all fit function variations described above, including fits that exclude one of the two currents entirely [67]. Sample individual results from the continuum extrapolation BMA for  $a_\mu^{\text{LL,LD}}(\text{conn})$  are shown in Fig. 2. Additional results and plots are shown in Supplemental Material.

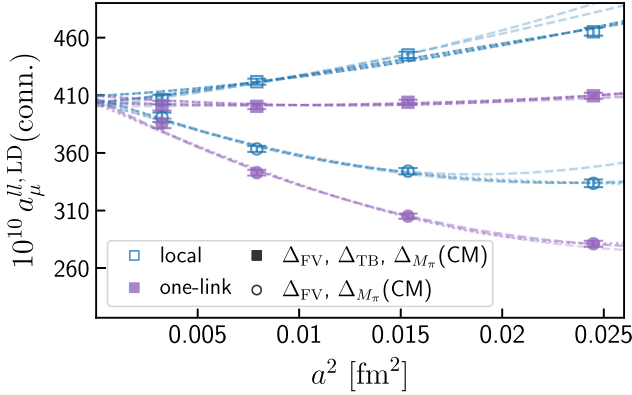


FIG. 2. Continuum extrapolations of the long-distance window observable  $a_\mu^{II,LD}(\text{conn.})$ . Data points shown are corrected for FV and  $M_\pi$  mistuning using the CM scheme. Blue unfilled (purple filled) points are from the local (one-link) current. Squares (circles) represent data corrected (uncorrected) for TB using the CM scheme. Different extrapolations are obtained from variations of the fit functions and ensembles included.

**Results**—Here, we report our main results, which use  $M_\Omega$  scale setting and the fit method for correlator reconstructions. Error budgets for all listed quantities below are given in Table II. Additional results for  $f_\pi$  scale setting or the bounding method are given in the End Matter. Note that Tables II and V list the scale error, which is useful for comparisons of results for scheme-dependent flavor-specific quantities from different lattice groups. Finally, the Supplemental Material (Table II) contains comparisons of all isospin-symmetric observables obtained here and in Ref. [67] with  $M_\Omega$ ,  $f_\pi$ , and  $w_0$  scale setting [109], including their correlated differences.

First, we take our results for  $a_\mu^{II,SD}(\text{conn})$  and  $a_\mu^{II,W}(\text{conn})$  from Ref. [67], which we restate here for convenience,

$$a_\mu^{II,SD}(\text{conn}) = 48.139(11)(91)[92] \times 10^{-10}, \quad (6)$$

TABLE II. Approximate absolute error budgets (in units of  $10^{-10}$ ) for the uncertainties reported in Eqs. (6)–(9). From left to right, the contributions to the error are Monte Carlo statistics and  $t_{\min}$  variations in the correlator reconstructions for  $a_\mu^{II,LD}$  and  $a_\mu^{II}$ , continuum extrapolation and TB corrections, FV and  $M_\pi$ -mistuning corrections, scale setting (due to the uncertainties in  $aM_\Omega$  and in the input  $M_\Omega$  mass in physical units), and current renormalization.

| Contribution    | Statistics,<br>$t_{\min}$ | $a \rightarrow 0$ ,<br>$\Delta_{\text{TB}}$ | $\Delta_{\text{FV}},$<br>$\Delta_{M_\pi}$ | $a$   | $Z_V$ | Total |
|-----------------|---------------------------|---------------------------------------------|-------------------------------------------|-------|-------|-------|
| $a_\mu^{II,SD}$ | 0.011                     | 0.060                                       | —                                         | 0.028 | 0.063 | 0.092 |
| $a_\mu^{II,W}$  | 0.14                      | 0.37                                        | 0.34                                      | 0.31  | 0.18  | 0.63  |
| $a_\mu^{II,LD}$ | 2.3                       | 2.8                                         | 1.4                                       | 1.9   | 0.2   | 4.3   |
| $a_\mu^{II}$    | 2.3                       | 2.9                                         | 1.7                                       | 1.8   | 0.3   | 4.5   |

$$a_\mu^{II,W}(\text{conn}) = 206.90(14)(62)[63] \times 10^{-10}. \quad (7)$$

The uncertainties are statistical (second column of Table II), systematic (third to sixth columns of Table II), and total. For the long-distance window we obtain

$$a_\mu^{II,LD}(\text{conn}) = 400.2(2.3)(3.7)[4.3] \times 10^{-10}. \quad (8)$$

Summing the LQC contribution in the three windows, Eqs. (6)–(8), with the correlation matrix given in Table IV, yields

$$a_\mu^{II}(\text{conn}) = 655.2(2.3)(3.9)[4.5] \times 10^{-10}. \quad (9)$$

Finally, from the independent analysis of the full integration region we obtain  $a_\mu^{II,\text{Full}}(\text{conn}) = 654.2(2.4)(4.7) \times [5.3] \times 10^{-10}$ , a result that is completely consistent with Eq. (9) with a correlated difference of  $1.0(5.5) \times 10^{-10}$ .

**Summary and outlook**—As part of our ongoing lattice QCD project to calculate the complete HVP with few-permille precision, we have calculated  $a_\mu^{II,LD}(\text{conn})$  at 1.1% precision, while we obtain  $a_\mu^{II}(\text{conn})$  with a total uncertainty of 0.69%, the most precise determination of this important quantity from lattice QCD to date [110]. These results are made possible by the high-statistics correlation function dataset we have generated to date [67], as well as the new high-precision scale setting using the  $\Omega^-$  baryon mass [75,76].

