

Exactly Solvable Model of Wave-Mean Field Interaction in Integrable Turbulence

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The kinetic theory of soliton gases (SG) is used to develop a solvable model for wave-mean field interaction in integrable turbulence. The waves are stochastic soliton ensembles that scatter off a critically dense SG or soliton condensate—the mean field. The derived two-fluid kinetic-hydrodynamic equations admit exact solutions predicting an induced mean field and SG filtering. The obtained SG statistical moments agree with ensemble averages of numerical simulations. The developed theory readily generalizes, with applications in fluids, nonlinear optics, and condensed matter.

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At the vanguard of quantum many-body physics [1], geophysical fluid dynamics [2–6], nonlinear optics [7–9], and soliton theory [10] is the evolution of random initial states according to an integrable or near-integrable system. Dubbed integrable turbulence (IT) [11], this class of multiscale nonlinear dynamics holds the tantalizing prospect of yielding solvable, physical turbulent models. In contrast, classical fluid turbulence is plagued by the closure problem in which statistical moments of the fluid state satisfying the Reynolds equations depend on successively higher order moments [12].

In classical turbulence, wave-mean flow is only solvable when the waves are linear [13]. In contrast, the IT setting does not involve small amplitude restrictions and allows for the description of fully nonlinear regimes including solitons—the localized nonlinear waves that exhibit particlelike properties such as elastic, pairwise interactions. Solitons can be viewed as natural “normal modes” of nonlinear wave fields in integrable systems [14,15]. This motivates the study of wave-mean field interaction for soliton-dominated IT, commonly called *soliton gas* (SG) [16,17].

The theory of SGs describes random ensembles of many interacting solitons via a kinetic equation for the evolution of the density of states (DOS), the distribution of solitons with respect to their amplitudes and spatial positions [18]. Closure is achieved in the thermodynamic limit ($N \rightarrow \infty$) of modulated N -phase (quasiperiodic) solutions [19], the existence of which is intimately connected to an infinite

number of independent conserved quantities. A related strand of research is generalized hydrodynamics (GHD) [1], in which integrable quantum and classical many-body systems are described by the out-of-equilibrium dynamics of interacting quasiparticles on large spatiotemporal scales. A correspondence between SG and GHD was established for soliton ensembles in the Korteweg-de Vries (KdV) equation [20].

A related counterpart to integrable turbulence is wave turbulence (WT) modeled by statistical ensembles of interacting small-amplitude dispersive waves [21,22]. While closure is achieved and analytical solutions are known, the theory is restricted to nearly linear, small-amplitude waves and does not account for the formation of localized, finite-amplitude coherent structures like solitons and breathers, which are ubiquitous in strongly nonlinear wave fields. WT, SG, and GHD theories also feature large-scale coherent states, or condensates [23–25] that can be viewed as mean fields. However, a theory that couples random waves or quasiparticles and a mean field is lacking.

In this Letter, we develop a two-fluid theory of SG for wave-mean field interaction (see a numerical realization in Fig. 1), a long-standing problem in geophysics [13]. We derive fully coupled kinetic-hydrodynamic equations for the KdV soliton DOS—describing the stochastic SG, an analog of the normal fluid in the two-fluid theory of superfluidity [26]—and a mean-field or soliton condensate [24]—describing the deterministic soliton superfluid. A key finding is the existence of a soliton induced mean field which provides the coupling. The SG-mean field equations are diagonalized and solutions are obtained. For a polychromatic SG [27] scattering off of a dynamic mean field as in Fig. 1, the theory predicts SG filtering, the spontaneous creation of pure superfluid states, stochastic rarefaction and dispersive shock waves, and contact shocks. The developed theory is general and applies to other integrable systems with applications in nonlinear physics.

