

Langer's Nucleation Rate Reproduced on the Lattice

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We show that Langer's rate of bubble nucleation is quantitatively correct up to small higher-loop corrections, in comparison to lattice simulations. These results are a significant advancement on decades of lattice studies showing only qualitative trends, and the first showing agreement for any conservative field theory. We confirm that the failure to fully thermalize the metastable phase explains discrepancies with recent lattice studies that found disagreement with Langer's rate. The key theoretical development is the translation of Langer's perturbative definition of a thermal metastable phase into a nonperturbative statement that can be implemented on the lattice. Our statistical and systematic errors are small enough to allow us to measure on the lattice the coefficient of the two-loop contribution, missing from the perturbative prediction. Our conclusions also exclude a possible systematic uncertainty in ^3He experiments.

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Introduction—The theory of bubble nucleation in homogeneous systems goes back at least 150 years to Gibbs [1], who introduced the notion of a critical bubble, for which the surface tension balances against the outward pressure. Important early developments in this field include Becker and Döring's discrete model for the dynamics of growing bubbles [2], Wigner's application of the transition state idea to chemical reactions [3], and Zeldovich's computation of the nucleation rate by modeling the bubble radius as a particle undergoing Brownian motion [4]. This latter result borrowed heavily from earlier work by Kramers on the escape rate of a classical thermal particle from a potential well [5]. A field theoretic description of critical bubbles was introduced by Cahn and Hilliard [6].

The modern theory of homogeneous nucleation was initiated by Langer in the late 1960s [7,8]. This constitutes a dynamical framework for studying nucleation, based on a description of the system in terms of stochastic Hamilton's equations. Langer's work generalizes Kramer's analysis of escape rates to an arbitrary number of degrees of freedom and to field theory [9]. This is sufficiently flexible to describe a wide range of systems.

For instance, consider a system in d spatial dimensions described by a scalar field $\phi(t, x)$, canonical momentum field $\pi(t, x)$, and the following Hamiltonian equations:

$$\partial_t \phi = \pi, \quad \partial_t \pi = -\frac{\delta H}{\delta \phi}, \quad (1)$$

where H is the effective Hamiltonian. The system is thermalized locally around a homogeneous metastable background ϕ_0 , normalized such that $H[\phi_0] \equiv H[\phi = \phi_0, \pi = 0] = 0$. The temperature is $T = 1/\beta$.

The rate per unit volume at which bubbles form was found to be [7,8]

$$\Gamma_{\text{Langer}} = \frac{V}{2\pi} \left(\frac{\beta H[\phi_{\text{cb}}]}{2\pi} \right)^{\frac{d}{2}} \left| \frac{\det H^{(2)}[\phi_0]}{\det^+ H^{(2)}[\phi_{\text{cb}}]} \right|^{\frac{1}{2}} e^{-\beta H[\phi_{\text{cb}}]} \quad (2)$$

in terms of the critical bubble ϕ_{cb} , a saddlepoint of the effective Hamiltonian. The superscript in $H^{(2)}$ denotes the second functional derivative, and the plus on $\det^+$ means that only positive eigenvalues are included. Equation (2) relies on the saddlepoint approximation to a path integral, valid if the rate is slow, and the system is weakly coupled.

Equation (2) was a major theoretical advance because it was the first complete expression of the nucleation rate to be derived from the underlying equations that describe the system and its fluctuations. As a consequence, the elements of Eq. (2) can all be computed self-consistently in terms of parameters in the Hamiltonian. While it relies on a saddlepoint approximation, this can be systematically improved [10]. More recently, Refs. [11,12] outlined the theory for matching to Eq. (2) from an underlying quantum field theory; see Supplemental Material Sec. I [13] for a summary.

Despite this provenance, and despite decades of effort, Langer's result has yet to be quantitatively tested. Experimental setups studying water, alcohols and other molecular liquids have in some cases found qualitative

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agreement on the nucleation rate, and in others disagreement by many orders of magnitude, including on the temperature dependence [28]. However, for such studies, the comparison to theory is obscured by uncertainties in modeling the physical system and in determining its microphysical parameters; see for instance Refs. [29,30]. Further, the possibility of impurities seeding nucleation raises persistent concerns.

A long-running experimental program on the A-B transition in liquid ^3He has sought to overcome these challenges, using electromagnetic trapping to avoid contact with vessel walls, but unfortunately remains inconclusive [31]. Recently, several experiments to test nucleation theory have been proposed which use analog systems [32–38]. Two such experiments have been performed [39,40], showing qualitative agreement between theory and experiment, though this required scaling and fitting, rather than direct comparison. Nevertheless, ongoing experimental efforts offer much promise.

In the absence of a quantitative experimental test, many researchers have investigated lattice simulations as a means to test homogeneous nucleation theory. The first exploratory studies were carried out in the late 1980s, and established the two main approaches. Grigoriev and Rubakov included thermal fluctuations at the initial time and then evolved under conservative Hamilton’s equations [41], while Bochkarev and de Forcrand simulated a Langevin equation, so sourcing thermal fluctuations throughout the evolution [42]. Investigating kink-antikink production, both found a Boltzmann-like exponential dependence on temperature, however Ref. [42], and later Ref. [43], found that the value of the exponent was significantly smaller than the saddlepoint prediction.

