Skeleton of Isometric Tensor Network States for Abelian String-Net Models

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We construct parametrized isometric tensor network states—referred to as "skeletons"—that allow us to explore phases of Abelian topological order and can be efficiently implemented on quantum processors. We obtain stable finite correlation length deformations of string-net fixed points, which are constructed both by conserving virtual symmetries of the tensor and by imposing local isometry constraints. They connect distinct topological phases via a shared critical point, thereby providing analytically tractable examples of phase transitions beyond anyon condensation. By mapping such classes of 2D tensor networks to 1D stochastic automata with local update rules, we show that expectation values of generalized Pauli strings of arbitrary weight can be efficiently computed using classical methods. Therefore these skeletons not only serve as an organizing principle for Abelian topological order but also provide a nontrivial test bed for quantum processors.

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Introduction—Tensor network states (TNSs) have emerged as an efficient tool to understand topological order [1–5]. Besides their strength as variational states, a number of exact wave function paths built from finite tensors are known that interpolate between different fixed points. In one dimension, skeletons of matrix product states have been identified that connect distinct symmetry-protected topological orders [6–8]. Such a transition has been implemented on a quantum device using a sequential circuit deduced from the matrix product state [9]. In higher dimensions, paths between topologically ordered and trivial phases can also be constructed [10–12]. However, these are more unwieldy than their 1D counterparts, as general 2D TNSs are less tractable [13–15] and do not have an efficient preparation protocol with unitary circuits [16]. Yet, fixed points of phases often exhibit more favorable properties: for instance, the fixed points of string-net models are isometric TNSs (isoTNSs) that allow for an efficient contraction of the bare tensor network and can be realized by means of a linear-depth sequential circuit [17–21]. Paths that keep these properties would thus serve as valuable tools to study topological phases and their transitions. Specific examples for such paths have already been introduced [22–24]. However, it remains open how such deformations can be systematically constructed.

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Here, we propose classes of exact continuous isoTNSs obtained from Abelian string-net models [25–31] that allow for analytical insights and precise numerical predictions. Leveraging the isometry condition on the tensors, we show that they belong to a set of quantum states that map to stochastic circuits, with Abelian string nets corresponding to update rules conserving paritylike quantities. This mapping ensures the presence of the topological order, as the gap of the associated circuit closes only when entries in the tensor are tuned to zero. By connecting different fixed points to such critical points, we find topological phase transitions beyond the anyon condensation paradigm [32–38]. This strategy allows us to map out the space of possible deformations as represented in Fig. 1: fixed points with the same local Hilbert space dimension form part of an interwoven network of stable directions with phase transitions at directly identifiable points; we call this network an "isoTNS skeleton." Furthermore, by utilizing that isoTNS skeletons map to stochastic circuits, we show that generalized Pauli strings of arbitrary weight can be evaluated efficiently via classical sampling methods. General operators, which are an exponential superposition of Pauli strings, can be measured classically only for sufficiently small support. Because of the relation between isoTNSs and linear-depth circuits, these states also serve as excellent higher-dimensional test beds for quantum devices, supporting finite correlation length and nontrivial many-body ordering, while still exhibiting a set of properties that admits efficient classical verification.

String-net models—A large number of gapped topologically ordered phases are represented by string-net models [25,30,31]. We consider Abelian string-net models, which

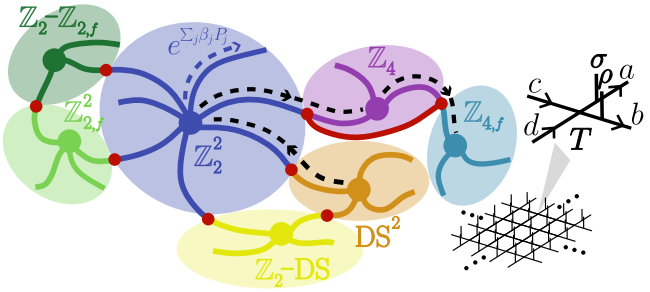


FIG. 1. Skeleton of isometric tensor network states. Starting from Abelian string-net models, we construct classes of isometric tensor network states within the topological phase of matter, connected by an imaginary time evolution. At some points (red dots), a critical state is reached from different topologically ordered phases, realizing a quantum phase transition. For local Hilbert space dimension $N = 4$, possible phases include pairs of toric code ($\mathbb{Z}_2$), pairs of double-semion (DS) states, or the $\mathbb{Z}_4$ toric code. The subscript f denotes phases with nontrivial symmetry fractionalization under an antiunitary symmetry. The dashed path is further investigated in Fig. 3. The isometric tensor network is illustrated on the right.

are fixed points for many Abelian topological orders. We explicitly discuss the case of the underlying Abelian group G being a cyclic group $G = \mathbb{Z}_N$; other finite Abelian groups can be treated equally as they are isomorphic to products of cyclic groups.