A comparison of our results for  $a_\mu^{II,LD}(\text{conn})$ , using both  $M_\Omega$  and  $f_\pi$  scale settings, with those of the RBC/UKQCD [63] and Mainz [65] collaborations as well as a data-driven evaluation [68] is shown in Fig. 3. Using the same scale setting in the comparisons, we find that our result is lower than RBC/UKQCD 24 ( $M_\Omega$ ) and Mainz 24 ( $f_\pi$ ) by  $1.7\sigma$  and  $3.6\sigma$ , respectively, while being higher than the data-driven evaluation of Ref. [68] by  $2.2\sigma$ , if using  $M_\Omega$ . In Fig. 4 we compare our result for  $a_\mu^{II}(\text{conn})$  with previous lattice-QCD calculations as well as the data-driven evaluations of Ref. [68]. Focusing on the three most precise

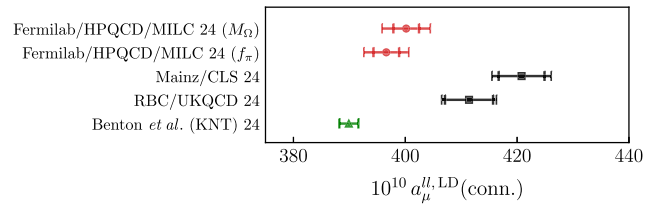


FIG. 3. Comparison of our lattice determinations for  $a_\mu^{II,LD}(\text{conn})$  (filled red circles) labeled “Fermilab/HPQCD/MILC 24” with  $n_f = 2 + 1$  (black squares) lattice-QCD calculations by Mainz/CLS 24 [65], RBC/UKQCD 24 [63]. The inner error bar shown for our result is from Monte Carlo statistics. Also shown is a data-driven evaluation of  $a_\mu^{II,LD}(\text{conn})$  using  $e^+e^-$  cross section data (green triangle) by Benton *et al.* 24 [68].

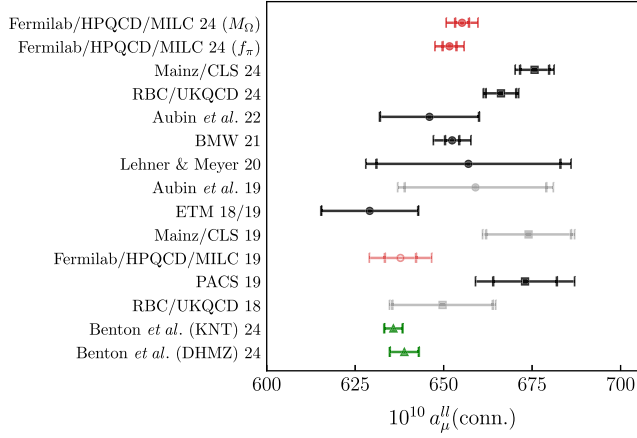


FIG. 4. Comparison of our lattice determinations for  $a_\mu^{\text{II}}(\text{conn})$  (filled red circles) labeled “Fermilab/HPQCD/MILC 24” with  $n_f = 2 + 1 + 1$  (black circles) and  $n_f = 2 + 1$  (black squares) lattice-QCD calculations by Mainz/CLS 24 [65], RBC/UKQCD 24 [63], Aubin *et al.* 22 [56], BMW 21 [55], Lehner & Meyer 20 [111], Aubin *et al.* 19 [112], ETM 18/19 [113], Mainz/CLS 19 [114], PACS 19 [115], and RBC/UKQCD 18 [69]. Our previous determination, Fermilab/HPQCD/MILC 19 [116], is shown with an unfilled red circle. Inner error bars indicate the reported statistical uncertainty. Also shown are data-driven evaluations of  $a_\mu^{\text{II}}(\text{conn})$  using  $e^+e^-$  cross section data (green triangles) by Benton *et al.* 24 [68].

lattice-QCD results, we find significances in the differences of  $0.4\sigma$  with BMW 21 ( $M_\Omega$ ),  $1.6\sigma$  with RBC/UKQCD 24 ( $M_\Omega$ ), and  $3.5\sigma$  with Mainz 24 ( $f_\pi$ ). We find a difference of  $1.4\sigma$  with our previous result of Ref. [106], if using the  $f_\pi$  scale. For a more direct comparison with Ref. [106] we take our result from the direct analysis of the full integration region obtained with  $f_\pi$  scale setting of  $a_\mu^{\text{II,Full}}(\text{conn}) = 650.5(2.4)(4.4)[5.0] \times 10^{-10}$ , which yields a significance of  $1.3\sigma$ . Finally, we find that our result in Eq. (9) differs from the data-driven evaluation of Ref. [68] by  $3.8\sigma$  ( $2.7\sigma$ ) if using the KNT (DHMZ) compilations, confirming the tensions between lattice-QCD and data-driven evaluations seen previously and in windowed HVP observables. More precise lattice-QCD calculations are clearly needed.