Extending this to nucleation from a metastable phase, Ref. [44] found saddlepoint and lattice results strongly disagreed in $2 + 1$ dimensions. Reference [45] improved agreement to within a factor of $e^{O(10)}$, using a loop-corrected effective potential. References [46,47] investigated nucleation in both $1 + 1$ and $2 + 1$ dimensions and were able to fit the saddlepoint form of the rate to the lattice results within statistical errors, by treating the prefactor as a constant nuisance parameter.

Moore, Rummukainen and Tranberg developed an alternative lattice approach for calculating the nucleation rate in very slow transitions, based on a split of the problem into statistical and dynamical parts [48,49]. Comparisons of the results of this method to perturbative predictions have shown qualitative agreement at best, with quantitative disagreement on the rate at the level of $e^{O(10)}$ to $e^{O(100)}$ [48–50].

In all these studies, the functional determinant in Eq. (2) was treated as an unknown fit parameter, or estimated crudely. Recent efforts to compute functional determinants have made it possible to directly evaluate Langer’s formula [21,51–53]. This was found to improve agreement in $3 + 1$ dimensions [54] from $\sim e^{50}$ to $\sim e^{10}$.

By far the closest quantitative agreement between lattice simulations and Langer’s formula was found in Refs. [21,53,55]. They considered the following scalar field model in $1 + 1$ dimensions:

$$H = \int dx \left(\frac{\pi^2}{2} + \frac{(\partial_x \phi)^2}{2} + \frac{m^2 \phi^2}{2} - \frac{\lambda \phi^4}{4} \right). \quad (3)$$

The metastable phase in this model is at $\phi_0 = 0$, and nucleation leads to escape to $\phi \rightarrow \pm\infty$. All m and λ dependence can be factored out by scaling $x \rightarrow (1/m)x$, $\phi \rightarrow (m/\sqrt{\lambda})\phi$ and $\pi \rightarrow (m^2/\sqrt{\lambda})\pi$. The sole remaining parameter which characterizes the system is then $\hat{T} \equiv \lambda T/m^3 \equiv 1/\hat{\beta}$.

References [21,53,55] found Eq. (2) to overestimate the measured rate by a factor of 8, arguing that this was due to a failure of the metastable phase to thermalize. The authors considered a modified system with damping and noise terms to aid thermalization. They found that, when these terms were sufficiently large, the lattice results agreed well with Langer’s generalized rate formula, which extends Eq. (2) to include damping. They also noted that the measured rate was time dependent, as previously observed in Ref. [56]. This implied a fundamental ambiguity in the definition of the nucleation rate.

We take the conservative Hamiltonian model and parameter points of Refs. [21,53] as our starting point. Its low dimensionality implies the toy-model aspects of strong superrenormalizability and slow thermalization of fluctuations, but simplifies the numerics allowing us to explore theoretical ideas. We aim to decisively test Langer’s formula by resolving the ambiguity in the definition of the nucleation rate, a key prerequisite for progress in more complicated and higher dimensional field theories.

Nucleation—The first step to understanding or computing the nucleation rate is defining it. While this might seem unambiguous, as we will argue below, some quantitative discrepancies in the literature result from different definitions of the nucleation rate.

We start from the observation that nucleation from a metastable phase is by assumption slow. It is relevant when the initial conditions are stable on short, microscopic timescales t_{micro} , but unstable on longer timescales, t_{global} , over which the field explores the full global potential and its multiple phases. Note that in our model with unbounded potential, escaping trajectories do not return so t_{global} is infinite. A natural microscopic timescale is that over which a slightly super-critical bubble grows, $1/\sqrt{|\lambda_-|}$, in terms of the single negative eigenvalue λ_- of fluctuations about the critical bubble.

Definition: The *escape rate* is the initial escape rate out of a region $\mathcal{R}$ in phase space, counting only those trajectories which do not return to $\mathcal{R}$ within any microscopic timescale. The initial probability density $\rho(t=0)$ is

taken to be nonzero within $\mathcal{R}$ and zero outside it. As an equation, this is

$$\Gamma \equiv - \int_{\mathcal{R}} \mathcal{D}\phi \mathcal{D}\pi \delta_{\text{NR}}(\mathcal{T}) \left. \frac{\partial \rho}{\partial t} \right|_{t=0}, \quad (4)$$

where $\mathcal{D}\phi \mathcal{D}\pi$ is the phase space integration measure. We have introduced a *no-return* (NR) projector, $\delta_{\text{NR}}(\mathcal{T})$, which removes configurations that return to $\mathcal{R}$ within a time $\mathcal{T}$. This can be defined as

$$\delta_{\text{NR}}(\mathcal{T}) \equiv \prod_{t=0}^{\mathcal{T}} \theta(\{\phi(t), \pi(t)\} \notin \mathcal{R}), \quad (5)$$

in terms of $\theta(\cdot)$, a Boolean step function which evaluates to 1 if the argument is true and 0 otherwise. The no-return projector ensures we only count the last escape of each evolution trajectory, and so avoid double counting. The timescale $\mathcal{T}$ is chosen such that $t_{\text{micro}} \ll \mathcal{T} \ll t_{\text{global}}$. An alternative approach would be to count all crossings, and then divide by the number of crossings [48,49].

