

One-Loop Corrections to Near-Extremal Kerr Thermodynamics from Semiclassical Virasoro Blocks

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We propose a method to perform an exact calculation of one-loop quantum corrections to black hole entropy in terms of Virasoro semiclassical blocks. We analyze in detail a four-dimensional Kerr black hole and show that in the near-extremal limit a branch of long-lived modes arises. We prove that the contribution of these modes accounts for a $(s - 1/2) \log T_{\text{Hawking}}$ correction to the entropy for massless particles of spin $s = 1, 2$. We show that in the full calculation performed in the exact Kerr background the leading contribution actually is sourced by the near-horizon region only, and as such has a universal validity for any asymptotic behavior at infinity.

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Introduction—Quantum corrections to near-extremal black hole (BH) entropy have been shown to be necessary in order to describe their low-temperature regime. Logarithmic corrections should indeed exist in any gravitational theory and characterize its infrared physics. These are expected to be determined solely by the low energy degrees of freedom and to be independent of the higher derivatives or massive spectrum corrections. On the other hand, they are not universal, as they depend on the specific low-energy spectrum of the theory. Therefore, once logarithmic corrections are fixed, any acceptable microstate description of quantum black holes shall reproduce them in the infrared (IR).

Actually, our best agnostic models of quantum black hole radiation are still semiclassical, and, as it has been largely discussed in the literature, these log-corrections arise from the one-loop analysis of quantum gravity path integral in the IR. Precise tests of this conjecture have been performed for supersymmetric BPS black holes in the context of AdS/CFT correspondence, where a microscopic calculation of the BH entropy is available [2–11].

Recent progress in the challenging nonsupersymmetric case has been obtained by analyzing the near-horizon BH

geometry in terms of Schwarzian modes that describe its reparametrization symmetry [12–16]. In order to confirm these results, it is clearly important to go beyond the near-horizon approximation and perform an analysis of the full BH background. This has recently been accomplished for the Bands-Teitelboim-Zanelli (BTZ) case in three dimensions in [17] and in [18], which also provides numerical evidence for the Reissner-Nordström AdS₄ case.

In [19] we proposed an exact formula for the one-loop effective action on BH backgrounds with enough symmetries such that the BH perturbations admit a separation of variables leading to ordinary differential operators. In these cases, the relevant determinants can be computed by a generalization of the Gelfand-Yaglom (GY) formula [20]. We can show that, when the involved differential operators display only regular singularities, these formulas reduce to the ones conjectured by Denef-Hartnoll-Sachdev (DHS) in [21] based on analytic properties of the effective action of Euclidean quantum gravity. In particular, this conjecture implies that the one-loop determinants have zeroes at the BH (anti)quasinormal modes (QNMs). The DHS formula has been the main tool to obtain the results in [17] where it has been proposed that the logarithmic corrections are due to particular modes in the black hole QNM spectrum developing a parametrically small negative imaginary part, called zero damping modes (ZDMs).

In this Letter, we extend the formalism of [19] to spectral problems for ordinary differential equations with irregular singularities, focusing on the confluent Heun equation, which is the relevant one for the study of perturbations around the Kerr BH. We provide an exact formula for the corresponding one-loop effective action. As it happens for Fuchsian differential equations [19], we find that the

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one-loop effective action can be expressed in terms of the connection coefficients between the local solutions around the singularities where the boundary conditions are imposed, since these are essentially the Wronskian between the appropriately normalized local solutions. We present all the technical details in Supplemental Material [22]. We notice that in the presence of irregular singularities the DHS conjectural formula does not apply, due to the presence of branch cuts in the frequency plane [see (13) for the Kerr case]. The exact calculation of QNM for Kerr BH [23,24] can be performed in terms of semiclassical (confluent) Virasoro conformal blocks, whose explicit expression can be computed from supersymmetric gauge theories [25]. In particular, we show how to isolate the sector containing the contributions of the ZDMs in the near-extremal limit and analyze the regularity conditions of the associated Euclidean wave functions.

