Strongly Interacting Fermions Are Nontrivial yet Nonglassy

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Random spin systems at low temperatures are glassy and feature computational hardness in finding low-energy states. We study the random all-to-all interacting fermionic Sachdev-Ye-Kitaev model and prove that, in contrast, the low-energy states have polynomial circuit depth, yet the annealed and quenched free energies agree to polynomially inverse low temperatures, ruling out a glassy phase transition in this sense. These results are derived by showing that fermionic and spin systems significantly differ in their "commutation index," which quantifies the noncommutativity of Hamiltonian terms. Our results suggest that low-temperature strongly interacting fermions, unlike spins, belong in a classically nontrivial yet quantumly easy phase.

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Introduction—Simulating ground and thermal state properties of quantum systems is a key application of future quantum computers [1–7]. Nevertheless, the search for particular, favorable instances that are quantumly easy and classically hard is not clear-cut [8]. A challenge is that current quantum computers are limited in quality and size, requiring the community to rely on theoretical arguments to give computational separations. However, the ground states for standard few-body quantum spin models can be QMA-hard (as classical spin models are NP-hard) in the worst case [9–11]; in the average case, random classical and quantum spin models exhibit glassy physics where computational hardness may arise [12,13]. To give an efficient quantum algorithm for low-temperature states, one must carefully avoid these instances.

Most chemical and condensed matter systems involve *fermionic* degrees of freedom, not only spins. Of particular importance in quantum chemistry is the *strongly interacting* regime, where Gaussian states do not give good approximations to the ground state and the Hartree-Fock method fails [14]. This has been proposed as a promising regime in which to apply quantum computers to achieve quantum advantage [3]. The Sachdev-Ye-Kitaev (SYK) Hamiltonian provides a natural model for strongly interacting fermions [15–17]. As a counterpart to random spins, it is a random Hamiltonian consisting of all-to-all q-body Majorana fermions,

$$\boldsymbol{H}_q^{\mathrm{SYK}} := i^{q/2}\binom{n}{q}^{-1/2}\sum_{j_1<\ldots<j_q} g_{j_1\ldots j_q}\gamma_{j_1}\cdots\gamma_{j_q}, \qquad (1)$$

where q is assumed to be even, $\gamma_i\gamma_j+\gamma_j\gamma_i=2\delta_{ij}$, and the $g_{j_1\ldots j_q}$ are independent identically distributed (i.i.d.) standard Gaussian random variables.

While the four-body fermionic ground state problem can be just as hard as spin models in the worst case (NP-hard) [18], average-case fermionic systems appear to have qualitatively different physics from and perhaps different computational complexity than spin systems [13,19,20]. Extensive heuristic calculations (such as large-N expansions) together with numerical evidence indicate that the SYK model resembles a thermalizing chaotic system, not a frozen spin glass as occurs with few-body quantum spin systems [13,21]. However, rigorous proofs that go beyond the physical arguments have been very limited [19,22].

In this Letter, we study the strongly interacting SYK model and give quantitative evidence that random, all-to-all connected fermionic systems have a "classically nontrivial" yet "nonglassy" thermal state at constant temperatures. In contrast, these two properties are false for disordered spin systems [23,24]. Remarkably, the proofs of both main results rely on the same quantity, the "commutation index" [25].

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This quantity pinpoints a crucial and often overlooked distinction between fermionic and spin Hamiltonians: low-degree fermionic monomials have a very different commutation structure than low-weight Pauli operators. The commutation index captures this difference, quantifying the fundamental distinction in the physics of local spin systems and local fermionic systems. This disparity, we argue, is the origin of a potential quantum advantage in simulating strongly interacting fermionic systems. Although the SYK model is our primary example, we also show that versions of our results—depending only on the commutation index—apply to *all* models with i.i.d. Gaussian couplings. This highlights the surprising importance of quantum uncertainty in governing both glassiness and the entanglement of low-energy states in disordered many-body systems.

More precisely, we first show that all low-energy states (including constant-temperature thermal states) of the SYK model have high circuit complexity (classically nontrivial). (We here consider disordered models where the distribution of $\boldsymbol{H}$ is identical to that of $-\boldsymbol{H}$, so the ground state energy is equivalent to the maximal energy.).