We consider a square lattice with a qudit of local Hilbert space dimension N on each edge. For a given N, we introduce the clock operator Z as $Z|j\rangle = e^{2\pi i j/N}|j\rangle$ and the shift operator X as $X|j\rangle = |(j+1) \bmod N\rangle$. Abelian string-net states can be written graphically as

$$|\psi\rangle \propto |\mathrm{vac}\rangle + \alpha_1 |\bigcirc\rangle + \alpha_2 |\bigcirc\!\!\bigcirc\rangle + \alpha_3 |\bigcirc\!\!\!\bigcirc\rangle + \cdots \qquad (1)$$

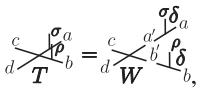
line styles indicate string types. At the fixed points, $|\alpha_c| = 1$ for every configuration c: the state is an equal-weight superposition of all closed-loop configurations, for which a parent Hamiltonian of commuting local projectors exists [25,39] [see Supplemental Material [40] (SM) Secs. A and C for a more thorough explanation].

String-net states can be expressed as TNSs [26,27,48]. Introducing a tensor $T^{\sigma\rho}_{abcd}$ at every vertex, with virtual legs a, b, c, d of bond dimension D and physical legs σ, ρ on two edges, the state is obtained by contracting the virtual legs,

$$|\psi\rangle = \sum_{\sigma_1,\ldots,\sigma_k} \mathrm{tTr}(\{T^{\sigma_1\sigma_2}, \ldots, T^{\sigma_{k-1}\sigma_k}\})|\sigma_1 \cdots \sigma_k\rangle, \qquad (2)$$

where tTr denotes this tensor contraction.

When the phases α_c depend on local properties as in the $\mathbb{Z}_N$ toric code, the state can be described by so-called "single-line TNS." The tensors lock physical and virtual legs and are decomposed as $T^{\sigma\rho}_{abcd} = \sum_{a'b'} \delta^{\sigma}_{aa'} \delta^{\rho}_{bb'} W_{(a'b')(cd)}$, or graphically as



(3)

where W is an $N^2 \times N^2$ matrix and δ is a "plumbing" tensor with $\delta^{\sigma}_{ab} = 1$ if $a = b = \sigma$ and zero otherwise. The bond dimension is equivalent to the physical dimension $D = N$.

A more general set that describes all Abelian string-net models are "double-line TNSs." They allow for the nonlocal phase factors of "nontoric code" string-net states [11,49,50]. The tensor has bond dimension $D = N^2$ and can be decomposed into a $N^4 \times N^4$ W matrix and tensors on the edges relating virtual and physical legs (see SM [40] Sec. A for a precise definition of the TNS). The TNS representation of every string-net fixed point preserves certain virtual symmetries that are associated with the underlying topological order. The virtual symmetries are implemented by operators that act solely within the total virtual Hilbert space, or more generally, virtual subspaces, spanned by the virtual legs of the tensor network (see SM [40] Sec. B for further details).

Single- and double-line representations of string nets are isoTNSs, which form an expressive subclass of TNSs with a contraction property similar to the canonical form of matrix product states; for single-line tensors, it is $\sum_{\sigma,\rho,c,d} T^{\sigma\rho}_{abcd} (T^*)^{\sigma\rho}_{a'b'cd} = \delta_{aa'}\delta_{bb'}$ [51]. A related structure holds for double-line tensors as well. IsoTNSs can be prepared using a linear-depth unitary circuit [19–22], making them accessible to current quantum devices [52,53].