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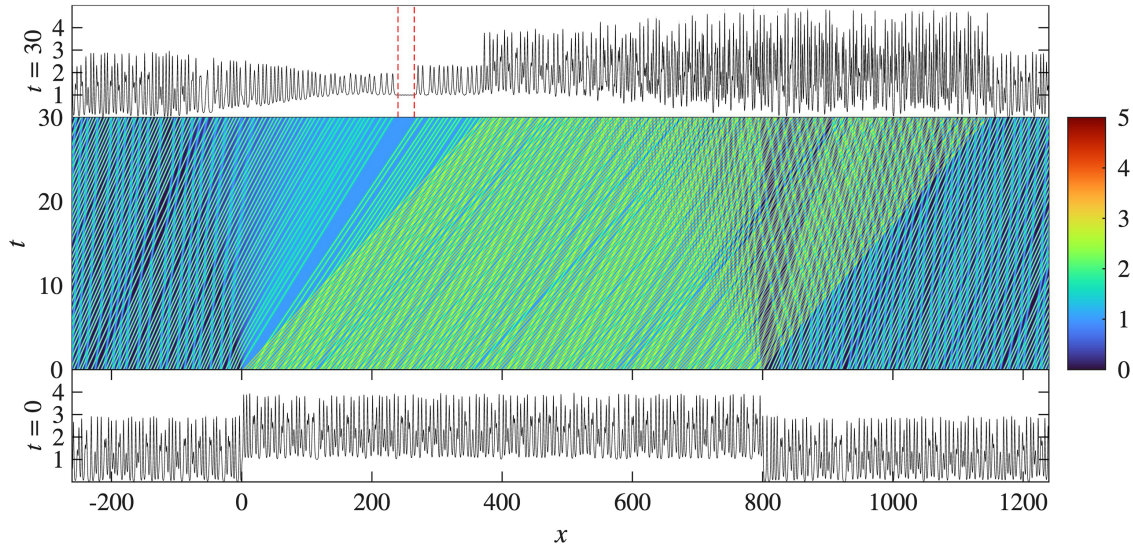


FIG. 1. Wave-mean field interaction in KdV soliton gas (SG) integrable turbulence. Lower panel: KdV initial condition for a wide, localized unit mean-field elevation added to a two-component SG with DOS $f(\eta) = 0.08\delta(\eta - 0.8) + 0.17\delta(\eta - 1.2)$ for 400 solitons. Middle panel: contour plot of the KdV solution $u(x, t)$ featuring stochastic rarefaction and dispersive shock waves regularizing the initial discontinuities at $x = 0$ and $x = 800$, respectively. Upper panel: the turbulent KdV wave field at $t = 30$. The vertical dashed lines delimit the region of pure condensate emerging due to slowing down of the SG by the mean.

The theory of SG utilizes the inverse scattering transform that maps discrete spectral points ζ_j of the associated Lax operator to solitons [14,15]. Here, we focus on the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ whose Lax operator is $-\partial_{xx} - u$. We transform its spectrum $\zeta_j = -\eta_j^2$, so that $\eta_j \in \mathbb{R}^+$ and the soliton's amplitude is $a_j = 2\eta_j^2$. The DOS $f(\eta; x, t)$ [we sometimes suppress the (x, t) dependence for simplicity] is the joint distribution of solitons with respect to their spectral parameters and positions; it satisfies the kinetic equation [17,19]

$$f_t + (sf)_x = 0, \quad (1)$$

where $s(\eta; x, t)$ is the effective soliton velocity in SG satisfying the integral equation of state

$$s(\eta) = s_0(\eta) + \int_{\Gamma} G(\eta, \mu) f(\mu) [s(\eta) - s(\mu)] d\mu. \quad (2)$$

Here, s_0 is the free soliton velocity, G the two-soliton scattering phase-shift kernel, and $\Gamma \subset \mathbb{R}^+$ is the DOS spectral support. For KdV,

$$s_0(\eta) = 4\eta^2, \quad G(\eta, \mu) = \frac{1}{\eta} \ln \left| \frac{\eta + \mu}{\eta - \mu} \right|. \quad (3)$$

The ensemble averages of the physical wave field can be expressed in terms of DOS moments, e.g., for KdV [18,19],

$$\langle u \rangle = \int_{\Gamma} 4\eta f(\eta) d\eta, \quad \langle u^2 \rangle = \int_{\Gamma} \frac{16}{3} \eta^3 f(\eta) d\eta. \quad (4)$$

Originally derived for KdV dilute [16] and dense [19] SG, the kinetic equation (1) and (2) universally describes a weakly inhomogeneous SG in integrable systems [10,18]. Independently, the same equation arose in GHD [1,28,29].

It was shown in [24] that coherent dispersive hydrodynamic mean fields such as a rarefaction or dispersive shock wave [30] represent nonequilibrium *soliton condensates*, critically dense SGs in which the individual soliton dynamics are superseded by the soliton interaction effects. Soliton condensates are characterized by a nonnegative integer N called the genus that is the number of gaps in the spectral support denoted Γ_N . Generally, SG includes two components that form a *composite SG* [31]: a condensate, modeling a coherent mean field or superfluid, and a non-condensate, modeling a stochastic SG or normal fluid. Then, wave-mean field interaction in integrable turbulence and a two-fluid theory of SG is described by their coupling. Our primary contribution in this Letter is a suitable partitioning of the DOS $f(\eta)$ and its spectral support Γ that realizes this two-fluid theory of wave-mean field interaction.