Theorem 1—(Low-energy states of SYK model are classically nontrivial [26]): Consider the degree-q SYK model $\boldsymbol{H}_q^{\mathrm{SYK}}$. With high probability, the maximum energy is $\lambda_{\max}(\boldsymbol{H}_q^{\mathrm{SYK}}) \geq \Omega_q(\sqrt{n})$, yet any state $\boldsymbol{\rho}$ such that

$$\mathrm{Tr}(\boldsymbol{\rho}\boldsymbol{H}_q^{\mathrm{SYK}}) \geq t\sqrt{n} \tag{2}$$

has circuit complexity

$$\tilde{\Omega}_q(n^{(q/2)+1}t^2). \tag{3}$$

The Ω_q notations assume a fixed q and growing n.

That is, low-energy states of the SYK model are highly entangled and require many parameters to describe; simple classical *Ansätze*, such as Gaussian states, must fail. In comparison, local quantum spin systems are known to have efficiently computable product state approximations to the ground state [24] and thus, in this sense, have "trivial" states that achieve a constant-factor approximation of the ground state energy.

Second, we show that the quenched free energy of the SYK model agrees with the annealed free energy even at very low temperatures (nonglassy), formalizing and strengthening previous results of this nature [13,21,27–29]. Here, the free energy is normalized such that $\beta = \mathcal{O}(1)$ corresponds to constant physical temperature.

Theorem 2—(SYK model is annealed at low temperatures): Consider the partition function of the degree-q SYK model $Z_\beta := \mathrm{Tr}\exp(-\beta\sqrt{n}\boldsymbol{H}_q^{\mathrm{SYK}})$. Then, we have

$$\frac{\mathbb{E}\log Z_\beta}{n} \leq \frac{\log \mathbb{E} Z_\beta}{n} \leq \frac{\mathbb{E}\log Z_\beta}{n} + \mathcal{O}_q(\beta^2 n^{-q/2}). \tag{4}$$

The $\mathcal{O}_q$ notations assume a fixed q and growing n.

The quantitative agreement of the two free energies at (inverse-polynomially) low temperatures strikes a stark contrast with disordered spin systems: the SYK model does not experience a "glass" phase transition in the sense of quenched vs annealed free energy. For classical spin Hamiltonians, it is known that the annealed free energy $n^{-1}\mathbb{E}\log Z_\beta$ fails to agree with the quenched free energy $n^{-1}\log \mathbb{E} Z_\beta$ at constant temperatures where the Hamiltonian is in its glassy phase and algorithmic hardness arises (see Supplemental Material [30], Sec. I); disordered quantum spin systems undergo a similar transition at constant temperature [23]. The lack of a glass transition for the SYK model suggests that there may be no algorithmic obstructions to preparing low-temperature states of the model on a quantum computer, but we do not prove this claim. We leave finding such an efficient quantum algorithm for future work.

Finally, we study the ground state energy of the SYK model as a function of the locality q, as an extension of Theorem 1 with potentially large q.

Theorem 3—(Lower bounding the norm with q dependence): For every q, n, it holds that the maximum energy of the degree-q SYK model is $\mathbb{E}\lambda_{\max} \geq \Omega(\sqrt{n}/q)$.

We show a similar $\Omega(\sqrt{n}/q)$ scaling for q-body quantum spin glasses. To our knowledge this is the first *lower bound* on the maximum eigenvalue, which scales as $\sqrt{n}/q$; for both models only the scaling at constant q was previously known [19,22,31,32]. Our lower bound technique relies on a measure of anticommutation that we call the commutation degree, which counts the maximum number of operators that anticommute with any given operator in the Hamiltonian. Interestingly, the commutation degree does not distinguish between local spin operators and local fermionic operators.

Background and related work: The SYK model is a canonical instance of a chaotic Hamiltonian [15–17] with related models studied as far back as [33,34]. For even $q = o(\sqrt{n})$, the SYK model has a Gaussian spectrum [22] and heuristics from physics indicate that the expected maximum energy of the SYK model scales as $(\sqrt{2n}/q)$ for even q [19,35,36]. However, the only rigorous result we are aware of with explicit constants is an upper bound of $\sqrt{(\log 2)n}$ [22]. Though Gaussian state approximation algorithms exist for fermionic systems [24,31], it is known that for the SYK model with $q \geq 4$, Gaussian states cannot achieve constant-factor approximations to the maximum energy [37]. Separate from the SYK model, so-called no low-energy trivial state (NLTS) theorems rule out constant-factor approximations to ground energies with low depth circuits in worst-case settings [38,39]. For random nonlocal Hamiltonians, [40] shows a circuit lower bound for sparse, sampled Pauli models using a similar technique as in the proof of Theorem 1, where the commutation index is much more straightforward to calculate.