Looking toward future improvements of our HVP calculations, we are generating correlators on ensembles with both a finer lattice spacing  $a \approx 0.04$  fm and with a larger spatial length  $L \approx 11$  fm at  $a \approx 0.09$  fm, both of which will help to address significant sources of uncertainty (as shown in Table II.) We also plan to add exclusive two-pion channel correlator data, which is expected to greatly improve the statistical precision for LD HVP, even for staggered fermions [102], as well as parallel improved calculations of subleading contributions due to quark-disconnected diagrams and isospin-breaking effects [67]. Taken all together, we expect that these improvements will allow us to meet the target precision set by the Fermilab  $g - 2$  experiment.

**Acknowledgments**—We are grateful to Yin Lin, Aaron Meyer, and Michael Wagman for their contributions to the  $\Omega^-$  scale-setting project, the results of which are used in this work. We thank Urs Heller for his contributions to our projects on gauge flow and the pion decay constant, also used in scale setting. Computations for this work were carried out in part with computing and long-term storage resources provided by the USQCD Collaboration, the National Energy Research Scientific Computing Center (Edison, Cori, Perlmutter), the Argonne Leadership Computing Facility (Mira, Theta) under the INCITE program, and the Oak Ridge Leadership Computing Facility (Summit, Frontier) under the Innovative and Novel Computational Impact on Theory and Experiment (INCITE) and the ASCR Leadership Computing Challenge (ALCC) programs. The Argonne Leadership Computing Facility (ALCF), is supported by the U.S. Department of Energy (USDOE) under contract DE-AC02-06CH11357. The National Energy Research Scientific Computing Center (NERSC) is located at Lawrence Berkeley National Laboratory and operated under USDOE Contract No. DE-AC02-05CH11231. (Our awards are HEP-ERCAP0032813, HEP-ERCAP0031326, and possible predecessors.) The Oak Ridge Leadership Computing Facility (OLCF) at the Oak Ridge National Laboratory is supported by the USDOE under Contract No. DE-AC05-00OR22725. ALCF, NERSC, and OLCF are all USDOE Office of Science User Facilities. This work used the Extreme Science and Engineering Discovery Environment (XSEDE) supercomputer Stampede 2 at the Texas Advanced Computing Center (TACC) through allocation TG-MCA93S002. The XSEDE program is supported by the National Science Foundation under grant number ACI-1548562. Computations on the Big Red II+, Big Red 3, and Big Red 200 supercomputers were supported in part by Lilly Endowment, Inc., through its support for the Indiana University Pervasive Technology Institute. The parallel file system employed by Big Red II+ was supported by the National Science Foundation under Grant No. CNS-0521433. This work utilized the RMACC Summit supercomputer, which is supported by the National Science Foundation (awards ACI-1532235 and ACI-1532236), the University of Colorado Boulder, and Colorado State University. The Summit supercomputer is a joint effort of the University of Colorado Boulder and Colorado State University. Some of the computations were done using the Blue Waters sustained-petascale computer, which was supported by the National Science Foundation (awards OCI-0725070 and ACI-1238993) and the State of Illinois. Blue Waters was a joint effort of the University of Illinois at Urbana-Champaign and its National Center for Supercomputing Applications. This work was supported in part by the U.S. Department of Energy, Office of Science, under Awards No. DE-SC0010005 (E. T. N. and J. W. S.), No. DE-SC0010120 (S.G.), No. DE-SC0015655

(A. X. K., S. L., M. L., A. T. L.), No. DE-SC0009998 (J. L.), the “High Energy Physics Computing Traineeship for Lattice Gauge Theory” No. DE-SC0024053 (J. W. S.), and the Funding Opportunity Announcement Scientific Discovery through Advanced Computing: High Energy Physics, LAB 22-2580 (D. A. C., C. T. P., L. H., M. L., S. L.); by the Exascale Computing Project (17-SC-20-SC), a collaborative effort of the U.S. Department of Energy Office of Science and the National Nuclear Security Administration (H. J.); by the National Science Foundation under Grants No. PHY20-13064 and No. PHY23-10571 (C. D., D. A. C., S. L., A. V.), No. PHY23-09946 (A. B.), No. 2139536 for Characteristic Science Applications for the Leadership Class Computing Facility (L. H., H. J.); by the Simons Foundation under their Simons Fellows in Theoretical Physics program (A. X. K.); by the Universities Research Association Visiting Scholarship awards 20-S-12 and 21-S-05 (S. L.); by MICIU/AEI/10.13039/501100011033 and FEDER (EU) under Grant PID2022-140440NB-C21 (E. G.); by Consejería de Universidad, Investigación e Innovación and Gobierno de España and EU–NextGenerationEU, under Grant AST22 8.4 (E. G.); by AEI (Spain) under Grant No. RYC2020-030244-I / AEI / 10.13039/501100011033 (A. V.); and by U.K. Science and Technology Facilities Council under Grant ST/T000945/1 (C. T. H. D). A. X. K. and E. T. N. are grateful to the Pauli Center for Theoretical Studies and the ETH Zürich for support and hospitality. A. X. K., A. S. K., and E. T. N. are grateful to the Kavli Institute for Theoretical Physics (KITP) for hospitality and support during the “What is Particle Theory?” program. The K. I. T. P. is supported in part by the National Science Foundation under Grant PHY-2309135. This document was prepared by the Fermilab Lattice, HPQCD, and MILC Collaborations using the resources of the Fermi National Accelerator Laboratory (Fermilab), a U.S. Department of Energy, Office of Science, HEP User Facility. Fermilab is managed by Fermi Research Alliance, LLC (FRA), acting under Contract No. DE-AC02-07CH11359.