Introducing the DOS spectral support

$$\Gamma = \Gamma_N \cup \tilde{\Gamma}, \quad \Gamma_N \cap \tilde{\Gamma} = \emptyset, \quad (5)$$

then the composite SG DOS

$$f(\eta) = \underbrace{f^{(N)}(\eta)}_{\text{condensate}} - \underbrace{f_{\text{IMF}}(\eta)}_{\text{noncondensate}} + \tilde{f}(\eta), \quad (6)$$

where the *core* condensate DOS $f^{(N)}(\eta)$ and the *induced mean field* condensate DOS $f_{\text{IMF}}(\eta)$ are both supported on Γ_N , while the noncondensate DOS $\tilde{f}(\eta)$ is supported on $\tilde{\Gamma}$, equaling zero at endpoints of $\tilde{\Gamma}$. The DOS decomposition (6) enables the effective separation of the mean field (condensate) from the stochastic, fluctuating solitons (noncondensate).

Using Eqs. (5) and (6) in the kinetic equation (1) and equation of state (2), we derive a general two-fluid theory for a composite SG corresponding to the interaction of a KdV SG with a genus N soliton condensate in the End Matter, where $f^{(N)}$ and f_{IMF} are defined by Eqs. (A1), (A3), and (A5), and $\tilde{f}$ satisfies Eqs. (10), (11), and (A7). Here, we highlight the simplest case with the genus 0 core soliton condensate DOS [24]

$$f^{(0)}(\eta) = \frac{\eta}{\pi\sqrt{\beta^2 - \eta^2}}, \quad \Gamma_0 = [0, \beta). \quad (7)$$

If $\tilde{f} \equiv 0$, the slow modulation of the condensate is enabled by allowing $\Gamma_0 = \Gamma_0(x, t)$ to evolve

$$\beta_t + 6\beta^2\beta_x = 0, \quad \tilde{f}(\eta) \equiv 0. \quad (8)$$

More generally, if $\tilde{f} \neq 0$ the induced mean field DOS is

$$f_{\text{IMF}}(\eta) = \int_{\beta}^{\infty} p^{(0)}(\eta, \mu) \tilde{f}(\mu) d\mu, \quad (9)$$

$$p^{(0)}(\eta, \mu) = \frac{2\eta\sqrt{\mu^2 - \beta^2}}{\pi(\mu^2 - \eta^2)\sqrt{\beta^2 - \eta^2}},$$

where $\tilde{\Gamma} = [\beta, \infty)$.

Letting $\tilde{f} = \tilde{f}(\eta; x, t)$ and substituting (5) and (6) into (1) we obtain the kinetic equation for the noncondensate component of the composite SG (see the End Matter for the derivation in the general case of genus $N \geq 0$),

$$\tilde{f}_t + (\tilde{s}\tilde{f})_x = 0, \quad (10)$$

where $\tilde{s}(\eta)$ satisfies the “dressed” equation of state

$$\tilde{s}(\eta) = \tilde{s}_0(\eta) + \int_{\beta}^{\infty} \tilde{G}(\eta, \mu) \tilde{f}(\mu) [\tilde{s}(\eta) - \tilde{s}(\mu)] d\mu, \quad (11)$$

$$\tilde{s}_0(\eta) = 2\beta^2 + 4\eta^2,$$

$$\tilde{G}(\eta, \mu) = G\left(\sqrt{\eta^2 - \beta^2}, \sqrt{\mu^2 - \beta^2}\right), \quad (12)$$

where the soliton free velocity $\tilde{s}_0$ and the two-soliton scattering shift $\tilde{G}$ are modified by the coupling to the condensate.

The condensate component in the composite SG satisfies the modulation equation [cf. Eq. (8) for pure condensate]

$$\beta_t + \gamma\beta_x = 0, \quad \gamma = \frac{6\beta^2 - \int_{\beta}^{\infty} \frac{2\tilde{s}(\mu)\tilde{f}(\mu)}{\sqrt{\mu^2 - \beta^2}} d\mu}{1 - \int_{\beta}^{\infty} \frac{2\tilde{f}(\mu)}{\sqrt{\mu^2 - \beta^2}} d\mu}. \quad (13)$$

Equations (5)–(13) comprise our main result: the kinetic-hydrodynamic two-fluid model for the DOS $f(\eta; x, t)$ and spectral support $\Gamma(x, t)$ of wave-mean field interaction in a composite SG.