The commutation index has connections to other areas of quantum information theory and Hamiltonian complexity.

In [41,42], the commutation index (there termed the "generalized radius") is used to study generalized Heisenberg uncertainty relations, and is shown to be a quantitative measure for the quantum complementarity present in a set of observables. Related to our work, [19] use the commutation index to analyze the performance of sum-of-squares relaxations of the SYK model and prove the $q = 4$ instance of Theorem 4, giving as well an algorithm verifying $\Omega(\sqrt{n})$ energy for $q = 4$. Reference [32] demonstrates that product states maximize the energy variance for random quantum spin Hamiltonians. Finally, the commutation index appears in quantum learning theory, where it provides a sample-complexity lower bound on how many copies of the state are required to learn the expectation values of a set of operators via shadow tomography [25,43].

Commutation structure of local operators—We define the commutation index $\Delta(\mathcal{S})$ of a given set of operators $\mathcal{S} = \{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m\}$ as follows:

$$\Delta(\mathcal{S}) \coloneqq \sup_{|\psi\rangle} \frac{1}{m} \sum_{i=1}^{m} \langle\psi|\boldsymbol{A}_i|\psi\rangle^2. \tag{5}$$

The commutation index is closely related to the minimum uncertainty of the operators $\boldsymbol{A}_i$. For instance, when the $\boldsymbol{A}_i$ are Pauli operators, the minimum quantum uncertainty of the $\boldsymbol{A}_i$ is given by

$$\inf_{|\psi\rangle} \frac{1}{m} \sum_{i=1}^{m} (\Delta_{|\psi\rangle}\boldsymbol{A}_i)^2 = 1 - \Delta(\mathcal{S}), \tag{6}$$

where $(\Delta_{|\psi\rangle}\boldsymbol{A}_i)^2$ is the variance of $\boldsymbol{A}_i$ in the state $|\psi\rangle$

$$(\Delta_{|\psi\rangle}\boldsymbol{A}_i)^2 = \langle\psi|\boldsymbol{A}_i^2|\psi\rangle - \langle\psi|\boldsymbol{A}_i|\psi\rangle^2 = 1 - \langle\psi|\boldsymbol{A}_i|\psi\rangle^2. \tag{7}$$

In this Letter, we will see that the commutation index has strong implications for the physics of the model $\boldsymbol{H} = m^{-1/2}\sum_{i=1}^{m} g_i\boldsymbol{A}_i$ with Gaussian coefficients g_i. When the commutation index is small and the operators have high quantum uncertainty, two properties follow: the model has highly entangled low-energy states, and the model is annealed at low temperatures.

Crucially, the commutation index controls the sensitivity of many physical properties when varying the couplings of the model. For instance, the norm of the energy gradient of a given state with respect to the disorder is bounded by

$$\|\nabla_{\boldsymbol{g}}\langle\phi|\boldsymbol{H}|\phi\rangle\|_2^2 = \frac{1}{m}\sum_{i=1}^{m}\langle\phi|\boldsymbol{A}_i|\phi\rangle^2 \le \Delta(\mathcal{S}). \tag{8}$$

Our key observation is that the commutation index of the set $\mathcal{S}_q^n$ of $\binom{n}{q}$ degree-q Majorana operators is very small.

Theorem 4—Let $\mathcal{S}_q^n$ be the set of degree-q Majorana operators on n fermionic modes. Then for any constant, even q,

$$\Delta(\mathcal{S}_q^n) = \Theta_q(n^{-q/2}). \tag{9}$$

TABLE I. The commutation index $\Delta(\mathcal{S})$ characterizes how noncommuting a set $\mathcal{S}$ of operators is. The commutation index reveals a key distinction between local spin operators and local fermionic operators: in the fermionic case, the commutation index decays polynomially with system size, while it is constant in the case of spins. The Θ_q notation assumes a fixed q and growing n.

Set $\mathcal{S}$	Commutation index $\Delta(\mathcal{S})$
Commuting	1
k-local Paulis	3^{-k} (Supplemental Material [30], Proposition II.1)
Degree-q Majoranas	$\Theta_q(n^{-q/2})$ (Theorem 4)
All Paulis	2^{-n} [43] [Lemma 5.8]

The decay with system size n is unique to the fermionic setting—for local Pauli operators, $\Delta(\mathcal{S})$ is constant with respect to n (see Table I). This behavior was first conjectured in [19] to our knowledge, and we establish the conjecture—including the setting when q scales with n—in Supplemental Material [30], Sec. II E.