Mapping to stochastic automata—In general, a single-line state defined as in Eq. (3) is an isoTNS if its W matrix is *normalized* [22] as

$$\sum_{c,d} |W_{(ab)(cd)}|^2 = 1, \quad \forall\ a, b. \qquad (4)$$

This allows for direct identification of the W matrix with the local update rule of a stochastic automaton: The value $|W_{(ab)(cd)}|^2$ expresses the probability $p(cd|ab)$ that two sites in the local states a and b change into states c and d. As physical and virtual legs are equivalent for a single-line tensor, the two-dimensional quantum state is mapped to a stochastic automaton circuit acting on a one-dimensional state as shown in Fig. 2: each tensor is an update step in a brickwork circuit, where space and time are identified with the diagonal directions of the original state. Starting from an open boundary, the quantum state is thus a superposition of all allowed trajectories weighed by the square root of their probability, possibly with complex phases (see also Ref. [54]).

The state at the boundary of the tensor network may be any matrix-product state, including the simplest case of a product state. For product states, local diagonal operators

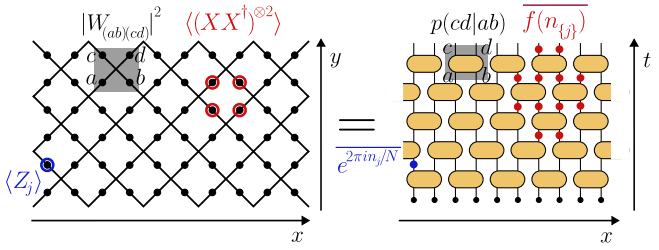


FIG. 2. Mapping to stochastic automata. The normalization property of the single-line W matrix (left) allows us to identify it with the two-local update rule of a stochastic automaton (right). From this, a tensor network state with an open boundary can be interpreted as a superposition of all paths of this stochastic process, where one spatial direction of the quantum state becomes the time direction of the automaton. Diagonal operators such as Z_j on site j can be evaluated with classical Monte Carlo methods from the local state n_j in the stochastic circuit. Generalized Pauli string operators such as loop operators $(XX^\dagger)^{\otimes 2}$ (red) can be mapped to functions of the classical state at different points in time and space $f(n_{\{j\}})$, and thereby efficiently evaluated as well.

can be directly evaluated using classical sampling methods. For example, the expectation value of the clock operator $Z(x,y)$ at a site given by coordinates x and y as shown in Fig. 2 can be evaluated using the identification $\langle Z(x,y)\rangle = \overline{e^{2\pi i n_x(t=y)/N}}$, where $n_x(t)$ is the value of the automaton at site x and time t and the average on the right-hand side is over realizations of the stochastic dynamics, which are initialized in the distribution given by the boundary state. This identification extends to all generalized Pauli strings formed by $Z^a X^b$, $\forall\ a,b \in \{0, N-1\}$, and $a+b>0$ (excluding the identity so that the number of nonidentity operators sets the weight of the Pauli string). This follows from absorbing the action of any nondiagonal Pauli operators (such as X) into Pauli operators at the virtual legs using the plumbing tensor,

$$\begin{array}{ccc} & X & \\ \sigma & & \delta \\ a & & b \end{array} = X^\dagger_a \begin{array}{ccc} & \sigma & \\ \delta & & \\ & b & \end{array} X^\dagger_b. \tag{5}$$

The virtual Pauli operators modify the local W matrices of the state, and thereby the coefficients of the wave function in the computational basis. Their action is equivalent to an operator diagonal in the computational basis; details of the procedure are presented in SM [40] Sec. D. In particular, we show how for any Pauli string O (of arbitrary weight) a diagonal operator O_{diag}, with support that is at most a constant multiple of that of the original operator O, can be explicitly constructed such that $\langle O\rangle = \langle O_{\text{diag}}\rangle$. Crucially, O_{diag} can always be decomposed into a product of local diagonal operators O_v, each of which acts on the four qubits adjacent to the vertex v, enabling efficient evaluation via the classical automaton (up to any fixed additive error). For example, a local loop operator of $(XX^\dagger)^{\otimes 2}$ corresponds to a function of 12 space-time points in the classical automaton; see Fig. 2. We emphasize that this property holds for all normalized single-line tensor-network states, including single-line TNSs in higher dimensions and on lattices with higher connectivity.