The *thermal nucleation rate* is the escape rate where $\mathcal{R}$ is identified with a metastable phase (meta) which is thermally occupied. So, the initial condition for the probability distribution is $\rho(t=0) = \rho_{\text{meta}}$, where

$$\rho_{\text{meta}} \equiv \frac{e^{-\beta H[\phi, \pi]} \theta(\phi \in \text{meta})}{Z_{\text{meta}}}, \quad (6)$$

and Z_{meta} normalizes the distribution. This definition defines the metastable phase at a given instant.

The reasoning behind defining the nucleation rate at the initial time is as follows. Over sufficiently long times, the system approaches global thermal equilibrium, where all net probability fluxes vanish. If we were to define a rate as a function of time, it would inevitably vary with time and approach zero as $t \rightarrow \infty$. To extract a well-defined and meaningful value for the nucleation rate, we must define it at a given time, or over a given time interval, for an appropriate initial distribution.

The exponential growth of super-critical bubbles ensures that bubbles which have remained outside the metastable phase for a time $\mathcal{T}$ are extremely unlikely to return in the near future, so that the rate should be exponentially insensitive to the precise choice of $\mathcal{T}$.

The next question arising is how to define the metastable phase, i.e., the set $\phi \in \text{meta}$. In mean field theory, the metastable phase is defined by the value of some order parameter at a local minimum of the free energy, for instance $\phi = \phi_0$. Perturbatively one can extend this definition to include small fluctuations. However, for lattice simulations a nonperturbative definition is needed, which allows to unambiguously identify any given lattice configuration as either *in* or *not in* the metastable phase. We propose the following.

Definition: The *metastable phase* is the set of field configurations that flow to the metastable minimum under gradient descent. That is, $\phi(\tau \rightarrow \infty) \rightarrow \phi_0$ under

$$\frac{d\phi}{d\tau} = -\nabla_{\phi} H[\phi, 0]. \quad (7)$$

This definition is nonperturbative, but nevertheless connects to mean field theory through the endpoint of the evolution. The boundary of the metastable phase is a separatrix for Eq. (7).

The phase separation in a first-order phase transition comes from a potential energy barrier. Applying gradient descent takes a configuration down the potential barrier either to the metastable phase, or not. A similar approach has been successful in nonperturbative computations of the sphaleron rate [57,58].

This definition of the metastable phase is well suited to our $1+1$ dimensional model, for which ultraviolet fluctuations yield only perturbatively small, finite contributions to $\delta H/\delta\phi$ in the continuum limit; see Supplemental Material Sec. II [13]. However, Eq. (7) flows to the minima of the bare potential so, in models where ultraviolet fluctuations are more important, this equation should be extended with a noise term, fixed by the fluctuation dissipation relation. While this would make the identification of the metastable phase statistical, such fluctuations could be incorporated in the statistical uncertainty of the final nucleation rate.

For perturbative fluctuations around the critical bubble, $\phi_{\text{cb}} + \delta\phi$, gradient descent leads to exponential growth in the unstable, negative eigenmode of $H^{(2)}[\phi_{\text{cb}}]$, $\delta\phi_{-}(\tau) \approx \delta\phi_{-} e^{-\lambda_{-}\tau}$. The sign of $\delta\phi_{-}$ therefore determines the phase, agreeing with Langer's definition of the transition surface: $\delta\phi_{-} = 0$ [7,8]. Nevertheless, Langer's definition of the nucleation rate differs from ours, which we expect to manifest at subleading orders. First, the factor $\delta_{\text{NR}}(\mathcal{T})$ is omitted, because multiple crossings of the separatrix do not occur at Langer's perturbative order. Second, the boundary conditions are modified to make the problem time independent, and hence analytically tractable: A source is introduced on the metastable side and a sink on the stable side to precisely compensate the probability flux due to nucleation.

Thermalizing the metastable phase—To sample the metastable phase, we combine a Hamiltonian (or hybrid) Monte Carlo [24] update with an accept-reject step for the condition $\phi \in \text{meta}$. For further details and a proof of detailed-balance for the algorithm, see Supplemental Material Sec. IV [13].

The configuration ϕ is in the metastable phase if gradient descent takes the configuration asymptotically to zero, $\phi(\tau \rightarrow \infty) \rightarrow 0$. This happens exponentially quickly, with the exponent given by $|\lambda_{-}|\tau = 3\tau$, so in practice only a relatively short period of gradient descent is needed.

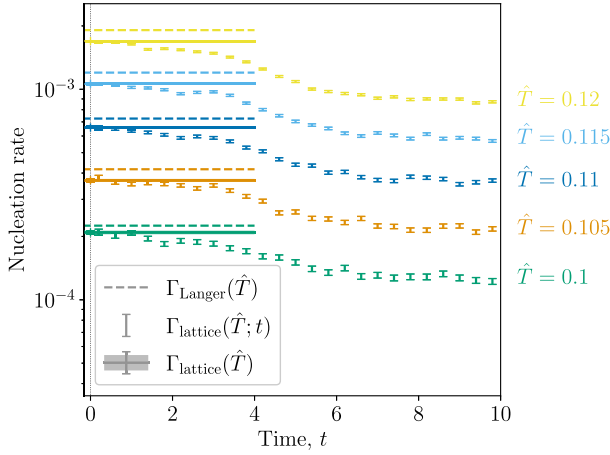


FIG. 1. Nucleation rates across different temperatures, showing numerically computed initial rates together with Langer's formula. Data points $\Gamma_{\text{lattice}}(\hat{T}; t)$ are the midpoints of histograms, and the bands labeled $\Gamma_{\text{lattice}}(\hat{T})$ are the results of linear extrapolation to $t \rightarrow 0$. Note the good agreement with Langer's rate up to $\sim 10\%$, the expected size of two-loop terms missing in the perturbative computation. We use units where $m = 1$.