The proof of Theorem 4 involves constructing the commutation graph $G(\mathcal{S})$, whose vertices correspond to operators $\boldsymbol{A}_i \in \mathcal{S}$ with edges between operators if and only if they anticommute. The commutation index can be upper bounded by $\Delta \le \vartheta[G(\mathcal{S})]/|G|$, where $\vartheta[G(\mathcal{S})]$ is the so-called "Lovász theta function" of the commutation graph. The Lovász theta function can be efficiently computed via a semidefinite program [44]. For the SYK Hamiltonian, $G(\mathcal{S}_q^n)$ is the graph of a certain Johnson association scheme [45].

Circuit lower bound—An almost direct consequence of a decaying commutation index is a lower bound on the complexity of any *Ansatz* in constructing near-ground states, including the *Ansatz* of quantum circuits.

Theorem 5—(Low-energy states are classically nontrivial): Consider the random Hamiltonian $\boldsymbol{H} = m^{-1/2}\sum_{i=1}^{m} g_i\boldsymbol{A}_i$ with i.i.d. Gaussian coefficients g_i. Let Δ be the commutation index of $\mathcal{S} = \{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m\}$ and let $\mathcal{C}$ be a set of fixed quantum states. Then the probability that $\mathcal{C}$ contains a low-energy state is exponentially small,

$$\mathbb{P}\left[\max_{|\psi\rangle\in\mathcal{C}}\langle\psi|\boldsymbol{H}|\psi\rangle \ge t\right] \le \exp\left(\log|\mathcal{C}| - \frac{t^2}{2\Delta}\right). \tag{10}$$

This theorem follows from a concentration argument and a union bound. The commutation index $\Delta(\{\boldsymbol{A}_i\}_{i=1}^m)$ characterizes the maximum variance of the energy $\langle\psi|\boldsymbol{H}|\psi\rangle$ for an arbitrary fixed state $|\psi\rangle$. Standard concentration bounds then imply that the probability a state $|\psi\rangle$ has energy t is bounded as $\exp[-\Omega(t^2/\Delta)]$. This concentration is so strong that one can bound the maximum energy over extremely large sets of states (or ϵ nets of infinite sets) $\mathcal{S}$ via a simple union bound argument with high probability over the disorder. In particular, we obtain a lower bound

TABLE II. To achieve energy scaling as $t\lambda_{\max}(\boldsymbol{H}_q^{\mathrm{SYK}})$ for the SYK Hamiltonian with high probability, *Ansatz* complexity (e.g., circuit depth) must scale polynomially with n. See Supplemental Material [30], Sec. V for proofs. The Ω_q notations assume a fixed q and growing n.

Ansatz	Circuit complexity[a]
Quantum circuit with G gates	$G \geq \tilde{\Omega}_q(n^{q/2+1}t^2)$
Matrix product state with bond dimension χ	$\chi \geq \Omega_q(n^{q/4+1/2}t)$
Neural network with W parameters	$W \geq \Omega_q(n^{q/2+1}t^2)$

[a]Min. complexity to achieve energy $t\lambda_{\max}(\boldsymbol{H}_q^{\mathrm{SYK}})$ with high probability.

$|\mathcal{S}| = \exp[\Omega(t^2/\Delta)]$ on the cardinality of the class of *Ansätze* needed to achieve a given energy t.

Specializing to the SYK model via Theorem 4, we summarize the implications of this result for various classes of states $\mathcal{S}$ in Table II. For instance, we show that all states that achieve a constant [i.e., $t = \Theta(1)$] approximation ratio with the SYK ground state energy have a quantum circuit depth of $\Omega_q(n^{q/2})$. In contrast, product states give constant-factor approximations to the ground state energy for any local spin Hamiltonian (see Supplemental Material [30], Sec. V for a short proof). Our argument also extends to classical *Ansätze*. For instance, tensor network methods require a bond dimension that grows polynomially with n to construct near-ground states [46,47]. Similarly, popular methods based on neural quantum states [48–52] need at least $\Omega(n^3)$ parameters to construct near-ground states for the standard $q = 4$ SYK model, implying a bounded depth fully connected network must have layer width that grows as $\Omega(n^{3/2})$.