These identifications do not hold for generic tensor networks, as different virtual indices may correspond to the same local physical state. In general, it is neither possible to evaluate diagonal operators in terms of stochastic processes nor to map an operator to a corresponding diagonal operator. The situation is, however, much more favorable for double-line states that preserve the virtual symmetries of a string-net model. The string-net fixed point $|\psi_{\text{fix}}\rangle = U|\mathbb{Z}_N\rangle$ is connected to the N-level qudit toric code $|\mathbb{Z}_N\rangle$ by a global diagonal unitary U. At the same time, a deformed state $|\psi\rangle = \mathcal{T}|\psi_{\text{fix}}\rangle$ conserving all double-line symmetries is connected to its respective fixed point by a diagonal imaginary time evolution $\mathcal{T}$, which is constructed from the physical perturbations associated to the tensor deformation (see SM [40] Sec. B). The expectation value of a diagonal operator O_{diag} can be rewritten as

$$\langle\psi|O_{\text{diag}}|\psi\rangle = \langle\mathbb{Z}_N|U^\dagger\mathcal{T}^\dagger O_{\text{diag}}\mathcal{T}U|\mathbb{Z}_N\rangle = \langle\psi'|O_{\text{diag}}|\psi'\rangle, \tag{6}$$

where $|\psi'\rangle = \mathcal{T}|\mathbb{Z}_N\rangle$ is a deformed $\mathbb{Z}_N$ toric code and can thus be expressed with single-line tensors. Here, we have used that all operators $\mathcal{T}, O_{\text{diag}}, U$ are diagonal and therefore commute. If the original double-line tensor of $|\psi\rangle$ further is an isoTNS, its double-line W matrix fulfills a normalization condition similar to Eq. (4). Then, the single-line W matrix of $|\psi'\rangle$ is also normalized and can be mapped to a stochastic circuit with an update rule that can be directly read off from the original double-line tensor (see SM [40] Sec. D). At the same time, we can use the closed-loop constraint to again map generalized Pauli strings to diagonal operators with the same expectation value. These can be efficiently evaluated classically; see SM [40] Sec. D. Therefore, for normalized double-line states that respect string-net virtual symmetries the same techniques are applicable as for normalized single-line states. For both classes of normalized TNSs, generic operators with a large support k are in general composed of N^{2k} generalized Pauli strings, and thereby remain difficult to evaluate classically; operators with small support can still be evaluated by treating each Pauli string separately.

As both classes of states discussed in this section, normalized single-line states (including normalized single-line string nets) and normalized double-line string nets, are isoTNSs by construction, they can serve as ideal benchmarking states for quantum processors. They occupy a sweet spot between classical and quantum accessibility and nontrivial features: they can be realized on a quantum device by an efficient quantum circuit, and allow at the same time for classical evaluation of a large class of

physically relevant operators, while still being able to represent topological order at finite correlation length, as we are going to discuss in the next section.

IsoTNS skeletons—General deformations of the string-net fixed point tensors that preserve the virtual symmetries lead to local physical perturbations [48]. Thus, topological order remains stable to such perturbations, provided they are weak enough. Determining up to which strength of the perturbation topological order is robust before undergoing a phase transition becomes a formidable task that typically needs to be solved numerically [14,15]. A simple example of such a transition that still allows for an exact solution is the toric code with string tension (generated by an imaginary time evolution with a field). This can be written as a single-line tensor that is not isometric away from the fixed point. Above a critical tension, the plaquette anyons condense and drive the state into a paramagnetic phase [10,55,56].

Our strategy is distinct: we want to find classes of states for which the presence of topological order is assured unless fine-tuned conditions are met. We achieve this by considering classes of tensors that preserve both (i) virtual symmetries and at the same time (ii) the isometry condition on the tensor. Property (i) is necessary for the robustness of topological order and property (ii) allows us to map such states to stochastic circuits, as discussed in the preceding section. Since the one-dimensional stochastic automata remain gapped when deforming their weights while keeping the symmetries of the fixed point of the circuit, the corresponding string-net fixed point will also remain in the same topologically ordered phase.

In the stochastic automaton, every virtual symmetry is expressed as conservation of parity (or a higher-modulo generalization) in the local updates. For instance, the matrix product operator symmetry of a $\mathbb{Z}_N$ toric code enforces conservation of the quantity $P_{N,j} = (\sum_{x\in j} n_x) \bmod N$ in every gate j. Accordingly, any update rule that respects such a paritylike conservation law provides a deformed single- or double-line tensor for each Abelian string net with the corresponding local branching constraint. Following this general strategy, these stochastic circuits provide us with classes of states with finite correlation length ξ, which belong to the same topological order as the respective fixed point.