It turns out that the ZDMs contributing to the leading log-correction in the temperature are the ones with azimuthal quantum number $m = 0$. By evaluating the contribution of these modes to the one-loop determinant, we find that the leading temperature factor for the massless spin $s = 1, 2$ mode is $T_H^{s-1/2}$, as conjectured in [12,26]. Our results are obtained by making use of explicit connection formulas for the Heun function given by successive analytic continuations ending in the near-horizon region. In particular, we show that the factor in the connection formulas that gives the ZDMs responsible for the temperature scaling arises in the step of the connection procedure performed locally around the black hole horizon, and it gets decoupled in the extremal limit. Therefore, the one-loop effective action of any black hole background admitting a near-extremal regime in which the inner and outer horizon approach each other will contain a contribution from the near-horizon region that decouples in the extremal limit, independently from the asymptotic geometry. We argue that this is the universal mechanism responsible for the temperature scaling in this extremal limit, known as *cold* regime. The ZDMs appearing in (32) for $m = 0$ should correspond to the extension to the full Kerr geometry of those analyzed in [13] in the near-horizon extremal Kerr throat approximation. It would be interesting to verify this in detail.

This separation of scales arises naturally from the Heun connection problem and is a characteristic feature of BH perturbation theory manifesting itself also for other physical quantities as scattering amplitudes and tidal deformations [27].

The QNMs of other BH geometries have been studied with similar techniques in [28–33]. In particular, the analysis of ZDMs for Kerr-Newman in (A)dS₄ and analogous higher-dimensional BHs can be performed using similar techniques applied here. More generally, our results point out that ZDMs generically arise when considering confluences of second-order Fuchsian differential operators.

Let us notice that log-corrections are reliable subleading terms in the black hole entropy only before the temperature becomes too low and nonperturbative corrections to the path integral become important [34]. At higher temperatures, instead, UV-generated contributions picture shall dominate over the classical result. The relevant mechanisms at scales at which these effects happen are model dependent, and their description is well beyond the scope of this Letter. Our analysis is limited to the cases $s = 1$ and $s = 2$. To extend it to massless $s = 0$, one should refine the analysis of the ZDM contribution to next order in the temperature. Moreover, it would be interesting to analyze the contribution of fermions using similar techniques. This should not be obtained by naively setting s to half-integer values in our formulas but requires further analysis. It would also allow the investigation of the supersymmetric case by studying the relevant BH geometries.

Kerr geometry—The four-dimensional asymptotically flat Kerr black hole metric in the Boyer-Lindquist coordinates is

$$ds^2 = -dt^2 + dr^2 + 2a_{\text{BH}}\sin^2\theta dr d\phi + (r^2 + a_{\text{BH}}^2\cos^2\theta)d\theta^2 + (r^2 + a_{\text{BH}}^2)\sin^2\theta d\phi^2 + \frac{2Mr}{r^2 + a_{\text{BH}}^2\cos^2\theta}(dt + dr + a_{\text{BH}}\sin^2\theta d\phi)^2, \quad (1)$$

where M is the mass of the black hole and a_{BH} is the parameter describing its angular momentum. The radial geometry admits two horizons: the event horizon $R_h = M + \sqrt{M^2 - a_{\text{BH}}^2}$ and an inner Cauchy horizon $R_i = M - \sqrt{M^2 - a_{\text{BH}}^2}$. The temperature and the angular velocity at the event horizon read

$$T_H = \frac{R_h - R_i}{8\pi MR_h}, \quad \Omega_H = \frac{a_{\text{BH}}}{2MR_h}. \quad (2)$$

Based on the Newman-Penrose formalism, Teukolsky [35] showed that in terms of curvature invariants, the perturbation equations decouple and separate for all Petrov type-D spacetimes. Let us consider the Fourier transform of a spin- s field $\Phi(t, r, \theta, \phi)$ and expand it in spin-weighted spherical harmonics as

$$\Phi = \frac{1}{2\pi} \int e^{-i\omega t} \sum_{\ell=|s|}^{\infty} \sum_{m=-\ell}^{\ell} e^{im\phi} S_{\ell m}(\theta, \phi) R_{\ell m}(r) d\omega, \quad (3)$$

where ω is the frequency of the perturbation, and ℓ and m are the angular quantum numbers. With the above separation of variables, both the radial and angular equations can be rewritten as confluent Heun equations [24]:

$$\mathcal{D}\psi \equiv \psi''(z) + \left[\frac{u - \frac{1}{2} + a_0^2 + a_1^2}{z(z-1)} + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} + \frac{\frac{1}{4} - a_0^2}{z^2} + \frac{\mu\epsilon}{z} - \frac{\epsilon^2}{4} \right] \psi(z) = 0. \quad (4)$$