For comparison, we consider a perturbative definition of the metastable phase: $\phi(t = 0, x) = \phi_0 + \delta\phi(t = 0, x)$, where $\phi_0 = 0$ is the mean-field metastable phase, and the fluctuations are Gaussian,

$$\rho_{\text{meta}}^{(G)} = \frac{e^{-\hat{\beta} \int dx (\frac{\pi^2}{2} + \frac{(\partial_x \delta\phi)^2}{2} + \frac{m_T^2 \delta\phi^2}{2})} \theta(\phi \in \text{meta})}{Z_{\text{meta}}^{(G)}}. \quad (8)$$

We have used the thermally corrected mass $m_T^2 = 1 - \frac{3}{2}\hat{T}$. As before, configurations which are outside the metastable phase of the full potential are removed from the initial ensemble. This accurately describes small fluctuations around the metastable minimum; see Supplemental Material Sec. V [13]. However, it significantly underrepresents larger fluctuations.

Results—Figure 1 summarizes our main results, and Fig. 2 shows a comparison with Gaussian initial fluctuations.

We have performed simulations for five temperatures $\hat{T} \in \{0.1, 0.105, 0.11, 0.115, 0.12\}$. In units where $m = 1$, we chose lattice extent $L = 50 \gg 1$, lattice spacing $\delta x = 0.1 \ll 1$ and time step $\delta t = 0.02 < \delta x$; for more details see Supplemental Material Sec. III [13]. We ensured that our results are independent of the precise choice of lattice parameters; see Supplemental Material Sec. V [13].

For each $\hat{T}$, we produced $N = 2 \times 10^7$ independent configurations in the metastable phase. Our final results use the gradient descent definition, but an alternative definition based on the eigenmodes of $\delta^2 H / \delta\phi^2$ gave consistent results.

The no-return criterion $\delta_{\text{NR}}(\mathcal{T})$ was implemented by evolving each configuration to a time $\mathcal{T} = 10 \gg 1/\sqrt{|\lambda_-|} = 1/\sqrt{3}$, under conservative Hamiltonian dynamics. If it is

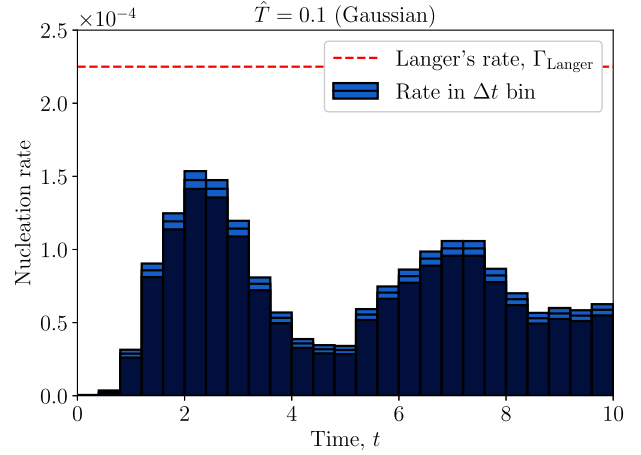


FIG. 2. Nucleation rate at $\hat{T} = 0.1$ for early times after Gaussian initialization around metastable phase minimum at time $t = 0$.

outside the metastable phase, we backtrack to the last step in the metastable phase, $t_{\text{last meta}}$ and define $t_{\text{escape}} = t_{\text{last meta}} + \delta t/2$ as the nucleation event. A configuration contributes to $N_{\text{meta}}(t)$ for $t < t_{\text{escape}}$.

We binned $N_{\text{meta}}(t)$ in time, choosing bin widths Δt via Scott's rule [59], and approximated the nucleation rate by mid-point differentiation $\Gamma_{\text{lattice}}(\hat{T}; t + \Delta t/2) = -[N_{\text{meta}}(t + \Delta t) - N_{\text{meta}}(t)]/\Delta t$.

Evaluating the critical bubble and functional determinant in this model [21], Langer's rate reads

$$\Gamma_{\text{Langer}}(\hat{T}) = L \frac{6}{\pi} \sqrt{\frac{2}{3\pi\hat{T}}} e^{-4/(3\hat{T})}, \quad (9)$$

in terms of the microscopic parameters. Subleading $l \geq 2$ loop corrections are suppressed relatively by $c_l \hat{T}^{l-1}$, with each $c_l = O(1)$.

