

Ramanujan's $1/\pi$ Series and Conformal Field TheoriesFaizan Bhat^{1,*} and Aninda Sinha^{1,2,†}¹Centre for High Energy Physics, *Indian Institute of Science*, C.V. Raman Avenue, Bangalore 560012, India²Department of Physics and Astronomy, *University of Calgary*, Calgary, Alberta T2N 1N4, Canada

(Received 10 April 2025; revised 14 August 2025; accepted 20 October 2025; published 2 December 2025)

In 1914, Ramanujan unveiled 17 extraordinary infinite series for $1/\pi$. In this Letter, we uncover their physics origin by relating them to 2D logarithmic conformal field theories (LCFTs), which emerge in diverse settings such as the fractional quantum Hall effect, percolation, polymers, and even holography. Through this LCFT connection, we reinterpret such infinite series in terms of fundamental CFT data—the operator spectrum and OPE coefficients. This perspective leads to novel physics-inspired approximations for $1/\pi$. Drawing lessons from Ramanujan's formulas, we construct a new family of bases for expanding LCFT correlators that converge far more rapidly than the standard conformal block decomposition. This is achieved using recently developed stringy/parametric crossing-symmetric dispersion relations. Remarkably, when working with these new expansions, the action of a certain differential operator (which arises naturally from the Ramanujan connection) dramatically enhances convergence, with the entire contribution collapsing to that of the logarithmic identity operator. This striking simplification hints at a universal property of LCFTs. Finally, we discuss a new holographic interpretation of this unexpected mathematics-physics connection.

DOI: 10.1103/c38g-fd2v

Introduction—In 1914, Ramanujan [1] recorded 17 remarkable formulas for π of the form

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (\sigma)_n (1-\sigma)_n}{n!^3} (a + bn) z_0^n = \frac{1}{\pi}. \quad (1)$$

Here, $\sigma \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$, while z_0, b , and a are algebraic numbers. At present, many such Ramanujan-like formulas are known [2]. These formulas not only laid the foundation for all subsequent fast-converging series for π but also led to several breakthroughs that continue to shape mathematics even today, especially analytic number theory and computational mathematics. In this Letter, we show that these formulas have a natural physics interpretation in terms of four-point correlators in 2D logarithmic conformal field theories (LCFTs) [3–5]. LCFTs are a fascinating class of nonunitary CFTs. They describe statistical physics systems such as critical dense polymers [3], percolation [6–8], and the fractional quantum Hall effect [9]. They also appear in the study of celestial CFTs [10] and in the context of holography, describing the dynamics of a scalar field of a

particular mass in the AdS Schwarzschild black brane background [11]—this last connection is expanded on in the Appendix. The fact that Ramanujan's extraordinary formulas naturally arise in LCFTs opens up avenues for further exploration at the intersection of physics and mathematics.

All Ramanujan-like formulas originate from the Legendre relation, which is given as [12,13]

$$z(z-1)(F_{\sigma}(z)\partial_z F_{\sigma}(1-z) - F_{\sigma}(1-z)\partial_z F_{\sigma}(z)) = \frac{\sin \pi \sigma}{\pi}, \quad (2)$$

where

$$F_{\sigma}(z) = {}_2F_1(\sigma, 1-\sigma, 1, z). \quad (3)$$

An important clue to arrive at the physics connection lies in the analytic structure of $F_{\sigma}(z)$, which exhibits a logarithmic singularity at $z = 1$. This points us toward 2D LCFTs, which are known for having logarithmic behavior in their correlation functions, rather than the typical power-law behavior. More precisely, we consider the four-point correlator of twist operators in the $c = -2$ LCFT, which is arguably the most well-studied LCFT in the literature [3,4,14–17]—we give more details about the $c = -2$ LCFT in the Appendix and comment on other central charges in [18]. The correlator has been explicitly calculated and is given as [3,4]

*Contact author: faizanbhat@iisc.ac.in

†Contact author: asinha@iisc.ac.in

Published by the American Physical Society under the terms of the *Creative Commons Attribution 4.0 International* license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

$$\begin{aligned} G(z, \bar{z}) &= \kappa(z, \bar{z}) (F_\sigma(z) F_\sigma(1 - \bar{z}) + F_\sigma(\bar{z}) F_\sigma(1 - z)), \\ &\equiv G_L(z, \bar{z}) + G_R(z, \bar{z}), \end{aligned} \quad (4)$$

where $\kappa(z, \bar{z}) = |z(1 - z)|^{2\sigma(1-\sigma)}$. We have used σ to denote the twist of the operators, their conformal dimension $\Delta = \sigma(\sigma - 1)$, and z is the 2D conformal cross ratio. Now, recall that the Witt algebra generators are given as $\ell_n \equiv -z^{n+1} \partial_z$, $\bar{\ell}_n \equiv -\bar{z}^{n+1} \partial_{\bar{z}}$. Defining the operator $\mathcal{L} = -(\ell_0 - \bar{\ell}_0) + (\ell_1 - \bar{\ell}_1)$, which is a linear combination of rotation and special conformal transformation, the Legendre relation in the LCFT language is simply

$$\mathcal{L} G_L(z, \bar{z}) \Big|_{\bar{z}=z} = \kappa(z, z) \frac{\sin(\pi\sigma)}{\pi} = -\mathcal{L} G_R(z, \bar{z}) \Big|_{\bar{z}=z}. \quad (5)$$