In particular, we are interested in the dictionary for the radial equation, given by [36]

$$\begin{aligned} z &= \frac{r - R_i}{R_h - R_i}, \quad \psi(z) = (r^2 - 2Mr + a_{\text{BH}}^2)^{\frac{s+1}{2}} R_{\ell m}(r), \\ a_0 &= i \frac{\omega - m\Omega_H}{4\pi T_H} - 2iM\omega - \frac{s}{2}, \\ a_1 &= i \frac{\omega - m\Omega_H}{4\pi T_H} + \frac{s}{2}, \\ \epsilon &= -16i\pi M\omega R_h T_H, \\ \mu &= s - 2iM\omega, \\ u &= -{}_sA_{\ell m} - s(s+1) + 7M^2\omega^2 \\ &\quad + 8\pi M\omega R_h T_H(2M\omega + is) + (4\pi M\omega R_h T_H)^2, \end{aligned} \quad (5)$$

where ${}_sA_{\ell m}$ is the separation constant, whose quantization can be obtained by solving the associated angular problem and imposing the regularity conditions at $\theta = 0$ and $\theta = \pi$. For example, the expansion of the separation constant around $a_{\text{BH}}\omega = 0$ reads [37]

$${}_sA_{\ell m} = (\ell - s)(\ell + s + 1) - \frac{2ms^2\omega a_{\text{BH}}}{\ell(\ell + 1)} + \mathcal{O}(a_{\text{BH}}^2\omega^2). \quad (6)$$

Notice that the angular problem does not depend at all on M , and so the expression of the separation constant is not affected by passing to the (near-)extremal limit. Moreover, we find it useful to introduce the following quantities, which are linear combinations of the parameters a_0 and a_1 [39]:

$$\begin{aligned} \mu_1 &= a_0 - a_1 = -s - 2iM\omega, \\ \mu_2 &= -a_0 - a_1 = 2iM\omega - i \frac{\omega - m\Omega_H}{2\pi T_H}. \end{aligned} \quad (7)$$

One-loop determinant—Here, we consider the relevant connection problem for the radial problem to obtain the quantization condition determining the QNMs. It is useful to rewrite the above equation with the redefinition

$$w(z) = e^{-\epsilon z/2} z^{-\gamma/2} (1-z)^{-\delta/2} \psi(z), \quad (8)$$

so that the confluent Heun differential equation becomes

$$\frac{d^2 w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon \right) \frac{dw}{dz} + \frac{\alpha z - q}{z(z-1)} w = 0, \quad (9)$$

with

$$\begin{aligned} \gamma &= 1 - 2a_0, \\ \delta &= 1 - 2a_1, \\ \alpha &= \mu\epsilon + \frac{\gamma + \delta}{2}\epsilon, \\ q &= \frac{1}{4} - u + \alpha - \frac{(\gamma + \delta - 1)^2}{4} - \frac{\delta\epsilon}{2}. \end{aligned} \quad (10)$$

The boundary conditions require the presence of only ingoing modes at the black hole horizon and only outgoing modes at radial infinity. In terms of the function w , these translate into the following requirements:

$$\begin{aligned} w(z) &\sim 1 \quad \text{for } z \sim 1, \\ w(z) &\sim e^{-\epsilon z} z^{\frac{\alpha}{2} - \gamma - \delta} \quad \text{for } z \sim \infty. \end{aligned} \quad (11)$$