*Data availability*—The data that support the findings of this Letter are openly available [93].

- [1] B. Abi *et al.* (Muon  $g-2$  Collaboration), *Phys. Rev. Lett.* **126**, 141801 (2021).
- [2] D. P. Aguillard *et al.* (Muon  $g-2$  Collaboration), *Phys. Rev. Lett.* **131**, 161802 (2023).
- [3] T. Aoyama *et al.*, *Phys. Rep.* **887**, 1 (2020).
- [4] G. W. Bennett *et al.* (Muon  $g-2$  Collaboration), *Phys. Rev. D* **73**, 072003 (2006).
- [5] M. Abe *et al.*, *Prog. Theor. Exp. Phys.* **2019**, 053C02 (2019).
- [6] J-PARC muon  $g-2$ /EDM experiment, <https://g-2.kek.jp/>.
- [7] G. Colangelo *et al.*, [arXiv:2203.15810](https://arxiv.org/abs/2203.15810).
- [8] K. Melnikov and A. Vainshtein, *Phys. Rev. D* **70**, 113006 (2004).
- [9] P. Masjuan and P. Sánchez-Puertas, *Phys. Rev. D* **95**, 054026 (2017).
- [10] G. Colangelo, M. Hoferichter, M. Procura, and P. Stoffer, *J. High Energy Phys.* **04** (2017) 161.
- [11] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold, and S. P. Schneider, *J. High Energy Phys.* **10** (2018) 141.
- [12] A. Gérardin, H. B. Meyer, and A. Nyffeler, *Phys. Rev. D* **100**, 034520 (2019).
- [13] J. Bijnens, N. Hermansson-Truedsson, and A. Rodríguez-Sánchez, *Phys. Lett. B* **798**, 134994 (2019).
- [14] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, and P. Stoffer, *J. High Energy Phys.* **03** (2020) 101.
- [15] V. Pauk and M. Vanderhaeghen, *Eur. Phys. J. C* **74**, 3008 (2014).
- [16] I. Danilkin and M. Vanderhaeghen, *Phys. Rev. D* **95**, 014019 (2017).
- [17] F. Jegerlehner, *The Anomalous Magnetic Moment of the Muon* (Springer, Cham, 2017), Vol. 274.
- [18] M. Knecht, S. Narison, A. Rabemananjara, and D. Rabetiarivony, *Phys. Lett. B* **787**, 111 (2018).
- [19] G. Eichmann, C. S. Fischer, and R. Williams, *Phys. Rev. D* **101**, 054015 (2020).
- [20] P. Roig and P. Sánchez-Puertas, *Phys. Rev. D* **101**, 074019 (2020).
- [21] J. Leutgeb, J. Mager, and A. Rebhan, *Phys. Rev. D* **107**, 054021 (2023).
- [22] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, and C. Lehner (RBC Collaboration), *Phys. Rev. Lett.* **124**, 132002 (2020).
- [23] E.-H. Chao, R. J. Hudspith, A. Gérardin, J. R. Green, H. B. Meyer, and K. Otnad, *Eur. Phys. J. C* **81**, 651 (2021).
- [24] J. Bijnens, N. Hermansson-Truedsson, L. Laub, and A. Rodríguez-Sánchez, *J. High Energy Phys.* **10** (2020) 203.
- [25] J. Lütke and M. Procura, *Eur. Phys. J. C* **80**, 1108 (2020).
- [26] J. Bijnens, N. Hermansson-Truedsson, L. Laub, and A. Rodríguez-Sánchez, *J. High Energy Phys.* **04** (2021) 240.
- [27] M. Hoferichter and P. Stoffer, *J. High Energy Phys.* **05** (2020) 159.
- [28] J. Leutgeb and A. Rebhan, *Phys. Rev. D* **104**, 094017 (2021).
- [29] M. Zanke, M. Hoferichter, and B. Kubis, *J. High Energy Phys.* **07** (2021) 106.
- [30] I. Danilkin, M. Hoferichter, and P. Stoffer, *Phys. Lett. B* **820**, 136502 (2021).
- [31] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, and P. Stoffer, *Eur. Phys. J. C* **81**, 702 (2021).
- [32] L. Cappiello, O. Catà, and G. D’Ambrosio, *Phys. Rev. D* **105**, 056020 (2022).
- [33] J. Bijnens, N. Hermansson-Truedsson, and A. Rodríguez-Sánchez, *J. High Energy Phys.* **02** (2023) 167.
- [34] M. Hoferichter, B. Kubis, and M. Zanke, *J. High Energy Phys.* **08** (2023) 209.
- [35] J. Lütke, M. Procura, and P. Stoffer, *J. High Energy Phys.* **04** (2023) 125.
- [36] P. Colangelo, F. Giannuzzi, and S. Nicotri, *Phys. Lett. B* **840**, 137878 (2023).