Figure 1 shows the nucleation rate at all five temperatures simulated. For each temperature, extrapolating to $t \rightarrow 0$ with linear fits to $t \in [0, 3]$ yields $\Gamma_{\text{lattice}}(\hat{T})$, and quadratic fits agree within errors. The lattice results (coloured bands) are shown alongside Langer's rate (dashed lines). For $\hat{T} = 0.1$, the results are

$$\Gamma_{\text{lattice}}(0.1) = 2.09(4) \times 10^{-4}, \quad (10)$$

$$\Gamma_{\text{Langer}}(0.1) = 2.25(23) \times 10^{-4}. \quad (11)$$

The lattice error is statistical and the perturbative error is from two-loop corrections. Discretization errors are expected to be of order 0.5%, for which see Supplemental Material Sec. V [13], and hence much smaller than statistical errors.

The time dependence of the rate is similar across temperatures. Crucially, the results show that the lattice rate is tracked by Langer's formula.

The field theory we have simulated has fluctuations at the lattice scale, so that taking the continuum limit requires some care. Fortunately, this theory is superrenormalizable, and hence quickly approaches a free theory at high energies. More specifically, the only divergent counterterm is a constant addition to the Hamiltonian, which affects neither the dynamics nor the statistics of the system; see Supplemental Material Sec. II [13].

Statistical errors on the lattice results are computed by multiplying the standard deviation by the square root of twice the integrated autocorrelation time of the observable under consideration. As expected, the computed errors on the binned nucleation rate are comparable to the statistical fluctuations between bins.

To assess whether the lattice data aligns with expected higher-order corrections, we compare two models for the $\hat{T}$ dependence: (i) one including two-loop corrections, $\Gamma = \Gamma_{\text{Langer}}(1 + c_2\hat{T})$, and (ii) a simple rescaling, $\Gamma = c\Gamma_{\text{Langer}}$. Fitting the data yields $c_2 = -0.94(4)$ with $\chi^2/\text{d.o.f.} = 1.1$, and $c = 0.894(6)$ with $\chi^2/\text{d.o.f.} = 1.8$, indicating a marginally better fit for the perturbative model. A Bayesian analysis supports this, using priors for c_2 that are approximately uniform for $O(1)$ values and suppressed for larger magnitudes. We compare this to two alternative models based on the hypothesis that Langer's rate is significantly modified: one with an $O(1)$ multiplicative factor (log-normal prior) and one with an $O(10)$ factor (log-uniform prior); see Supplemental Material Sec. VI [13]. The resulting Bayes factors, 4.0 and 9.5 respectively, provide substantial to strong support for the perturbative model. This stems from philosophical differences between the models; the perturbative model assigns more prior weight in near agreement with Langer's rate.

Figure 2 shows that Gaussian initialization leads to a strongly time-dependent rate which starts from zero. This is because the Gaussian initialization gives exponentially smaller weight to configurations which are near the edge of the metastable phase, where $\frac{1}{2}m_T^2\phi^2 > V(\phi)$. The rate then peaks around $t \approx 2.5$, corresponding to configurations that escape in a single motion. Subsequent oscillations reflect configurations that bounce back and escape in later cycles. All of these time-dependent features thwart extracting a precise nucleation rate for such initial conditions.

One might hope that after some initial transitory behavior, the nucleation rate would settle to an approximately constant value. This has been investigated in Refs. [21,53], where they showed that over long times the rate decreases and remains significantly below Langer's formula at this parameter point. Figure 7 in Ref. [21] shows that the rate continues to decrease over times $O(10^3)$.

The rates shown in Fig. 1 are also time dependent, decreasing by a factor half over the time $\mathcal{T} = 10$. However, unlike the rate following Gaussian initialization, it has a well-defined, nonzero limit for $t \rightarrow 0$, allowing us to extract the thermal rate following Eq. (4).

Conclusions—For the first time, we have demonstrated the quantitative correctness of Eq. (2) for the nucleation rate in lattice simulations. The discrepancy is dramatically smaller than found in all previous lattice studies, the closest of which found a multiplicative discrepancy of ~ 8 for the same parameter point [21,53], though using a different definition of the nucleation rate. Most previous studies in other models have found multiplicative discrepancies between $e^{O(10)}$ and $e^{O(100)}$.

The precise agreement found between lattice and perturbation theory puts homogeneous nucleation theory on firm footing, and excludes the mooted possibility of the failure of Langer's theory to describe homogeneous nucleation. Thus, some other reason must explain the many orders of magnitude discrepancy between slower semiclassical and faster measured nucleation rates in ^3He .

The residual 10% difference between the lattice and Langer's semiclassical formalism is well explained by two-loop corrections, including the expected functional form. Thus, a two-loop computation of the nucleation rate, following Ref. [10], see also [60], would extend the test of agreement down to 1% and include new physical effects, such as curvature of the boundary between phases and back-reaction of the bubble on the thermal fluctuations.

A key development of this work has been a precise nonperturbative definition of the nucleation rate, that can be implemented on the lattice. This also reveals the *meaning* of nucleation implied by the perturbative expression. The rate depends sensitively on the initial distribution, and hence in turn relies on a precise definition of the metastable phase. This has important implications for simulations based on the truncated-Wigner approximation [61–63], that only accounts for Gaussian initial fluctuations, but also for analog experiments which will require a precise initial state.

In agreement with previous works, we find that observations of a time-dependent nucleation rate are a consequence of incomplete thermalization of the full metastable phase. This we confirmed by factorizing the simulation into first thermalization, and then conservative evolution. A completely thermalized metastable phase nucleates with Langer's rate, as we have demonstrated.