We remark that, by definition, the interval $r \in [R_h, \infty[$ corresponds to the real interval $z \in [1, \infty[$. The connection formula from the irregular singularity at $z = \infty$ to the regular singularity at $z = 1$, performing the analytical continuation along the real axis and for small values of $\epsilon \ll 1$, reads [40]:

$$\begin{aligned} &e^{-\epsilon z} z^{\frac{\alpha}{2} - \gamma - \delta} \text{HeunC}_{\infty}[q - \gamma\epsilon, \alpha - \epsilon(\gamma + \delta), \gamma, \delta, -\epsilon; z] \\ &= \epsilon^{\frac{1}{2} - \mu} e^{-\frac{1}{2}\partial_{\mu} F} \times \left[\sum_{\sigma=\pm} \frac{\Gamma(-2\sigma a) \Gamma(1 - 2\sigma a) \Gamma(2a_1) \epsilon^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F + \frac{1}{2}\partial_{a_1} F}}{\Gamma(\frac{1}{2} - \mu_1 - \sigma a) \Gamma(\frac{1}{2} - \mu_2 - \sigma a) \Gamma(\frac{1}{2} - \mu - \sigma a)} \times \text{HeunC}(q - \alpha, -\alpha, \delta, \gamma, -\epsilon; 1 - z) \right. \\ &\quad \left. + \sum_{\sigma=\pm} \frac{\Gamma(-2\sigma a) \Gamma(1 - 2\sigma a) \Gamma(-2a_1) \epsilon^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F - \frac{1}{2}\partial_{a_1} F}}{\Gamma(\frac{1}{2} + \mu_1 - \sigma a) \Gamma(\frac{1}{2} + \mu_2 - \sigma a) \Gamma(\frac{1}{2} - \mu - \sigma a)} \times (1 - z)^{1 - \delta} \text{HeunC}\{\tilde{q}, -\alpha - (1 - \delta)\epsilon, 2 - \delta, \gamma, -\epsilon; 1 - z\} \right], \end{aligned} \quad (12)$$

where $\tilde{q} = q - \alpha - (1 - \delta)(\epsilon + \gamma)$ and a is the composite monodromy parameter around the singularities at $z = 0$ and $z = 1$. We refer to Supplemental Material [22] for the relevant notations.

As computed in Eq. (24) in Supplemental Material [22], the contribution of QNMs to the determinant [41] at fixed quantum numbers ℓ, m, s is

$$\det_{(\ell, m, s)}^{[\text{Kerr}]} = \sqrt{2\pi} \times \sum_{\sigma=\pm} \frac{\Gamma(-2\sigma a) \Gamma(1-2\sigma a) \epsilon^{\frac{1}{2}-\mu+\sigma a} e^{-\frac{1}{2}(\sigma\partial_a+\partial_{a_1}+\partial_\mu)F}}{\Gamma(\frac{1}{2}+\mu_1-\sigma a) \Gamma(\frac{1}{2}+\mu_2-\sigma a) \Gamma(\frac{1}{2}-\mu-\sigma a)}. \quad (13)$$

Let us remark that the above formula displays a branch cut in ω due to the presence of the term $\epsilon^{\frac{1}{2}-\mu+\sigma a}$ and the dictionary (5).

The exact quantization condition for QNMs follows by studying the zeros of (13). This corresponds to

$$\frac{\Gamma(1-2a)^2 \Gamma(\frac{1}{2}+\mu_1+a) \Gamma(\frac{1}{2}-\mu+a)}{\Gamma(1+2a)^2 \Gamma(\frac{1}{2}+\mu_1-a) \Gamma(\frac{1}{2}-\mu-a)} e^{-\partial_a F} \times \frac{\Gamma(\frac{1}{2}+\mu_2+a)}{\Gamma(\frac{1}{2}+\mu_2-a)} \epsilon^{2a} = 1. \quad (14)$$