- [37] M. Hoferichter, P. Stoffer, and M. Zillinger, *J. High Energy Phys.* **04** (2024) 092.
- [38] M. Hoferichter, P. Stoffer, and M. Zillinger, *Phys. Rev. Lett.* **134**, 061902 (2025).
- [39] M. Hoferichter, P. Stoffer, and M. Zillinger, *J. High Energy Phys.* **02** (2025) 121.
- [40] J. Bijmans, N. Hermansson-Truedsson, and A. Rodríguez-Sánchez, *J. High Energy Phys.* **03** (2025) 094.
- [41] S. Holz, C. Hanhart, M. Hoferichter, and B. Kubis, *Eur. Phys. J. C* **82**, 434 (2022); **82**, 1159(A) (2022).
- [42] S. Holz, M. Hoferichter, B.-L. Hoid, and B. Kubis, *Phys. Rev. Lett.* **134**, 171902 (2025).
- [43] J. Leutgeb, J. Mager, and A. Rebhan, *arXiv:2411.10432*.
- [44] P. Colangelo, F. Giannuzzi, and S. Nicotri, *Phys. Rev. D* **109**, 094036 (2024).
- [45] E. J. Estrada, J. M. Márquez, D. Portillo-Sánchez, and P. Roig, *arXiv:2411.07115*.
- [46] A. S. Miramontes, K. Raya, A. Bashir, P. Roig, and G. Paredes-Torres, *arXiv:2411.02218*.
- [47] E. J. Estrada, S. González-Solís, A. Guevara, and P. Roig, *J. High Energy Phys.* **12** (2024) 203.
- [48] N. Asmussen, E.-H. Chao, A. Gérardin, J. R. Green, R. J. Hudspith, H. B. Meyer, and A. Nyffeler, *J. High Energy Phys.* **04** (2023) 040.
- [49] E.-H. Chao, R. J. Hudspith, A. Gérardin, J. R. Green, and H. B. Meyer, *Eur. Phys. J. C* **82**, 664 (2022).
- [50] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner, and C. Tu, *Phys. Rev. D* **111**, 014501 (2025).
- [51] A. Gérardin, W. E. A. Verplanke, G. Wang, Z. Fodor, J. N. Guenther, L. Lellouch, K. K. Szabo, and L. Varnhorst, *Phys. Rev. D* **111**, 054511 (2025).
- [52] C. Alexandrou *et al.* (Extended Twisted Mass Collaboration), *Phys. Rev. D* **108**, 094514 (2023).
- [53] T. Lin, M. Bruno, X. Feng, L.-C. Jin, C. Lehner, C. Liu, and Q.-Y. Luo, *arXiv:2411.06349*.
- [54] Z. Fodor, A. Gerardin, L. Lellouch, K. K. Szabo, B. C. Toth, and C. Zimmermann, *arXiv:2411.11719*.
- [55] S. Borsanyi *et al.* (BMW Collaboration), *Nature (London)* **593**, 51 (2021).
- [56] C. Aubin, T. Blum, M. Golterman, and S. Peris, *Phys. Rev. D* **106**, 054503 (2022).
- [57] C. Alexandrou *et al.* (Extended Twisted Mass Collaboration), *Phys. Rev. D* **107**, 074506 (2023).
- [58] M. Cè *et al.*, *Phys. Rev. D* **106**, 114502 (2022).
- [59] A. Bazavov *et al.* (Fermilab Lattice, HPQCD, MILC Collaborations), *Phys. Rev. D* **107**, 114514 (2023).
- [60] T. Blum *et al.* (RBC, UKQCD Collaborations), *Phys. Rev. D* **108**, 054507 (2023).
- [61] A. Boccaletti *et al.*, *arXiv:2407.10913*.
- [62] S. Kuberski, M. Cè, G. von Hippel, H. B. Meyer, K. Ottnad, A. Risch, and H. Wittig, *J. High Energy Phys.* **03** (2024) 172.
- [63] T. Blum *et al.* (RBC, UKQCD Collaborations), *Phys. Rev. Lett.* **134**, 201901 (2025).
- [64] S. Spiegel and C. Lehner, *arXiv:2410.17053*.
- [65] D. Djukanovic, G. von Hippel, S. Kuberski, H. B. Meyer, N. Miller, K. Ottnad, J. Parrino, A. Risch, and H. Wittig, *J. High Energy Phys.* **04** (2025) 098.
- [66] C. Alexandrou *et al.* (Extended Twisted Mass Collaboration (ETMC)), *Phys. Rev. D* **111**, 054502 (2025).
- [67] A. Bazavov *et al.* (Fermilab Lattice, HPQCD, MILC Collaborations), *Phys. Rev. D* **111**, 094508 (2025).
- [68] G. Benton, D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, and S. Peris, *Phys. Rev. D* **111**, 034018 (2025).
- [69] T. Blum, P. A. Boyle, V. Gülpers, T. Izubuchi, L. Jin, C. Jung, A. Jüttner, C. Lehner, A. Portelli, and J. T. Tsang (RBC, UKQCD Collaborations), *Phys. Rev. Lett.* **121**, 022003 (2018).
- [70] G. Colangelo, A. X. El-Khadra, M. Hoferichter, A. Keshavarzi, C. Lehner, P. Stoffer, and T. Teubner, *Phys. Lett. B* **833**, 137313 (2022).
- [71] G. Benton, D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, and S. Peris, *Phys. Rev. Lett.* **131**, 251803 (2023).
- [72] M. Davier, Z. Fodor, A. Gerardin, L. Lellouch, B. Malaescu, F. M. Stokes, K. K. Szabo, B. C. Toth, L. Varnhorst, and Z. Zhang, *Phys. Rev. D* **109**, 076019 (2024).
- [73] G. Benton, D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, and S. Peris, *Phys. Rev. D* **109**, 036010 (2024).
- [74] M. Della Morte, A. Francis, V. Gülpers, G. Herdoíza, G. von Hippel, H. Horsch, B. Jäger, H. B. Meyer, A. Nyffeler, and H. Wittig, *J. High Energy Phys.* **10** (2017) 020.
- [75] A. Bazavov *et al.*, *Proc. Sci. LATTICE2023* (2024) 292 [*arXiv:2401.06522*].
- [76] A. Bazavov *et al.* (Fermilab Lattice, MILC) (to be published).
- [77] T. Blum, *Phys. Rev. Lett.* **91**, 052001 (2003).
- [78] D. Bernecker and H. B. Meyer, *Eur. Phys. J. A* **47**, 148 (2011).
- [79] A. Bazavov *et al.* (Fermilab Lattice, HPQCD, MILC) (to be published).
- [80] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/d583-yhfs>, which includes Refs. [81–84], for additional details on the noise-reduction strategy, lattice corrections, continuum extrapolation, and Bayesian model averaging procedures.
- [81] S. Borsanyi *et al.* (Budapest-Marseille-Wuppertal Collaboration), *Phys. Rev. Lett.* **121**, 022002 (2018).
- [82] G. P. Lepage *et al.*, *Nucl. Phys. B, Proc. Suppl.* **106**, 12 (2002).
- [83] C. M. Bouchard, G. P. Lepage, C. Monahan, H. Na, and J. Shigemitsu, *Phys. Rev. D* **90**, 054506 (2014).
- [84] A. Bazavov *et al.* (Fermilab Lattice, MILC Collaborations), *Phys. Rev. D* **93**, 113016 (2016).
- [85] Y. Aoki *et al.* (Flavour Lattice Averaging Group (FLAG) Collaboration), *arXiv:2411.04268*.
- [86] A. Bazavov *et al.* (Fermilab Lattice, MILC Collaborations), *Phys. Rev. D* **90**, 074509 (2014).
- [87] B. Chakraborty *et al.* (Fermilab Lattice, HPQCD, MILC Collaborations), *Phys. Rev. Lett.* **120**, 152001 (2018).
- [88] A. Bazavov, C. Bernard *et al.* (Fermilab Lattice, MILC) (to be published).
- [89] R. L. Workman *et al.* (Particle Data Group), *Prog. Theor. Exp. Phys.* **2022**, 083C01 (2022).
- [90] A. Bazavov *et al.* (MILC Collaboration), *Phys. Rev. D* **82**, 074501 (2010).