Substituting (5)–(7) and (9) in (4) yields

$$\langle u \rangle = \beta^2 + \int_{\beta}^{\infty} 4\sqrt{\eta^2 - \beta^2} \tilde{f}(\eta) d\eta, \quad (14)$$

$$\langle u^2 \rangle = \beta^4 + \int_{\beta}^{\infty} \frac{8}{3}(2\eta^2 + \beta^2) \sqrt{\eta^2 - \beta^2} \tilde{f}(\eta) d\eta.$$

In the absence of a noncondensate, stochastic SG component ($\tilde{f} \equiv 0$), the mean field is $\langle u \rangle = \beta^2$ and the variance $\text{Var}(u) = \langle u^2 \rangle - \langle u \rangle^2 = 0$; i.e., the flow (8) corresponds to a pure, deterministic superfluid for the mean field β^2 . The addition of a nonzero, noncondensate SG in Eq. (14) results in positive variance and a positive induced mean field since $\langle u \rangle > \beta^2$.

We now use these results to describe the interaction of a dense KdV SG with a genus zero soliton condensate in a stochastic rarefaction wave initiated by step Riemann data [24]. This problem generalizes soliton-mean field interaction [32,33], and its accompanying soliton transmission through or trapping inside the mean field, to the complex nonlinear interactions of a dense SG with a mean field as in Fig. 1.

For analytical results, we focus on a multicomponent (polychromatic) SG for the noncondensate DOS

$$\tilde{f}(\eta; x, t) = \sum_{j=1}^M w_j(x, t) \delta(\eta - \xi_j), \quad (15)$$

where $\xi_j \in \tilde{\Gamma}$, $j = 1, \dots, M$. The delta-function ansatz (15) is a mathematical idealization of a narrow spectral distribution around ξ_j , valid for ξ_j sufficiently separated from β . In this case, the kinetic-hydrodynamic system (10)–(13) reduces to the hydrodynamic conservation laws (cf. Ref. [34])

$$w_{j,t} + (c_j w_j)_x = 0, \quad j = 1, \dots, M, \quad (16)$$

$$c_j = 2\beta^2 + 4\xi_j^2 + \sum_{k \neq j} \tilde{G}_{jk} w_k (c_j - c_k),$$

with $\tilde{G}_{jk} = \tilde{G}(\xi_j, \xi_k)$ that is coupled to the mean field evolution

$$\beta_t + \gamma\beta_x = 0, \quad \gamma = \frac{6\beta^2 - \sum_{j=1}^M \frac{2c_j w_j}{\sqrt{\xi_j^2 - \beta^2}}}{1 - \sum_{j=1}^M \frac{2w_j}{\sqrt{\xi_j^2 - \beta^2}}}. \quad (17)$$

The combined system (16) and (17) exhibits important mathematical properties with physical implications. The system (16) with $\beta \equiv 0$ has been shown to be a linearly degenerate, integrable, hyperbolic system of hydrodynamic type [17,34]. Equation (16) with β satisfying (17) remains linearly degenerate [35], implying the absence of wave breaking and that only contact shock discontinuities moving with characteristic speed c_j are admissible. Equation (17) for the mean flow is not strictly genuinely nonlinear $\partial_\beta \gamma \neq 0$ [39], which affects solutions of the Riemann problem. The system (16) can be diagonalized [35],

$$\begin{aligned} q_{j,t} + c_j q_{j,x} &= 0, \quad j = 1, \dots, M, \\ q_j &= \left(1 - \sum_{k \neq j} \tilde{G}_{jk} w_k \right) \sqrt{\xi_j^2 - \beta^2 / w_j} \\ &\quad - \ln(1 - \beta^2 / \xi_j^2), \end{aligned} \quad (18)$$

so that the coupled system for $\{\beta\} \cup \{q_j\}_{j=1}^M$ is diagonal and hyperbolic. We prove in [35] that the system (16) and (17) is integrable via the generalized hodograph transform [40,41] for $M = 2$.