In the confluent limit corresponding to the extremal Kerr case in which the horizons R_h and R_i coalesce, $\epsilon \rightarrow 0$, and $\mu_2 \rightarrow \infty$, in such a way that $\Lambda \equiv \epsilon\mu_2$ is finite [42]. In this limit,

$$\frac{\Gamma(\frac{1}{2}+\mu_2+a)}{\Gamma(\frac{1}{2}+\mu_2-a)} \epsilon^{2a} \sim \left[\left(\frac{1}{2} + \mu_2 - a \right) \epsilon \right]^{2a} = \Lambda^{2a} + \mathcal{O}(\epsilon), \quad (15)$$

where we used the Stirling approximation $\Gamma(z+\alpha)/\Gamma(z) \sim z^\alpha$ as $z \rightarrow \infty$ and $\alpha \in \mathbb{C}$. The expansion of the parameter a is determined up to an overall sign and reads

$$a = \pm \sqrt{\frac{1}{4} - u} + \mathcal{O}(\epsilon). \quad (16)$$

The choice of the branch in (16) is a symmetry of the theory, and we choose the positive one to ensure that $\epsilon^a \rightarrow 0$ in the small ϵ expansion.

As we see from (15), in the extremal limit an entire branch of solutions associated to the poles of the Γ function in the second line of (14) decouples from the spectrum. From (13), it is clear that one of the two channels of the connection problem gets suppressed, and (14), up to corrections of order ϵ^{2a} , is solved by the poles of the Γ function in the remaining channel, which is precisely the one appearing in the numerator of (15):

$$\frac{1}{2} + a + \mu_2 = -n, \quad n \in \mathbb{Z}_{\geq 0}. \quad (17)$$

The physical properties of this branch of modes become clear by using the dictionary (5) to the gravitational quantities.

Scaling with the temperature—Let us consider the following small temperature expansions

$$\begin{aligned} a &= \sum_{j \geq 0} a^{(j)} T_H^j, \\ {}_s A_{\ell m} &= \sum_{j \geq 0} {}_s A_{\ell m}^{(j)} T_H^j, \\ \omega_{n, \ell, m}^* &= \sum_{j \geq 0} \frac{\beta_j}{a_{\text{BH}}} T_H^j, \end{aligned} \quad (18)$$

where we denoted with $\omega_{n, \ell, m}^*$ the series expansion in the temperature of the frequency solutions to (17). Since μ_2 diverges for $a_{\text{BH}} \rightarrow M$, the leading order of the lhs of (17) is finite if [see (7)]

$$\beta_0 = m/2. \quad (19)$$

Then, inserting (18) and (19) in (16), we find

$$a^{(0)} = \frac{1}{2} \sqrt{4 {}_s A_{\ell m}^{(0)} - 7m^2 + (1+2s)^2}, \quad (20)$$

where

$${}_s A_{\ell m}^{(0)} = {}_s A_{\ell m} |_{\omega=m/(2a_{\text{BH}})}. \quad (21)$$

With the substitutions (18) and (19) in (7), we find

$$\mu_2 = -i\sqrt{2M}\beta_1 + \mathcal{O}(T_H). \quad (22)$$

Hence, the quantization condition (17) gives at leading order

$$\beta_1 = -i \frac{2n+1 + \sqrt{4 {}_s A_{\ell m}^{(0)} - 7m^2 + (1+2s)^2}}{2\sqrt{2M}}. \quad (23)$$

Substituting (19) and (23) in (18), we get

$$\omega_{n, \ell, m}^* = \frac{m}{2a_{\text{BH}}} - i\pi T_H \left(2n+1 + \sqrt{4 {}_s A_{\ell m}^{(0)} - 7m^2 + (1+2s)^2} \right) + \mathcal{O}(T_H^2), \quad (24)$$

which has a parametrically small negative imaginary part, in agreement with formula (59) of [43], proportional to the temperature T_H . These modes have a small imaginary part in the near-extremal limit and are called zero-damping modes. As displayed in Supplemental Material [22], our analysis in the leading order matches the one in [44]. In particular, the modes with $m < 0$ in (24) are understood as those arising from the symmetry $\omega_{n, \ell, m}^* \rightarrow -\bar{\omega}_{n, \ell, -m}^*$. Our results align with the studies in [44–49].