However, in a dynamical scenario, will the system thermalize sufficiently quickly to nucleate at Langer's rate? It has been argued that this occurs only if [64]

$$t_{\text{th}}\sqrt{|\lambda_-|} < \beta H[\phi_{\text{cb}}], \quad (12)$$

in terms of the thermalization time t_{th} , and the negative eigenvalue λ_- of fluctuations around the critical bubble. When this condition fails, configurations near the critical bubble are unable to thermalize.

For the parameter points we have studied, the observed decrease of the nucleation rate over relatively short times suggests that this condition is not met. Thus, we expect

Eq. (4) overestimates the time-dependent rate obtained in a dynamical case.

More generally, the appropriate value for $\beta H[\phi_{\text{cb}}]$ depends on the temperature at which the system is transitioning, typically when an $O(1)$ fraction is in each phase. This reflects both microscopic and macroscopic properties of the system, such as how quickly the temperature changes through the transition, with slower changes leading to transitions at larger $\beta H[\phi_{\text{cb}}]$. By contrast, the left-hand side of Eq. (12) is determined purely by microphysics, being larger for more weakly coupled fields.

In Big Bang cosmology, the slow cooling due to Hubble expansion leads to large values of $\beta H[\phi_{\text{cb}}]$, growing with $\sim 4 \log(M_{\text{Planck}}/T)$. For instance $\beta H[\phi_{\text{cb}}] \sim 40$ at grand unification, ~ 140 at the electroweak scale, and ~ 170 at the QCD scale [65]. A Boltzmann analysis predicts the thermalization time is approximately $t_{\text{th}} \sim 1/(\alpha^2 T)$ for a relativistic plasma in $3 + 1$ dimensions, with α the relevant fine-structure constant [66]. (Thermalization from light bosonic fields during nucleation is parametrically underestimated by the Boltzmann analysis [12].)

Putting this together for a first-order transition at the electroweak scale, and assuming a one-step transition with $\sqrt{|\lambda_-|} \sim \alpha^{3/4} T$ [67], suggests that models with couplings $\alpha \gtrsim 1/50$ should thermalize, and hence nucleate with Langer's rate. More weakly coupled fields will fail to thermalize, and hence will nucleate more slowly. However, this point deserves further scrutiny.

Finally, a key direction for future study is the generalization to higher dimensions, particularly in light of Ref. [54], which found much larger discrepancies between lattice and perturbative nucleation rates in $3 + 1$ dimensions. In higher dimensions, effects from high energy thermal fluctuations are more pronounced, but even in $3 + 1$ dimensions these are superrenormalizable and hence tractable. Thus, extending our approach to higher dimensions offers clear promise.

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Data availability—Data shown in Figs. 1 and 2 in this Letter are available at Ref. [68].

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- [1] J. W. Gibbs, On the equilibrium of heterogeneous substances, *Trans. Connecticut Acad. Arts Sci.* **3**, 343 (1874).
 - [2] R. Becker and W. Döring, Kinetische behandlung der keimbildung in übersättigten dämpfen, *Ann. Phys. (Berlin)* **416**, 719 (1935).

- [3] E. Wigner, The transition state method, *Trans. Faraday Soc.* **34**, 29 (1938).
- [4] Y. B. Zeldovich, On the theory of new phase formation. Cavitation, in *Selected Works of Yakov Borisovich Zeldovich, Volume I: Chemical Physics and Hydrodynamics*, Vol. 1, edited by J. P. Ostriker, G. I. Barenblatt, R. A. Sunyaev, A. Granik, and E. Jackson (Princeton University Press, Princeton, NJ, 1992), pp. 120–137.
- [5] H. A. Kramers, Brownian motion in a field of force and the diffusion model of chemical reactions, *Physica (Utrecht)* **7**, 284 (1940).
- [6] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system. III. Nucleation in a two-component incompressible fluid, *J. Chem. Phys.* **31**, 688 (1959).
- [7] J. S. Langer, Theory of the condensation point, *Ann. Phys. (N.Y.)* **41**, 108 (1967); *Ann. Phys. (N.Y.)* **281**, 941(E) (2000).
- [8] J. S. Langer, Statistical theory of the decay of metastable states, *Ann. Phys. (N.Y.)* **54**, 258 (1969).
- [9] A. Berera, J. Mabillard, B. W. Mintz, and R. O. Ramos, Formulating the Kramers problem in field theory, *Phys. Rev. D* **100**, 076005 (2019).
- [10] A. Ekstedt, Bubble nucleation to all orders, *J. High Energy Phys.* **08** (2022) 115.
- [11] O. Gould and J. Hirvonen, Effective field theory approach to thermal bubble nucleation, *Phys. Rev. D* **104**, 096015 (2021).
- [12] J. Hirvonen, Real-time nucleation and off-equilibrium effects in high-temperature quantum field theories, *Phys. Rev. D* **111**, 116020 (2025).
- [13] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/c75g-xbw4> for discussion of the connection to high-temperature quantum field theory, the continuum limit, lattice details, the thermalization algorithm, the numerical tests we performed, and the priors for model fits, which includes Refs. [12,14–27].
- [14] G. Aarts and J. Smit, Finiteness of hot classical scalar field theory and the plasmon damping rate, *Phys. Lett. B* **393**, 395 (1997).
- [15] G. Aarts and J. Smit, Classical approximation for time dependent quantum field theory: Diagrammatic analysis for hot scalar fields, *Nucl. Phys. B* **511**, 451 (1998).
- [16] D. Bodeker, Classical real time correlation functions and quantum corrections at finite temperature, *Nucl. Phys. B* **486**, 500 (1997).
- [17] C. Greiner and B. Muller, Classical fields near thermal equilibrium, *Phys. Rev. D* **55**, 1026 (1997).
- [18] J.-P. Blaizot and E. Iancu, The Quark gluon plasma: Collective dynamics and hard thermal loops, *Phys. Rep.* **359**, 355 (2002).
- [19] S. Jeon, The Boltzmann equation in classical and quantum field theory, *Phys. Rev. C* **72**, 014907 (2005).
- [20] M. Laine, Exact relation of lattice and continuum parameters in three-dimensional $SU(2) + \text{Higgs}$ theories, *Nucl. Phys. B* **451**, 484 (1995).
- [21] D. Pirvu, A. Shkerin, and S. Sibiryakov, Thermal false vacuum decay in $(1 + 1)$ -dimensions: Evidence for non-equilibrium dynamics, *Int. J. Mod. Phys. A* **39**, 2445007 (2024).
- [22] E. Forest and R. D. Ruth, Fourth order symplectic integration, *Physica (Amsterdam)* **43D**, 105 (1990).