Our circuit lower bound is related to the study of NLTS Hamiltonians, whose existence was conjectured in [53] and resolved in [38,39]. However, the settings are not strictly comparable: our instances are random (average-case), whereas NLTS is formalized for worst-case bounded interaction instances of Hamiltonians. The randomness allows us to prove stronger statements in two ways. First, our circuit lower bounds hold for states at *any* constant temperature, rather than for states below some energy threshold. Second, we can achieve arbitrary polynomial circuit depth lower bounds, whereas current constructions of NLTS only give a logarithmic depth lower bound. See Supplemental Material [30], Sec. V for more discussion.

Annealed approximation—The commutation index also has direct implications for the concentration of various physical properties of interest around their disordered expectation. One manifestation of this is in the relation between the "quenched" and "annealed" free energies,

$$\underbrace{\frac{1}{n}\mathbb{E}\ln Z_\beta}_{\text{quenched}} \leq \underbrace{\frac{1}{n}\ln \mathbb{E} Z_\beta}_{\text{annealed}}, \tag{11}$$

where Z_β is the partition function of the model $\sqrt{n}\boldsymbol{H}$ at an inverse temperature β. The quenched free energy assumes the disorder induced by the random couplings is fixed when averaging over thermal fluctuations; the annealed free energy treats these fluctuations on an equal footing. While the two quantities agree at high temperature, at low temperature the latter is incapable of accounting for frustration induced by the disorder of the random couplings that can induce a spin glass phase [54,55]. Their disagreement is thus indicative of the presence of a spin glass phase (see Supplemental Material [30], Sec. I). Motivated by this, our second main result bounds the difference in quenched and annealed free energies as a function of the temperature and the commutation index of the model.

Theorem 6—(Annealed at low temperatures): Consider the partition function $Z_\beta := \mathrm{Tr}\exp(-\beta\sqrt{n}\boldsymbol{H})$ of the random Hamiltonian $\boldsymbol{H} = m^{-1/2}\sum_{i=1}^m g_i \boldsymbol{A}_i$ with i.i.d. Gaussian coefficients g_i. Let Δ be the commutation index of $\mathcal{S} = \{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m\}$. Then the quenched and annealed free energies are bounded by

$$\frac{1}{n}\mathbb{E}\ln Z_\beta \leq \frac{1}{n}\ln \mathbb{E} Z_\beta \leq \frac{1}{n}\mathbb{E}\ln Z_\beta + 4\beta^2\Delta. \tag{12}$$

For the SYK model, this directly implies Theorem 2. Informally, this bound is due to controlling the growth of the moment generating function of $\ln(Z_\beta) - \mathbb{E}[\ln(Z_\beta)]$ using the commutation index Δ. We formally prove Eq. (12) in Supplemental Material [30], Sec. III. We there also prove concentration bounds for observable expectations as well as two-point correlators, again following from bounding how sensitive these quantities are when varying the disorder. We summarize some of these results when applied to the SYK model in Table III. We also emphasize that this general theorem applies to any model for which the commutation index is known. In the case of the k-body Pauli models, this indicates that the annealed approximation remains valid for β growing exponentially with the locality k as previously predicted [56].

Lower bound for the SYK maximum eigenvalue—Finally, we show how commutation properties of local Hamiltonians can give rise to lower bounds on the

TABLE III. Concentration bounds for functions f of the Hamiltonian around its mean, i.e., $\mathbb{P}[|f - \mathbb{E}[f]| \geq t] \leq 4\exp(-Kt^2)$. $\lambda_{\max}$ denotes the largest eigenvalue and $\boldsymbol{X}$ is an arbitrary bounded operator. $\boldsymbol{\rho}_\beta$ is the thermal state of $\sqrt{n}\boldsymbol{H}$ at an inverse temperature β.

Quantity f	Rate K
$\lambda_{\max}(\boldsymbol{H}_q^{\mathrm{SYK}})$	$\Omega_q(n^{q/2})$
$\mathrm{Tr}(\boldsymbol{X}\boldsymbol{\rho}_\beta)$	$\Omega_q(\beta^{-2}n^{q/2-1})$
$\mathrm{Tr}(\boldsymbol{H}_q^{\mathrm{SYK}}\boldsymbol{\rho}_\beta)$	$\Omega_q(\min(1,\ \beta^{-2}n^{-2})n^{q/2})$[a]

[a]For t order of $\|\boldsymbol{H}_q^{\mathrm{SYK}}\| = \mathcal{O}(\sqrt{n})$.