These stochastic automata become gapless only when fine-tuned to higher-symmetry points, which determine critical end points of the topologically ordered phases. Accordingly, the critical isoTNS must have a higher virtual symmetry than the topological isoTNS it directly connects with; this happens only when some transition amplitudes or tensor entries $p(cd|ab) = |W_{(ab)(cd)}|^2$ vanish. We identify critical points whose enhanced symmetries include the virtual symmetries of multiple phases: at these points, a continuous transition between these phases occurs. With that we construct an interwoven web of parametrized paths that allow us to systematically connect distinct quantum phases, including both topologically ordered and trivial phases. As noted, we refer to this web as an isoTNS skeleton; see Fig. 1.

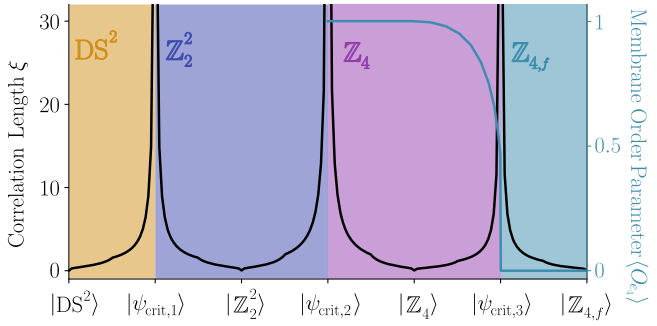


FIG. 3. Path crossing multiple string-net phases. We track the correlation length ξ along the path shown in Fig. 1; the tensor network passes from a pair of decoupled double-semion models (yellow) to a pair of $\mathbb{Z}_2$ toric codes (violet), which are subsequently coupled to a $\mathbb{Z}_4$ toric code (purple). If we further conserve a global antiunitary $\mathbb{Z}_2^T$ symmetry, this phase is distinguished from another phase with the same topological order but nontrivial symmetry fractionalization on the anyons labeled $\mathbb{Z}_{4,f}$ (turquoise). The correlation length is zero at the fixed points and diverges at the critical points, where a conservation law emerges in the associated stochastic circuit, leading to algebraic decay of correlations in one direction. We also show a membrane order parameter $\langle O_{e_4}\rangle$ which distinguishes $\mathbb{Z}_4$ and $\mathbb{Z}_{4,f}$.

We see that deformed isoTNSs host continuous quantum phase transitions without anyon condensation. Critical points are realized when the gap of the transfer matrix shared by the quantum state and the stochastic automaton closes. (It should be noted that this is the transfer matrix in the timelike direction y, i.e., it is rotated by 45° compared to the unit vectors of the square lattice.) As an example, we construct a path that crosses multiple fixed points with the same local Hilbert space dimension $N = 4$; see Fig. 3 and Sec. E of SM [40]. The path includes pairs of $\mathbb{Z}_2$ string-net layers as well as the intrinsic $\mathbb{Z}_4$ string nets. In total, four distinct phases are explored: We start with a phase transition from a pair of double-semion models to a pair of $\mathbb{Z}_2$ toric codes, where the models differ only in the braiding statistics of their excitations (yellow to violet) [11]. This transition needs a double-line description as the sign structure of the double-semion model is nonlocal. From the fixed point of two decoupled toric code layers, we pass a phase transition to the $\mathbb{Z}_4$ toric code (violet to purple). This is an example of a more general coupling of k-layer $\mathbb{Z}_N$ toric code states to a $\mathbb{Z}_{N^k}$ toric code, where the class of allowed loop configurations, i.e., the local branching constraint, changes across the phase transition. Finally, we implement a phase transition between phases with the same topological order, but different fractionalization patterns for the anyonic excitations under a global symmetry [22,57,58]. These are distinct phases as long as the symmetry is preserved, and is referred to as symmetry

enriched topological order. We can tune from the $\mathbb{Z}_4$ toric code with trivial fractionalization pattern for all excitations under a global antiunitary $\mathbb{Z}_2^T$ symmetry $\{1, \mathscr{K}X^2\}$, where $\mathscr{K}$ is complex conjugation, to another state where the vertex excitation e_4 exhibits nontrivial symmetry fractionalization (purple to turquoise). They are distinguished by a membrane order parameter $\langle O_{e_4} \rangle$ which is zero for nontrivial fractionalization [51,59,60].