A particular sub-branch of modes can be obtained for $m = 0$. In this case, the expansion of the separation constant ${}_sA_{\ell m}$ around $a_{\text{BH}}\omega = 0$ in (6) becomes relevant. Indeed, in the near-extremal limit,

$$a_{\text{BH}}\omega_{n,\ell,m}^* = \frac{m}{2} + \mathcal{O}(T_H), \quad (25)$$

and this scales with the temperature for $m = 0$. It follows that (24) simplifies to

$$\omega_{n,\ell,0}^* = -i(n + \ell + 1)2\pi T_H + \mathcal{O}(T_H^2), \quad (26)$$

which were considered also in [50].

Let us underline that the Γ function that gives the modes (24) arises in the step of the connection procedure, which gets performed locally around the black hole horizon {for the full details about the confluent Heun connection problem, we refer to formulas (3.2.11) and (3.2.12) in [40]}. Therefore, the existence of a branch of modes with small imaginary part decoupling in the confluence procedure does not depend on the near-infinity region, and applies to any black hole background admitting a near-extremal regime in which the inner and outer horizon approach each other, independently from their asymptotic geometry. This proves the universality of the mechanism responsible for the temperature scaling in these extremal limits. We intend to address the asymptotically (anti-)de Sitter case in future research.

Let us now focus on the contribution to the effective action of the Γ function from which ZDMs can arise, and compute the corresponding leading contribution in the Hawking temperature.

Let us start by studying the normalization of (13). From (15), we have

$$\frac{\epsilon^{-a}\Lambda^a}{\Gamma(\frac{1}{2} + \mu_2 + a)} \sim \frac{\epsilon^a\Lambda^{-a}}{\Gamma(\frac{1}{2} + \mu_2 - a)} [1 - \Lambda^{-2a}\mathcal{O}(\epsilon)], \quad (27)$$

and this permits us to rewrite (13) in the factorized form

$$\det_{(\ell,m,s)}^{[\text{NEK}]} = \det_{(\ell,m,s)}^{[\text{ZDM}]} \left[\det_{(\ell,m,s)}^{[\text{EK}]} + \mathcal{O}(\epsilon) \right], \quad (28)$$

where

$$\det_{(\ell,m,s)}^{[\text{ZDM}]} = \frac{\sqrt{2\pi}\epsilon^{\frac{1}{2}-\mu}(\epsilon/\Lambda)^a}{\Gamma(\frac{1}{2} + \mu_2 - a)}, \quad (29)$$

$$\det_{(\ell,m,s)}^{[\text{EK}]} = \sum_{\sigma=\pm} \frac{\Gamma(-2\sigma a)\Gamma(1-2\sigma a)\Lambda^{\sigma a}e^{-\frac{1}{2}(\sigma\partial_a+\partial_{a_1}+\partial_\mu)F}}{\Gamma(\frac{1}{2} + \mu_1 - \sigma a)\Gamma(\frac{1}{2} - \mu - \sigma a)}. \quad (30)$$

Equation (29) is the factor that decouples in the extremal limit and that contains the contribution of the ZDMs,

whereas (30) is the QNM determinant of the doubly confluent Heun equation, encoding the spectral problem of the extremal Kerr case. Thanks to (27), the definition of the normalization (29) is independent of the branch of (16) up to orders in ϵ . In (29), for definiteness, we chose the positive branch. The terms in (29) characterize the corrections of the near-extremal geometry, including the normalization factors of the confluent Heun solutions. Indeed, when considering the extremal Kerr limit one must redefine the z variable. As can be seen from (5), in this limit, $z \rightarrow \infty$ and $\epsilon \rightarrow 0$ with the local new variable ϵz finite. The redefinition of the local variable implies a Jacobian factor that cancels the term $\epsilon^{\frac{1}{2}-\mu+a}$ in the numerator of (29) [51]. The gauge quantities a and F in (30) are redefined as the corresponding quantities in the $N_f = 2$ gauge theory. In (30) the expressions for μ and μ_1 remain the same as appearing in (5) and (7), and $\Lambda = -8M^2\omega(\omega - m\Omega_H^{\text{ext}})$, with $\Omega_H^{\text{ext}} = 1/(2M)$.