maximum eigenvalue. In particular, for any random Hamiltonian $\boldsymbol{H} = m^{-1/2}\sum_{i=1}^{m} g_i \boldsymbol{A}_i$, where all $\boldsymbol{A}_i^2 = \mathbf{I}$, the scaling of the maximum eigenvalue is related to what we call the "commutation degree" $h_{\mathrm{comm}}(\{\boldsymbol{A}_i\}_{i=1}^m)$ of the operators comprising the Hamiltonian,

$$h_{\mathrm{comm}}(\{\boldsymbol{A}_i\}_{i=1}^m) := \frac{1}{2}\sup_i \sum_{j=1}^{m} \left\| [\boldsymbol{A}_i, \boldsymbol{A}_j] \right\|, \tag{13}$$

where $\|\cdot\|$ denotes the operator norm. The term commutation degree is derived from the fact that it is the maximal degree of the commutation graph associated with the set $\mathcal{S} = \{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m\}$ (see Supplemental Material [30], Sec. II. B). The commutation degree can be interpreted as controlling the maximal amount of operator spreading of any $\boldsymbol{A}_i$ under the dynamics of $\boldsymbol{H}$. Using this intuition we are able to control how sensitive the expected partition function is to the inverse temperature,

$$\frac{\partial}{\partial\beta}\mathbb{E}\mathrm{Tr}[e^{\beta \boldsymbol{H}}] \geq \beta\left(1 - \frac{c_1\beta^2 h_{\mathrm{comm}}}{m}\right)\mathbb{E}\mathrm{Tr}[e^{\beta \boldsymbol{H}}], \tag{14}$$

where $c_1 > 0$ is an absolute constant. Using the lower bound of $\mathrm{Tr}[e^{\beta \boldsymbol{H}}] \leq \exp\left(\beta O(n) + \beta\lambda_{\max}(\boldsymbol{H})\right)$, we prove in Supplemental Material [30], Sec. IV that, for all β,

$$\mathbb{E}\exp\left(\beta\lambda_{\max}(\boldsymbol{H})\right) \geq \exp\left(\frac{\beta^2}{2}\left(1 - \frac{c_1\beta^2 h_{\mathrm{comm}}}{2m}\right)\right). \tag{15}$$

Using the fact that the maximal eigenvalue concentrates (as in, e.g., Table III) and maximizing the bound over β then implies

$$\mathbb{E}\lambda_{\max}(\boldsymbol{H}) \geq \frac{\sqrt{m}}{4\sqrt{c_1 h_{\mathrm{comm}}}}(1 - 16\Delta). \tag{16}$$

By the same concentration results this bound also holds for $\lambda_{\max}(\boldsymbol{H})$ with high probability over the disorder, not just in expectation. Intriguingly, the overall scaling depends only on the commutation degree h_{comm}. This quantity (when normalized by m) agrees to leading order for both the q-body quantum spin glass model $\boldsymbol{H}_q^{\mathrm{SG}}$ and the SYK model $\boldsymbol{H}_q^{\mathrm{SYK}}$. As long as $\Delta < 1/16$—which is true for the former when $q \geq 3$, and always for the latter for sufficiently large n—this implies that for *both* models,

$$\lambda_{\max}(\boldsymbol{H}_q^{\mathrm{SYK}}) \geq \Omega\left(\frac{\sqrt{n}}{q}\right), \quad \lambda_{\max}(\boldsymbol{H}_q^{\mathrm{SG}}) \geq \Omega\left(\frac{\sqrt{n}}{q}\right). \tag{17}$$

We here allow the locality q to potentially grow with n as $n \to \infty$. This strengthens the previously known $\Omega(\sqrt{n})$ scaling for both when q is constant [19,31,32]. For $q = \omega(\sqrt{n})$ it is known that both models exhibit a phase transition in their spectrums to a semicircle law [22,57], which is consistent with our results.

Discussion—Our results identify a single quantity—the commutation index—that controls both the entanglement of low-energy states and the ergodicity of quantum systems. As the commutation index is a quantitative measure of noncommutativity, our results provide the potential for new connections between quantum contextuality, quantum learning theory, and quantum spin glass theory. Furthermore, the commutation index can be efficiently bounded via a semidefinite program [41], and thus our methods can be used to efficiently identify if a system has highly entangled low-energy states or is ergodic at low temperatures. Our results suggest that studying the low-temperature properties of systems with highly noncommuting terms is a fertile testing ground for quantum experiments probing beyond what is classically simulable.

Note added—We recently became aware of Ref. [58], which also establishes the necessary results on the Lovász theta function of Johnson association schemes. Our results use different proof techniques and determine the explicit q dependence of the constant in Eq. (9), which was not derived in [58].

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