Prompted by the simulation in Fig. 1, we consider a two-component (bichromatic) noncondensate SG with densities $w_1(x, t)$, $w_2(x, t)$ satisfying Eq. (16) and a genus 0 soliton condensate $\beta(x, t)$ with the step initial condition

$$(\beta, w_1, w_2)|_{t=0} = \begin{cases} (0, w_1^0, w_2^0), & x < 0, \\ (1, 0, 0), & x > 0. \end{cases} \quad (19)$$

This represents the collision of a pure normal fluid for $x < 0$ with a pure superfluid for $x > 0$. The solution of this Riemann problem describes the wave field to the left of the vertical dashed red lines in Fig. 1. Figures 2(a) and 2(b) display representative numerical realizations for $t \gg 1$ of two distinct scenarios in which $1 < \xi_1 < \xi_2$ and $0 < \xi_1 < 1 < \xi_2$, respectively. Details of the numerical method are given in [35]. A visual examination reveals that the bichromatic, stochastic SG penetrates through the mean field rarefaction wave to emerge on the unit background in Fig. 2(a); i.e., the SG is transmitted. The two components of the stochastic SG then separate due to their different speeds c_j , yielding two contact discontinuities. The stochastic SG does not propagate past the rightmost discontinuity, highlighted by a red dashed line in Fig. 2(a), and a pure condensate emerges as identified in Fig. 1. Conversely, in Fig. 2(b), the mean field acts as a SG filter allowing one component of the bichromatic SG to transmit while the other remains trapped inside the varying mean; in that case only one contact discontinuity separates the transmitted component from the pure condensate.

We compare the analytical solutions to the Riemann problem (16), (17), and (19) [see Eqs. (B2) and (B4) in End Matter for details] with ensemble averages over 100

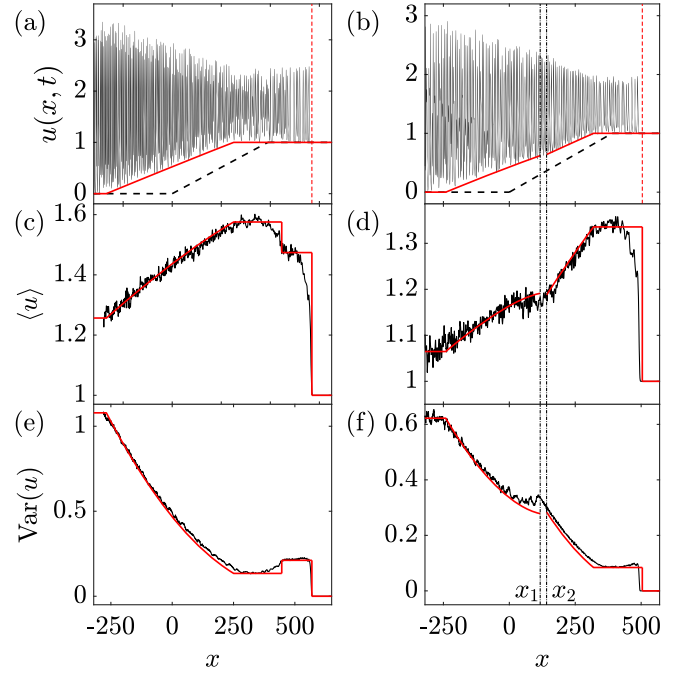


FIG. 2. SG interaction with a mean-field rarefaction wave (cf. left half of Fig. 1). (a),(c),(e) Bichromatic SG transmission with $(\xi_1, \xi_2) = (1.2, 1.3)$ and $(w_{10}, w_{20}) = (0.1, 0.14)$. (b),(d),(f) Bichromatic SG filtering with $(\xi_1, \xi_2) = (0.8, 1.2)$ and $(w_{10}, w_{20}) = (0.08, 0.17)$. (a),(b) One realization of the soliton gas at $t = 65$. The solid red curves vs dashed black curves show the backflow of the condensate due to the interaction with the stochastic SG component. The vertical red dashed line separates the composite SG from the region of pure condensate also identified in Fig. 1. (c)–(f) Comparison between the analytical results (14) (red curves) and ensemble averages over 100 realizations (black curves). The black dashed vertical lines in (b),(d),(f) show the region of the local breakdown of genuine nonlinearity of the characteristic velocity γ field. See main text for additional details.

numerical realizations of the SG for the transmission and filtering cases. Figures 2(c)–2(f) depict the mean and variance computed from the numerical ensemble averages (black curves) and from the analytical expressions Eqs. (14) and (15) with solutions (B2) and (B4) substituted (red curves). Contact discontinuities in the analytical solution appear as discontinuities of the averages. In the interval $x \in [x_1, x_2]$ denoted by vertical dashed lines, $\beta(x, t)$ and ξ_1 are no longer well separated, not compatible with the delta function ansatz (15). This leads to the local loss of genuine nonlinearity $\partial_\beta \gamma = 0$ and a singularity in γ . It can be resolved by replacing $\delta(\eta - \xi_1)$ in (15) with a smooth, narrow distribution centered at ξ_1 and solving (10)–(13).