Thus, we see that the σ appearing in Ramanujan's formulas [1] is mapped to the twists or, equivalently, to the conformal dimensions of the twist operators in the LCFT. In the Appendix, we discuss a physical interpretation of the Legendre relation via holography.

Having expressed the Legendre relation in terms of an LCFT correlator, we can explicitly specify the conformal data that goes into it by performing a conformal block decomposition. Doing so leads us to new approximations for π of the form given in (18). We have not encountered such approximations in the literature (at least they do not appear to be commonly reported—see, e.g., Ref. [2]). While novel and interesting, these formulas do not converge fast. This is because the blocks $g_{\Delta, \ell}(z)$ [see Eq. (14)] form a good basis only near $z, \bar{z} \sim 0$, while the correlator contains both $F_\sigma(z)$ and $F_\sigma(1 - z)$. As a result, for any $0 \leq z \leq 1$, convergence is slow. Ramanujan encountered the same issue: directly plugging the series expansion for $F_\sigma(z)$ and $F_\sigma(1 - z)$ into the Legendre relation does not yield fast-converging series for $1/\pi$. To resolve this, he came up with ingenious manipulations that exploited special properties of elliptic integrals and modular equations, which we will review later. The key idea of his techniques is to rewrite the Legendre relation, so only $F_\sigma(z)$ appears, and convergence becomes fast for $z \sim 0$.

Ramanujan's result motivates the following question: *Can one emulate his procedure to derive a new, rapidly converging basis for expanding LCFT correlators in terms of conformal data?* We answer this question in the affirmative. The key idea, motivated by lessons derived from the derivation of Ramanujan's formulas, is to employ the recently derived stringy dispersion relation [25] to express the full LCFT correlator solely in terms of its discontinuity across the logarithmic branch cut. Unlike the full correlator, this discontinuity admits a rapidly converging conformal block expansion around $z, \bar{z} = 0$, yielding an efficient basis for reconstructing the entire correlator. Remarkably, in this new basis, the Legendre relation in (5) converges incredibly fast; in fact, *it is reproduced*

entirely from the contribution of the identity operator alone. We emphasize that this striking simplification arises only after the application of the $\mathcal{L}$ operator—such a structure would remain hidden without casting the Legendre relation in LCFT language. This result does not depend on the details of the LCFT, but only on the fact that the correlators possess logarithmic singularities, hinting at a universal feature of LCFTs.

The mathematics behind Ramanujan's $1/\pi$ formulas— Let us briefly review the derivation of Ramanujan's formulas, as it is very instructive. As mentioned before, directly plugging the series representations for $F_\sigma(z)$ and $F_\sigma(1 - z)$ into the Legendre relation does not lead to fast-converging formulas for $1/\pi$. Any series that would arise from there would, first, be a double sum, and secondly, be very slowly convergent, since one sum needs $z \sim 0$, and the other $z \sim 1$ for fast convergence. The first step, therefore, is to recast the Legendre relation so that only $F_\sigma(z)$ appears. This is achieved using the so-called modular equations. According to these, for any rational n , whenever z and $\bar{z}$ satisfy a “modular” condition $z = f_n(\bar{z})$ [18,26], the following holds:

$$F_\sigma(1 - z) F_\sigma(\bar{z}) = n F_\sigma(z) F_\sigma(1 - \bar{z}). \quad (6)$$

Here, n is called the degree of the modular equation, and we shall only consider $n \in \mathbb{N}$. The case $n = 1$ is trivial, and the modular condition is just $z = \bar{z}$. As a nontrivial example, the modular condition for $n = 3$ is

$$(z\bar{z})^{1/4} + [(1 - z)(1 - \bar{z})]^{1/4} = 1. \quad (7)$$

Next, we solve the modular condition for singular values z_0 defined as $z_0 = f_n(1 - z_0)$. From (6), we then get

$$\sqrt{n} F_{\frac{1}{2}}(z_0) = F_{\frac{1}{2}}(1 - z_0). \quad (8)$$

Plugging this into the Legendre relation, the LHS becomes

$$\frac{z_0(1 - z_0)}{2} \left(\sqrt{n} \partial_z F_\sigma^2(z) \Big|_{z=z_0} + \frac{1}{\sqrt{n}} \partial_z F_\sigma^2(z) \Big|_{z=1-z_0} \right). \quad (9)$$