- [23] H. Yoshida, Construction of higher order symplectic integrators, *Phys. Lett. A* **150**, 262 (1990).
- [24] S. Duane, A. D. Kennedy, B. J. Pendleton, and D. Roweth, Hybrid Monte Carlo, *Phys. Lett. B* **195**, 216 (1987).
- [25] M. Tuckerman, *Statistical Mechanics: Theory And Molecular Simulation* (Oxford University Press, New York, 2001), pp. 121–124.
- [26] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, Equation of state calculations by fast computing machines, *J. Chem. Phys.* **21**, 1087 (1953).
- [27] W. K. Hastings, Monte carlo sampling methods using markov chains and their applications, *Biometrika* **57**, 97 (1970).
- [28] D. W. Oxtoby, Homogeneous nucleation: Theory and experiment, *J. Phys. Condens. Matter* **4**, 7627 (1992).
- [29] J. S. Langer and A. J. Schwartz, Kinetics of nucleation in near-critical fluids, *Phys. Rev. A* **21**, 948 (1980).
- [30] R. B. Nellas, S. J. Keasler, J. I. Siepmann, and B. Chen, Exploring the discrepancies between experiment, theory, and simulation for the homogeneous gas-to-liquid nucleation of 1-pentanol, *J. Chem. Phys.* **132** (2010).
- [31] M. Hindmarsh *et al.* (QUEST-DMC Collaboration), A-B transition in superfluid ^3He and cosmological phase transitions, *J. Low Temp. Phys.* **215**, 495 (2024).
- [32] O. Fialko, B. Opanchuk, A. I. Sidorov, P. D. Drummond, and J. Brand, Fate of the false vacuum: Towards realization with ultra-cold atoms, *Europhys. Lett.* **110**, 56001 (2015).
- [33] O. Fialko, B. Opanchuk, A. I. Sidorov, P. D. Drummond, and J. Brand, The universe on a table top: Engineering quantum decay of a relativistic scalar field from a metastable vacuum, *J. Phys. B* **50**, 024003 (2017).
- [34] T. P. Billam, K. Brown, and I. G. Moss, False-vacuum decay in an ultracold spin-1 Bose gas, *Phys. Rev. A* **105**, L041301 (2022).
- [35] Y. Tian *et al.*, Supercooling of the A phase of ^3He , *Nat. Commun.* **14**, 148 (2023).
- [36] A. C. Jenkins, J. Braden, H. V. Peiris, A. Pontzen, M. C. Johnson, and S. Weinfurter, Analog vacuum decay from vacuum initial conditions, *Phys. Rev. D* **109**, 023506 (2024).
- [37] A. C. Jenkins, I. G. Moss, T. P. Billam, Z. Hadzibabic, H. V. Peiris, and A. Pontzen, Generalized cold-atom simulators for vacuum decay, *Phys. Rev. A* **110**, L031301 (2024).
- [38] S. Darbha, M. Kornjača, F. Liu, J. Balewski, M. R. Hirsbrunner, P. L. S. Lopes, S.-T. Wang, R. Van Beeumen, D. Camps, and K. Klymko, False vacuum decay and nucleation dynamics in neutral atom systems, *Phys. Rev. B* **110**, 155103 (2024).
- [39] B. Song, S. Dutta, S. Bhawe, J.-C. Yu, E. Carter, N. Cooper, and U. Schneider, Realizing discontinuous quantum phase transitions in a strongly correlated driven optical lattice, *Nat. Phys.* **18**, 259 (2022).
- [40] A. Zenesini, A. Berti, R. Cominotti, C. Rogora, I. G. Moss, T. P. Billam, I. Carusotto, G. Lamporesi, A. Recati, and G. Ferrari, Observation of false vacuum decay via bubble formation in ferromagnetic superfluids, *Nat. Phys.* **20**, 558 (2023).
- [41] D. Y. Grigoriev and V. A. Rubakov, Soliton pair creation at finite temperatures. Numerical study in $(1 + 1)$ -dimensions, *Nucl. Phys. B* **299**, 67 (1988).
- [42] A. I. Bochkarev and P. de Forcrand, The study of Kink—Antikink pair production at finite temperature by means of the Langevin equation, *Phys. Rev. Lett.* **63**, 2337 (1989).
- [43] M. G. Alford, H. Feldman, and M. Gleiser, Thermal nucleation of kink—anti-kink pairs, *Phys. Rev. Lett.* **68**, 1645 (1992).
- [44] O. T. Valls and G. F. Mazenko, Nucleation in a time-dependent Ginzburg-Landau model: A numerical study, *Phys. Rev. B* **42**, 6614 (1990).