Returning to Figs. 2(a) and 2(b), the rarefaction wave solution β^2 of the mean field equation (8) without the stochastic SG component $(w_1^0, w_2^0) = (0, 0)$ is plotted with a dashed curve. Due to the coupling of the composite SG when $(w_1^0, w_2^0) \neq (0, 0)$, the solitons constituting the

condensate are phase-shifted leftward owing to their interaction with the solitons in the bichromatic SG. This results in an overall backflow of the condensate as displayed by the solution of (17) plotted as the solid red curve.

In conclusion, we have developed a two-fluid theory of composite soliton gases (SG) in the KdV equation consisting of deterministic condensate and stochastic noncondensate components. The condensate corresponds to a mean field, analogous to a superfluid, whereas the noncondensate is analogous to a normal fluid. In contrast to deterministic soliton steering by the mean field, a one-way interaction in soliton-mean field theory [32,33], both condensate and noncondensate components influence one another, leading to an induced mean field and nontrivial interactions such as SG filtering by the mean. Another effect is the spontaneous emergence of a zero variance state or pure superfluid due to the slowing down of the SG by the mean, identified in Fig. 1 by the vertical dashed lines. We obtained the kinetic solution and statistics for a stochastic rarefaction wave describing Fig. 1 for $x < 200$. For $x > 200$, the initial condition is a step down in the mean that, in the absence of a stochastic SG, gives rise to a dispersive shock wave (DSW). An important extension of this Letter will be to stochastic DSWs. Also, the derived kinetic-hydrodynamic system and its hydrodynamic reductions represent a new class of integrable systems worthy of further investigation. Importantly, the approach taken here can be generalized to other quantum and classical integrable systems such as the Lieb-Liniger model and the nonlinear Schrödinger equation with applications in condensed matter, optical, and nonlinear physics. Bridging the developed two-fluid theory of SG with generalized hydrodynamics is another promising direction of further research.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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End Matter

Two-fluid kinetic description—The DOS $f^{(N)}$ and the corresponding spectral flux density $g^{(N)}$ of a soliton condensate are defined by the nonlinear dispersion relations (NDRs) [24]

$$\begin{aligned} \int_{\Gamma_N} G(\eta, \mu) f^{(N)}(\mu) d\mu &= 1, & \eta \in \Gamma_N, \\ \int_{\Gamma_N} G(\eta, \mu) g^{(N)}(\mu) d\mu &= s_0(\eta), & \eta \in \Gamma_N, \end{aligned} \quad (\text{A1})$$

with $\Gamma_N = [0, \beta_1) \cup (\beta_2, \beta_3) \dots \cup (\beta_{2N}, \beta_{2N+1})$, N being the condensate genus; for simplicity we dropped the index i in β_i when $N = 0$ in the main text. Explicit solutions $f^{(N)}(\eta)$, $g^{(N)}(\eta)$ to the condensate NDRs that are parameterized by the β_i 's were found in [24,42] with the kinetic equation (1) mapping to the N -phase Whitham modulation equations for the spectral band edges [43]

$$\beta_{i,t} + v_i^{(N)} \beta_{i,x} = 0, \quad i = 1, \dots, 2N + 1, \quad (\text{A2})$$

where $v_i^{(N)} = \partial_{\beta_i} g^{(N)}(\eta) / \partial_{\beta_i} f^{(N)}(\eta)$ is independent from η [24,35]. One has in particular $v^{(0)} = 6\beta^2$.