To determine the temperature contribution to the partition function, we therefore focus on the Γ function in (29), and, inspired by the analysis in [52], we substitute the gravitational dictionary (5) evaluated in the Matsubara frequencies. Moreover, we focus on the case $s > 0$, specifying in the end how the analysis applies to the case $s < 0$. By imposing the exponent of the local solution in Eq. (3) in Supplemental Material [22] to be a non-negative integer, we find the following quantization for the Matsubara frequencies associated with the QNMs:

$$2a_1 = -k, \quad k \geq 0, \quad (31)$$

from which, using (5),

$$\omega_k^{(M)} = m\Omega_H + 2\pi iT_H(k + s). \quad (32)$$

This selects the frequencies with good Euclidean continuation in the Kerr geometry, and is analogous to the results found in [52] for $s = 2$ perturbations around the rotating BTZ black hole.

By using the gravitational dictionary (5), the expression of the Matsubara frequencies (32), and the first-instanton expansion for a :

$$a = \sqrt{\frac{1}{4} - u} - \frac{u + a_0^2 - a_1^2}{4u\sqrt{\frac{1}{4} - u}}\mu\epsilon + \mathcal{O}(\epsilon^2), \quad (33)$$

we find

$$\mu_2|_{\omega=\omega_k^{(M)}} = k + s + 2iM[m\Omega_H + 2\pi iT_H(k + s)] \quad (34)$$

so that

$$\begin{aligned} \frac{1}{2} + \mu_2 - a|_{\omega=\omega_k^{(M)}} &= \frac{1}{2} + k + s + im \\ &\quad - \frac{1}{2} \sqrt{4_s A_{\ell m}^{(0)} - 7m^2 + (2s+1)^2} \\ &\quad + c_1(m)T_H + \mathcal{O}(T_H^2), \end{aligned} \quad (35)$$

where $c_1(m)$ is given in Eq. (30) in Supplemental Material [22]. When $m = 0$, by comparison with the expansion for the separation constant ${}_s A_{\ell 0}$ around $a_{\text{BH}}\omega = 0$ in (6), we obtain

$$c_1(0) = -4\pi M(k+s), \quad (36)$$

where we used that, in the near-extremal limit,

$$a_{\text{BH}}\omega_k^{(M)} = \mathcal{O}(T_H) \quad (37)$$

for $m = 0$. We remark that for $m = 0$ and $\omega = \omega_k^{(M)}$ the parameter $\epsilon \sim \mathcal{O}(T_H^2)$ in the small T_H expansion. We finally get

$$\begin{aligned} \frac{1}{2} + \mu_2 - a|_{\omega=\omega_k^{(M)}, m=0} \\ = k + s - \ell - 4\pi M T_H(k+s) + \mathcal{O}(T_H^2). \end{aligned} \quad (38)$$

Taking the product over the quantum numbers k, ℓ, m , we thus find that the inverse of the determinant coming from (29), up to factors independent of the temperature, is equal to

$$\begin{aligned} \prod_{\ell \geq s} \prod_{m=-\ell}^{\ell} \prod_{k \geq 0} \left[\frac{1}{\det_{(\ell, m, s)}^{\text{[ZDM]}}} \right]_{\omega=\omega_k^{(M)}} &\sim \prod_{\ell \geq s} \prod_{m=-\ell}^{\ell} \prod_{k \geq 0} \Gamma\left(\frac{1}{2} + \mu_2 - a\right)_{\omega=\omega_k^{(M)}} \\ &= \prod_{\ell \geq s} \prod_{m=-\ell}^{\ell} \prod_{k \geq 0} \Gamma\left(\frac{1}{2} + k + s + im - \frac{1}{2} \sqrt{4_s A_{\ell m}^{(0)} - 7m^2 + (2s+1)^2} + \mathcal{O}(T_H)\right) \\ &= \prod_{\ell \geq s} \prod_{0 \neq m=-\ell}^{\ell} \prod_{k \geq 0} \Gamma\left(\frac{1}{2} + k + s + im - \frac{1}{2} \sqrt{4_s A_{\ell m}^{(0)} - 7m^2 + (2s+1)^2} + \mathcal{O}(T_H)\right) \\ &\quad \times \prod_{\ell \geq s} \prod_{k \geq 0} \Gamma[k + s - \ell - 4\pi M T_H(k+s) + \mathcal{O}(T_H^2)], \end{aligned} \quad (39)$$