- [45] S. Borsanyi, A. Patkos, J. Polonyi, and Z. Szepl, Fate of the classical false vacuum, *Phys. Rev. D* **62**, 085013 (2000).
- [46] M. G. Alford, H. Feldman, and M. Gleiser, Thermal activation of metastable decay: Testing nucleation theory, *Phys. Rev. D* **47**, R2168 (1993).
- [47] M. G. Alford and M. Gleiser, Metastability in two-dimensions and the effective potential, *Phys. Rev. D* **48**, 2838 (1993).
- [48] G. D. Moore and K. Rummukainen, Electroweak bubble nucleation, nonperturbatively, *Phys. Rev. D* **63**, 045002 (2001).
- [49] G. D. Moore, K. Rummukainen, and A. Tranberg, Non-perturbative computation of the bubble nucleation rate in the cubic anisotropy model, *J. High Energy Phys.* **04** (2001) 017.
- [50] O. Gould, S. Güyer, and K. Rummukainen, First-order electroweak phase transitions: A nonperturbative update, *Phys. Rev. D* **106**, 114507 (2022).
- [51] A. Ekstedt, Higher-order corrections to the bubble-nucleation rate at finite temperature, *Eur. Phys. J. C* **82**, 173 (2022).
- [52] A. Ekstedt, O. Gould, and J. Hirvonen, BubbleDet: A Python package to compute functional determinants for bubble nucleation, *J. High Energy Phys.* **12** (2023) 056.
- [53] D. Pírvu, A. Shkerin, and S. Sibiryakov, Thermal false vacuum decay is not what it seems, [arXiv:2407.06263](https://arxiv.org/abs/2407.06263).
- [54] O. Gould, A. Kormu, and D. J. Weir, Nonperturbative test of nucleation calculations for strong phase transitions, *Phys. Rev. D* **111**, L051901 (2025).
- [55] A. Shkerin and S. Sibiryakov, Simple third order operator-splitting schemes for stochastic mechanics and field theory, [arXiv:2504.03844](https://arxiv.org/abs/2504.03844).
- [56] L. Batini, A. Chatrchyan, and J. Berges, Real-time dynamics of false vacuum decay, *Phys. Rev. D* **109**, 023502 (2024).
- [57] S. Y. Khlebnikov and M. E. Shaposhnikov, The statistical theory of anomalous fermion number nonconservation, *Nucl. Phys. B* **308**, 885 (1988).
- [58] G. D. Moore, Measuring the broken phase sphaleron rate nonperturbatively, *Phys. Rev. D* **59**, 014503 (1999).
- [59] D. W. Scott, On optimal and data-based histograms, *Biometrika* **66**, 605 (1979).
- [60] M. Carosi and B. Garbrecht, False vacuum decay beyond the quadratic approximation: Summation of nonlocal self-energies, *Phys. Rev. D* **111**, 085002 (2025).
- [61] J. Braden, M. C. Johnson, H. V. Peiris, A. Pontzen, and S. Weinfurter, New semiclassical picture of vacuum decay, *Phys. Rev. Lett.* **123**, 031601 (2019) *Phys. Rev. Lett.* **129**, 059901(E) (2022).
- [62] T. P. Billam, R. Gregory, F. Michel, and I. G. Moss, Simulating seeded vacuum decay in a cold atom system, *Phys. Rev. D* **100**, 065016 (2019).

- [63] M. P. Hertzberg, F. Rompineve, and N. Shah, Quantitative analysis of the stochastic approach to quantum tunneling, *Phys. Rev. D* **102**, 076003 (2020).
- [64] P. Hanggi, P. Talkner, and M. Borkovec, Reaction-rate theory: Fifty years after kramers, *Rev. Mod. Phys.* **62**, 251 (1990).
- [65] K. Enqvist, J. Ignatius, K. Kajantie, and K. Rummukainen, Nucleation and bubble growth in a first order cosmological electroweak phase transition, *Phys. Rev. D* **45**, 3415 (1992).
- [66] A. Kurkela and G. D. Moore, Thermalization in weakly coupled nonabelian plasmas, *J. High Energy Phys.* **12** (2011) 044.
- [67] A. Ekstedt, O. Gould, and J. Löfgren, Radiative first-order phase transitions to next-to-next-to-leading order, *Phys. Rev. D* **106**, 036012 (2022); *Phys. Rev. D* **110**, 019901(E) (2024).
- [68] J. Hirvonen and O. Gould, Langer’s nucleation rate reproduced on the lattice, dataset, 2025, [10.5281/zenodo.15424939](https://zenodo.org/record/15424939).