The induced mean field DOS in (6) can be represented in the form

$$f_{\text{IMF}}(\eta) = \int_{\tilde{\Gamma}} p^{(N)}(\eta, \mu) \tilde{f}(\mu) d\mu, \quad (\text{A3})$$

where the kernel $p^{(N)}(\eta, \mu)$ results from the requirement that the induced mean field only contributes to the condensate component of the composite SG; i.e., the SG remains a condensate for $\eta \in \Gamma_N$ [cf. Eq. (A1)],

$$\int_{\Gamma} G(\eta, \mu) f(\mu) d\mu = 1, \quad \eta \in \Gamma_N. \quad (\text{A4})$$

Substituting (5), (6), and (A3) in (A4), and using (A1) yields the integral equation for $p^{(N)}(\eta, \mu)$,

$$\int_{\Gamma_N} G(\eta, z) p^{(N)}(z, \mu) dz = G(\eta, \mu), \quad (\text{A5})$$

for $(\eta, \mu) \in \Gamma_N \times \tilde{\Gamma}$, which represents the induced mean field NDR additional to (A1); the solution of (A5) for $N = 0$ is derived in [35].

We introduce a counterpart of (6) for the spectral flux density

$$g(\eta) = g^{(N)}(\eta) + \tilde{g}(\eta) - \int_{\tilde{\Gamma}} p^{(N)}(\eta, \mu) \tilde{g}(\mu) d\mu, \quad (\text{A6})$$

where $\tilde{g}(\eta) = \tilde{s}(\eta) \tilde{f}(\eta)$, and substitute (5), (6), and (A6) into the kinetic equation (1). Then, restricting to $\eta \in \tilde{\Gamma}$, we obtain the kinetic equation (10) for the noncondensate component of the composite SG where $\tilde{s}(\eta)$ satisfies the dressed equation of state (11) (with the replacement $\int_{\beta}^{\infty} \rightarrow \int_{\tilde{\Gamma}}$ in the general case), and $\tilde{s}_0$ and $\tilde{G}$ the free soliton velocity and phase shift kernel modified by the mean field

$$\begin{aligned} \tilde{s}_0(\eta) &= \frac{s_0(\eta) - \int_{\Gamma_N} G(\eta, \mu) g^{(N)}(\mu) d\mu}{1 - \int_{\Gamma_N} G(\eta, \mu) f^{(N)}(\mu) d\mu}, \\ \tilde{G}(\eta, \mu) &= \frac{G(\eta, \mu) - \int_{\Gamma_N} G(\eta, z) p^{(N)}(z, \mu) dz}{1 - \int_{\Gamma_N} G(\eta, z) f^{(N)}(z) dz}, \end{aligned} \quad (\text{A7})$$

which simplifies to (12) for $N = 0$.

For $\eta \in \Gamma_N$ in (1), the condensate component modified by the induced mean field satisfies the modulation equations [cf. Eq. (A2)]

$$\beta_{i,t} + \gamma_i \beta_{i,x} = 0, \quad i = 1, \dots, 2N + 1, \quad (\text{A8})$$

$$\gamma_i = \frac{v_i^{(N)} - \int_{\tilde{\Gamma}} a_i^{(N)}(\mu) \tilde{g}(\mu) d\mu}{1 - \int_{\tilde{\Gamma}} a_i^{(N)}(\mu) \tilde{f}(\mu) d\mu}, \quad (\text{A9})$$

where $a_i^{(N)}(\mu) = \partial_{\beta_i} p^{(N)}(\eta, \mu) / \partial_{\beta_i} f^{(N)}(\eta)$ is independent of η [35].

Simple wave solution—Suppose first that $1 < \xi_1 < \xi_2$ as in Fig. 2(a) (the transmission case). The stochastic rarefaction wave solution of (16), (17), and (19) is given implicitly by the simple wave

$$\begin{aligned} \gamma(\beta, w_1, w_2) &= x/t, \\ q_j(\beta, w_1, w_2) &= q_j(0, w_1^0, w_2^0), \quad j = 1, 2, \end{aligned} \quad (\text{B1})$$

with $\beta(x, t) \in [0, 1]$, γ is defined in (17) and q_j in (18). Note that this solution is admissible if (17) remains genuinely nonlinear ($\partial_{\beta} \gamma \neq 0$) along the characteristic $\gamma = x/t$, which is the case for the solution (B1) plotted in Fig. 2(a) with the red curve. The transmitted SG emerging to the right of the simple wave is uniform and the corresponding densities w_j^+ are determined by the conservation of the two Riemann invariants