- [91] A. Bazavov *et al.* (MILC Collaboration), *Phys. Rev. D* **87**, 054505 (2013).
- [92] A. Bazavov *et al.* (Fermilab Lattice, MILC Collaborations), *Phys. Rev. D* **98**, 074512 (2018).
- [93] MILC, GitHub (2021), <https://github.com/milc-qcd/sharing/wiki/LatticeSharing>.
- [94] The 0.09 fm ensemble was, in part, generated by the CalLat Collaboration [95] using retuned values of the quark masses determined by MILC [86].
- [95] N. Miller *et al.*, *Phys. Rev. D* **103**, 054511 (2021).
- [96] D. Hatton, C. T. H. Davies, G. P. Lepage, and A. T. Lytle (HPQCD Collaboration), *Phys. Rev. D* **100**, 114513 (2019).
- [97] D. Hatton, C. T. H. Davies, B. Galloway, J. Koponen, G. P. Lepage, and A. T. Lytle (HPQCD Collaboration), *Phys. Rev. D* **102**, 054511 (2020).
- [98] T. A. DeGrand and S. Schaefer, *Comput. Phys. Commun.* **159**, 185 (2004).
- [99] L. Giusti, P. Hernandez, M. Laine, P. Weisz, and H. Wittig, *J. High Energy Phys.* **04** (2004) 013.
- [100] T. Blum, T. Izubuchi, and E. Shintani, *Phys. Rev. D* **88**, 094503 (2013).
- [101] G. P. Lepage, in *Theoretical Advanced Study Institute in Elementary Particle Physics* (Cornell, Ithaca, 1989), <https://lib-extopc.kek.jp/preprints/PDF/1990/9003/9003479.pdf>.
- [102] S. Lahert, C. DeTar, A. X. El-Khadra, S. Gottlieb, A. S. Kronfeld, and R. S. Van de Water, *arXiv:2409.00756*.
- [103] B. Chakraborty, C. T. H. Davies, P. G. de Oliveira, J. Koponen, G. P. Lepage, and R. S. Van de Water (HPQCD Collaboration), *Phys. Rev. D* **96**, 034516 (2017).
- [104] E. T. Neil and J. W. Sitison, *Phys. Rev. D* **109**, 014510 (2024).
- [105] E. T. Neil and J. W. Sitison, *Phys. Rev. E* **108**, 045308 (2023).
- [106] C. T. H. Davies *et al.* (Fermilab Lattice, HPQCD, MILC Collaborations), *Phys. Rev. D* **101**, 034512 (2020).
- [107] G. Lepage, C. Gohlke, and D. Hackett, *gplepage/gvar* v11.10 (2022).
- [108] W. I. Jay and E. T. Neil, *Phys. Rev. D* **103**, 114502 (2021).
- [109] We follow the Muon g-2 Theory Initiative's white paper scheme (WP25) and take  $w_0$  from Ref. [55] as a fixed value,  $w_0 = 0.17236$  fm.
- [110] Our results for  $a_\mu^{ll,LD}(\text{conn})$  and  $a_\mu^{ll}(\text{conn})$  obtained from  $f_\pi$  scale setting are at 1.0% and 0.63% precision, respectively.
- [111] C. Lehner and A. S. Meyer, *Phys. Rev. D* **101**, 074515 (2020).
- [112] C. Aubin, T. Blum, C. Tu, M. Golterman, C. Jung, and S. Peris, *Phys. Rev. D* **101**, 014503 (2020).
- [113] D. Giusti, F. Sanfilippo, and S. Simula, *Phys. Rev. D* **98**, 114504 (2018).
- [114] A. Gérardin, M. Cè, G. von Hippel, B. Hörz, H. B. Meyer, D. Mohler, K. Otnad, J. Wilhelm, and H. Wittig, *Phys. Rev. D* **100**, 014510 (2019).
- [115] E. Shintani and Y. Kuramashi (PACS Collaboration), *Phys. Rev. D* **100**, 034517 (2019).
- [116] C. E. DeTar *et al.* (Fermilab Lattice, HPQCD, MILC Collaborations), *Proc. Sci. LATTICE2019* (2019) 070.
- [117] G. Wang, T. Draper, K.-F. Liu, and Y.-B. Yang ( $\chi$ QCD Collaboration), *Phys. Rev. D* **107**, 034513 (2023).