where in the second equality we split the contributions of $m \neq 0$ and $m = 0$. Equation (39) gives small contributions in the low-temperature limit only for the poles of the Γ functions. For this reason, the factors corresponding to $m \neq 0$ give finite contributions in the small temperature limit, their argument being different from nonpositive integers. Indeed, ${}_s A_{\ell m}^{(0)}$ evaluated in the leading order in T_H of (32), that is $m\Omega_H$, is real, and so is the term under the

square root. Then, if the square root is real, in the argument of the Γ function there is an imaginary part given by im , whereas if the square root is imaginary, the argument of the Γ function has a positive real part $\frac{1}{2} + k + s$. Therefore, the leading factors in the low-temperature behavior come from the last line. By neglecting subleading corrections in the temperature, we have

$$\begin{aligned} \prod_{\ell \geq s} \prod_{k \geq 0} \Gamma[k + s - \ell - 4\pi M T_H(k+s)] &= \prod_{\ell' \geq 0} \prod_{k \geq 0} \Gamma[k - \ell' - 4\pi M T_H(k+s)] \\ &= \prod_{k \geq 0} \prod_{r \in \mathbb{Z}} \Gamma[r - 4\pi M T_H(k+s)] \sim \prod_{k \geq 0} \prod_{r \in \mathbb{Z}} \prod_{n \geq 0} [n + r - 4\pi M T_H(k+s)]^{-1} \\ &= \prod_{r \in \mathbb{Z}} \prod_{n+r \neq 0} \prod_{k \geq 0} [n + r - 4\pi M T_H(k+s)]^{-1} \prod_{n \geq 0} \prod_{k \geq 0} [-4\pi M T_H(k+s)]^{-1}. \end{aligned} \quad (40)$$

In the above computation, we first introduced $\ell' = \ell - s$, then $r = k - \ell'$, and we used the infinite product representation of the Γ function. Finally, we can select the last infinite product in (40) corresponding to $r = -n$ as the only one scaling with the temperature, and rewrite it as

$$\begin{aligned} \prod_{n \geq 0} \prod_{k \geq 0} [-4\pi M T_H(k+s)]^{-1} &\sim \prod_{n \geq 0} \Gamma_1(4\pi s M T_H | 4\pi M T_H) \\ &= \prod_{n \geq 0} \frac{\Gamma(s)}{\sqrt{2\pi}} (4\pi M T_H)^{s-\frac{1}{2}} \sim T_H^{\frac{s}{2}-\frac{1}{4}} \end{aligned} \quad (41)$$

where we used

$$\Gamma_1(z|a) = \frac{a^{a^{-1}z-\frac{1}{2}}}{\sqrt{2\pi}} \Gamma(a^{-1}z), \quad (42)$$

and in the last line we extracted the explicit T_H dependence by a further ζ regularization $\prod_{n \geq 0} T_H = T_H^{1/2}$.

The gravitational perturbations are described by the Teukolsky equation with $s = \pm 2$, which correspond to the two helicity states of the on-shell graviton. They give both the same contributions thanks to the symmetry of the separation constant

$$-sA_{\ell m} = sA_{\ell m} + 2s \quad (43)$$

and the fact that $s \rightarrow -s$ swaps QNMs and anti-QNMs in the analysis of Matsubara frequencies [53]. Taking into account that the one-loop partition function is proportional to the inverse square root of the determinant, we conclude

$$Z_{1\text{-loop}}^{\text{NEK}} \sim \frac{1}{\sqrt{|\det[\text{ZDM}]|^2}} \frac{1}{\sqrt{|\det[\text{EK}]|^2}} \sim T_H^{\frac{3}{2}} Z_{1\text{-loop}}^{\text{EK}}, \quad (44)$$

where $Z_{1\text{-loop}}^{\text{EK}}$ is the one-loop partition function of the extremal Kerr geometry. Analogously, the temperature scaling we find for $s = 1$ is $T_H^{\frac{1}{2}}$.

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