$q_j(1, w_1^+, w_2^+) = q_j(0, w_1^0, w_2^0)$, $j = 1, 2$ so that the complete solution is piecewise defined,

$$(\beta, w_1, w_2) = \begin{cases} (0, w_1^0, w_2^0), & \frac{x}{t} \leq \gamma^0, \\ (\beta, w_1, w_2) \left(\frac{x}{t} \right), & \gamma^0 < \frac{x}{t} < \gamma^+, \\ (1, w_1^+, w_2^+), & \gamma^+ \leq \frac{x}{t}, \\ (1, 0, w_2^*), & c_1^+ \leq \frac{x}{t} < c_2^+, \\ (1, 0, 0), & c_2^+ \leq \frac{x}{t}, \end{cases} \quad (\text{B2})$$

with $\gamma^0 = \gamma(0, w_1^0, w_2^0)$, $\gamma^+ = \gamma(1, w_1^+, w_2^+)$, $c_1^+ = c_1(1, w_1^+, w_2^+)$, $c_2^+ = 2 + 4\xi_2^2$ [cf. Eq. (16)]. The density in the single component region w_2^* is obtained using the continuity of c_1 at the first contact discontinuity: $c_1(1, w_1^+, w_2^+) = c_1(1, 0, w_2^*)$, see e.g., Ref. [17]. Since $w_j^0 = 0$ in Eq. (B1) corresponds to a rarefaction wave, the solution with nonzero w_j^0 describes the interaction of a SG with a rarefaction wave, which becomes a stochastic rarefaction wave. To the right of the stochastic rarefaction wave are two contact discontinuities moving with speeds c_1^+ and c_2^+ . Substituting (15) and (B2) in (14) yields the variations of the ensemble averages $\langle u \rangle$, $\langle u^2 \rangle$ displayed in Figs. 2(c) and 2(e).

Suppose now that $\xi_1 < 1 < \xi_2$ as in Figs. 1 and 2(b) (the filtering case). The modulation of (β, w_1, w_2) is given implicitly by the simple wave solution (B1) in the genuinely nonlinear region $x < x_1(t)$ where $x_1(t)$ corresponds to the position where $\partial_{\beta} \gamma = 0$, inside the region indicated by vertical black dashed lines in Figs. 2(b), 2(d), and 2(f). The loss of genuine nonlinearity is due to the logarithmic divergence of γ in the region (x_1, x_2) , where β is in the neighborhood of ξ_1 ; see the explicit formula in [35]. The logarithmic divergence originates from the idealization of the stochastic SG DOS by a sum of delta functions and should be regularized by considering an adequate continuous distribution. We suppose that the solution (B1) is still valid in the region $x > x_2(t)$, where $\beta \in [\xi_1, 1]$ and $w_1 = 0$: the modulation of $(\beta, w_2) = (\beta', w_2')$ is then given by the simple wave

$$\begin{aligned} \gamma(\beta', w_1 = 0, w_2') &= x/t, \\ q_2(\beta', w_1 = 0, w_2') &= q_2(0, w_1^0, w_2^0), \end{aligned} \quad (\text{B3})$$

for $x_2(t) < x < \gamma(1, 0, w_2^+)t$ with w_2^+ determined by $q_2(1, 0, w_2^+) = q_2(0, w_1^0, w_2^0)$. This latter assumption is validated numerically by the good agreement between the analytical description and the numerical simulation in the region $x > x_2$ displayed in Figs. 2(d) and 2(f). The complete solution is piecewise defined,

$$(\beta, w_1, w_2) = \begin{cases} (0, w_1^0, w_2^0), & \frac{x}{t} \leq \gamma^0, \\ (\beta, w_1, w_2) \left(\frac{x}{t} \right), & \gamma^0 < \frac{x}{t} \leq \frac{x_1}{t}, \\ (\beta', 0, w_2') \left(\frac{x}{t} \right), & \frac{x_2}{t} \leq \frac{x}{t} < \gamma^+, \\ (1, 0, w_2^+), & \gamma^+ \leq \frac{x}{t} < c_2^+, \\ (1, 0, 0), & c_2^+ \leq \frac{x}{t}, \end{cases} \quad (\text{B4})$$

with $\gamma^0 = \gamma(0, w_1^0, w_2^0)$, $\gamma^+ = \gamma(1, 0, w_2^+)$, $c_2^+ = 2 + 4\xi_2^2$, and (β, w_1, w_2) , (β', w_2') given by (B1) and (B3), respectively. This solution corresponds to SG filtering in which only one component of the stochastic SG is transmitted by the mean field [cf. the solution (B2) for the full transmission case].