## End Matter

Here, we collect additional results for the various scale-setting schemes and correlator-reconstruction strategies considered. We compare our updated results for

$a_\mu^{ll,SD}(\text{conn})$  and  $a_\mu^{ll,W}(\text{conn})$  to previous determinations in Fig. 5. Final results for the scale-setting and correlator-reconstruction variations considered are listed in Table III.

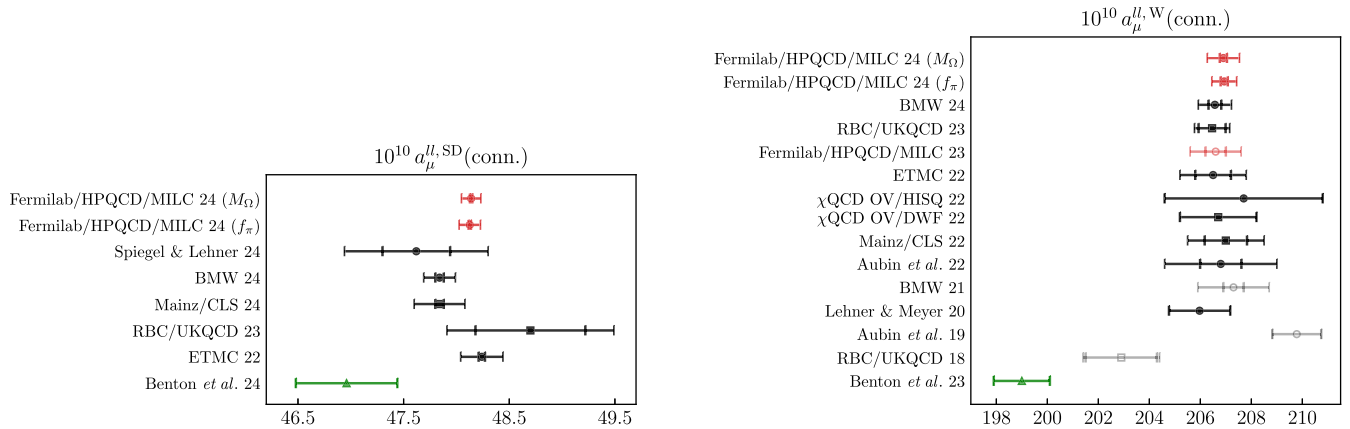


FIG. 5. Comparison of our lattice determinations for  $a_\mu^{ll,SD}(\text{conn})$  (left) and  $a_\mu^{ll,W}(\text{conn})$  (right) (red circles) labeled “Fermilab/HPQCD/MILC 24 ( $M_Q$ )” and “Fermilab/HPQCD/MILC 24 ( $f_\pi$ )” to  $n_f = 2 + 1 + 1$  (black circles) and  $n_f = 2 + 1$  (black squares) lattice-QCD calculations by Spiegel & Lehner 24 [64], BMW 24 [61], Mainz/CLS 24 [62], RBC/UKQCD 23 [60], Mainz/CLS 22 [58], Aubin *et al.* 22 [56],  $\chi$ QCD 22 [117], ETMC 22 [57], and Lehner & Meyer 20 [111]. Our previous result, Fermilab/HPQCD/MILC 23, is shown in light red. BMW 21 [55], Aubin *et al.* 19 [112], and RBC/UKQCD 18 [69], shown in gray, have been superseded. The inner error bar shown for our result is from Monte Carlo statistics. Also shown is a data-driven evaluation of  $a_\mu^{ll,SD}(\text{conn})$  and  $a_\mu^{ll,W}(\text{conn})$  using  $e^+e^-$  cross-section data (green triangles) by Benton *et al.* 24 [68] and Benton *et al.* 23 [68,71], respectively.