The correlation length ξ in the timelike direction y diverges at each critical point, at which certain transition probabilities $p(cd|ab)$ of the stochastic circuit vanish and a higher symmetry is realized; see Fig. 3. In particular, for the considered deformations we find that the stochastic circuit exhibits a conserved U(1) symmetry [22,24]. This allows for a hydrodynamic description [61]: In $1+1$D circuits, the local density q_j of the associated charge exhibits Gaussian scaling, showing algebraic decay of equal-site correlations in time t. As this charge can be constructed from the automata values as $q_j = \sum_{x \in j} (n_x \bmod k)$ for some $k \leq N/2$, correlations of the associated power of Z operators in y direction exhibit critical scaling $\langle Z^k(x,y)(Z^\dagger)^k(x,y+r) \rangle \sim r^{-1/2}$ as well, leading to the observed diverging correlation length ξ. More generally, other correlators, such as loop-loop correlators, will also follow an algebraic decay in this direction, as we showed that they map to diagonal operators in the automaton. In SM [40] Sec. F, we illustrate how conservation of higher multipole moments in the stochastic evolution inspire isoTNSs with modified exponents, and how they lie at the endpoints of paths of critical states. Other classes of critical points exist as well. They do not have to possess emergent U(1) conservation laws, but could also describe decoupled GHZ states [23]; in this case, only the fixed point of the automaton, not necessarily the update rule itself, respects an additional symmetry.

From the parametrized isoTNS, we construct the parent Hamiltonian (see SM [40], Sec. C), which in general is not unique. We can construct continuous parent Hamiltonians for all paths between different phases as described by our approach. For critical points separating phases of the same topological order but with distinct symmetry fractionalization ($|\psi_{\mathrm{crit},3}\rangle$ in the path we consider in Fig. 3), this is already achieved by the imaginary time evolution of the fixed point Hamiltonians; for two distinct types of topological order ($|\psi_{\mathrm{crit},1}\rangle$, $|\psi_{\mathrm{crit},2}\rangle$), it involves a slightly more involved procedure as detailed in SM [40] Sec. C. In all cases, the Hamiltonian stays topological and exhibits a nontrivial closing of the gap at the critical point.

Discussion and outlook—We have constructed skeletons of isometric tensor network states (isoTNSs) that provide an organizing principle for Abelian string-net phases and can be efficiently realized on quantum processors. They contain analytically tractable examples of continuous phase transitions beyond the anyon condensation framework. By mapping to classical automata we showed that polynomially many generalized Pauli strings (of arbitrary weight) can be efficiently measured classically for these isoTNS skeletons. The classical simulability is surprising as these states are generically neither generated by Clifford circuits nor match gates, which are familiar examples that admit efficient classical simulation. The analytical insights we derive for this class of complex quantum many-body states render them as intricate test beds for quantum processors.

Our framework suggests several generalizations, including modified variants of $\mathbb{Z}_N$ toric code states [62]. Working out how to connect non-Abelian string nets to stochastic processes will further enable the study of novel deformations of the fixed points as well as phase transitions between non-Abelian topological orders [15]; the known isometric tensor networks and sequential quantum circuits for the fixed points of general string-net models may serve as useful starting points [18,21]. Higher-dimensional stabilizer codes for topological and fracton order can be similarly treated [24]. Our class of states with guaranteed topological order can also be efficiently variationally optimized and may be useful in ground-state searches for perturbed string-net Hamiltonians. We have also constructed continuous parent Hamiltonians in the vicinity of the phase transitions between different topological orders, which could inspire novel field theoretic constructions to describe topological criticality.

Furthermore, the reverse mapping shows that any stochastic process with local rules defines an isoTNS, providing a systematic route to new, efficiently preparable quantum states with nontrivial correlations [63–65]. Extending this approach to nonequilibrium stochastic processes, which violate detailed balance and can exhibit genuine nonequilibrium phase transitions, could uncover a richer class of novel critical phenomena [66,67]. The equivalence discussed here is a special case of a more general relation between isoTNSs and trace-preserving quantum operations; the virtual symmetries discussed in the context of topological order correspond to strong symmetries of the quantum channel, which enforce nontrivial steady-state degeneracy [68]. This perspective promises to shed more light on the remarkable stability of topological order in isoTNSs [69].

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Data availability—Data, data analysis, and simulation codes are available upon reasonable request on Zenodo [70].

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