TABLE III. Final results for  $a_\mu$  (in units of  $10^{-10}$ ). The first column lists the specific LQC contribution. Columns two and three give results using the fit method [or just raw data in the case of  $a_\mu^{ll,SD}(\text{conn})$  and  $a_\mu^{ll,W}(\text{conn})$ ] based on scale setting via  $M_\Omega$  and  $f_\pi$ , respectively. The correlated difference between these values is given in column four. The final two columns list the results corresponding to columns two and three using the bounding method. The small differences in continuum-extrapolated results between the fit and bounding methods are predominantly driven by a fluctuation in the finest  $a \sim 0.06$  fm data, which is discussed further in Supplemental Material.

| Contribution      | Fit (or raw data)    |                      |                            | Bounding             |                      |
|-------------------|----------------------|----------------------|----------------------------|----------------------|----------------------|
|                   | $M_\Omega$           | $f_\pi$              | $\Delta_{M_\Omega, f_\pi}$ | $M_\Omega$           | $f_\pi$              |
| $a_\mu^{ll,SD}$   | 48.139(11)(91)[92]   | 48.126(15)(100)[101] | 0.013(78)                  | —                    | —                    |
| $a_\mu^{ll,W}$    | 206.90(14)(61)[63]   | 206.94(15)(46)[49]   | −0.03(43)                  | —                    | —                    |
| $a_\mu^{ll,LD}$   | 400.2(2.3)(3.7)[4.3] | 396.6(2.2)(3.3)[4.0] | 3.6(3.9)                   | 405.1(3.6)(4.0)[5.3] | 401.0(3.4)(3.6)[5.0] |
| $a_\mu^{ll,Full}$ | 654.2(2.4)(4.7)[5.3] | 650.5(2.4)(4.4)[5.0] | 3.7(4.5)                   | 659.5(3.8)(5.3)[6.5] | 655.3(3.7)(5.2)[6.4] |
| $a_\mu^{ll}$      | 655.2(2.3)(3.9)[4.5] | 651.7(2.2)(3.5)[4.1] | 3.5(4.0)                   | 660.1(3.1)(4.5)[5.4] | 656.0(3.1)(4.0)[5.0] |

TABLE IV. Correlation matrices for the LQC contributions to the window observables using  $M_\Omega$  (left) and  $f_\pi$  (right) scale setting and (where applicable) the fit method correlator-reconstruction strategy.

|                   | $a_\mu^{ll,SD}$ | $a_\mu^{ll,W}$ | $a_\mu^{ll,LD}$ | $a_\mu^{ll,Full}$ |                   | $a_\mu^{ll,SD}$ | $a_\mu^{ll,W}$ | $a_\mu^{ll,LD}$ | $a_\mu^{ll,Full}$ |
|-------------------|-----------------|----------------|-----------------|-------------------|-------------------|-----------------|----------------|-----------------|-------------------|
| $a_\mu^{ll,SD}$   | 1.00            | 0.09           | 0.01            | 0.01              | $a_\mu^{ll,SD}$   | 1.00            | 0.19           | 0.02            | 0.02              |
| $a_\mu^{ll,W}$    | 0.09            | 1.00           | 0.18            | 0.18              | $a_\mu^{ll,W}$    | 0.19            | 1.00           | 0.16            | 0.15              |
| $a_\mu^{ll,LD}$   | 0.01            | 0.18           | 1.00            | 0.33              | $a_\mu^{ll,LD}$   | 0.02            | 0.16           | 1.00            | 0.31              |
| $a_\mu^{ll,Full}$ | 0.01            | 0.18           | 0.33            | 1.00              | $a_\mu^{ll,Full}$ | 0.02            | 0.15           | 0.31            | 1.00              |

TABLE V. Approximate absolute error budgets (in units of  $10^{-10}$ ) for the uncertainties reported in Table III for the  $f_\pi$  results using the fit method. From left to right, the contributions to the error are Monte Carlo statistics and  $t_{\min}$  variation in the correlator fits (where applicable), continuum extrapolation and TB corrections, FV and  $M_\pi$ -mistuning corrections, scale setting (due to the uncertainties in  $af_\pi$  and the input  $f_\pi$  value in physical units), and current renormalization.

| Contribution    | Statistics, $t_{\min}$ | $a \rightarrow 0$ , $\Delta_{TB}$ | $\Delta_{FV}$ , $\Delta_{M_\pi}$ | $a$   | $Z_V$ | Total |
|-----------------|------------------------|-----------------------------------|----------------------------------|-------|-------|-------|
| $a_\mu^{ll,SD}$ | 0.015                  | 0.072                             | —                                | 0.009 | 0.069 | 0.101 |
| $a_\mu^{ll,W}$  | 0.15                   | 0.21                              | 0.34                             | 0.14  | 0.19  | 0.49  |
| $a_\mu^{ll,LD}$ | 2.2                    | 2.7                               | 1.4                              | 1.4   | 0.2   | 4.0   |
| $a_\mu^{ll}$    | 2.2                    | 2.7                               | 1.7                              | 1.4   | 0.3   | 4.1   |

For the correlated differences in Table III, we take the BMA systematic contributions to the uncertainties to be 100% correlated for a given window observable across each scale.

For the fit method, we give the correlation matrices for these results using  $M_\Omega$  and  $f_\pi$  scale setting in Table IV, and error budgets for results using  $f_\pi$  are given in Table V.