The job is not done yet, as the two terms are still being evaluated at z_0 and $1 - z_0$. Using (6) again, after some lengthy algebra [26], we arrive at the final form:

$$\sqrt{n} z_0 (1 - z_0) \left(\frac{d}{dz} - \frac{dM_n(\bar{z})}{d\bar{z}} \Big|_{\bar{z}=1-z_0} \right) F_{\frac{1}{2}}(z)^2 \Big|_{z=z_0} = \frac{1}{\pi}, \quad (10)$$

where we have defined the multiplier $M_n(\bar{z})$ as $M_n^2(\bar{z}) = \bar{z}(1 - \bar{z})/(z(1 - z)) dz/d\bar{z}$, where $z = f_n(\bar{z})$. For $n = 1$, $M_1(\bar{z}) = 1$. For $n = 3$, one can work out $dM_3/d\bar{z}|_{\bar{z}=1-z_0} = -4/\sqrt{3}$ with $z_0 = (2 - \sqrt{3})/4 \approx 0.067$. At this point,

we would get fast-converging formulas for $1/\pi$ by plugging the series representation for $F_\sigma(z)$, but it would still be a double sum. To get to the single sum that appears in Ramanujan's formulas, we use Clausen's identity given as (e.g., Ref. [12])

$$F_\sigma(z)^2 = {}_3F_2\left[\sigma, 1-\sigma, \frac{1}{2}; 1, 1; 4z(1-z)\right], \quad (11)$$

for $z < 1/2$. Plugging this into Eq. (10) leads to Ramanujan-like formulas for all n . For example, for $n = 3$, we get

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} (6k+1) \left(\frac{1}{4}\right)^k \frac{\left(\frac{1}{2}\right)_k^3}{k!^3}, \quad (12)$$

which is among the original 17 formulas Ramanujan presented. The fastest of his formulas arises from $n = 58$. In general, convergence improves as n increases because the singular values z_0 approach 0 along the line $\bar{z} = 1 - z$. We show this in Fig. 1 using the results in [26].

The derivation of Ramanujan's formulas offers valuable insights, with two key lessons. First, it demonstrates that the Legendre relation can be transformed—through a sequence of highly nontrivial manipulations—into a form involving only a power series expansion around $z = 0$, resulting in remarkably fast convergence. Second, the appearance of a parameter n , corresponding to the order of the modular equations, introduces a family of such formulas. Each value of n yields a distinct series representation for $1/\pi$, with the rate of convergence improving significantly as n increases. In light of the LCFT connection, how do we leverage these lessons to learn more about the LCFT correlators? Or from a different perspective, can we give a physics explanation for the existence of a one-parameter family of fast-converging Ramanujan-like formulas? In the following sections, we address these questions.

The CFT Legendre relation—In (5), we recast the Legendre relation in terms of the four-point correlator of twist operators in the well-studied $c = -2$ LCFT. From the

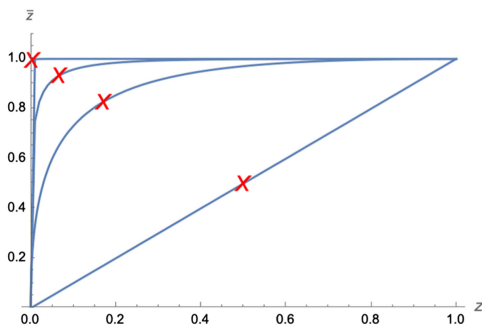


FIG. 1. Slices in the $z, \bar{z}$ plane corresponding to solutions of the modular equations for $n = 1, 2, 3, 7$ from right to left. Red crosses indicate the singular values z_0 .

CFT perspective, the natural basis for expanding correlators is in terms of the (global) conformal blocks, labeled by the scaling dimensions and spins of the primary operators in the theory. As explained in [27], in the case of LCFTs, the correct version of the conformal blocks also involves derivatives of the usual blocks, leading to the following “logarithmic” conformal block decomposition:

$$G(z, \bar{z}) = (z\bar{z})^{\sigma(1-\sigma)} \sum_{\Delta, \ell} \left(c_{\Delta, \ell}^{(1)} + c_{\Delta, \ell}^{(2)} \partial_{\Delta} \right) g_{\Delta, \ell}(z, \bar{z}), \quad (13)$$

where $g_{\Delta, \ell}(z, \bar{z})$ are the 2D conformal blocks given as

$$g_{\Delta, \ell}(z, \bar{z}) = k_{\Delta+\ell}(z) k_{\Delta-\ell}(\bar{z}) + k_{\Delta+\ell}(\bar{z}) k_{\Delta-\ell}(z),$$

$$k_{\beta}(x) = x^{\frac{\beta}{2}} {}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta, x\right). \quad (14)$$

Our goal is to write down the conformal block expansion of the CFT Legendre relation. Note that the Legendre relation only involves $G_{L/R}(z, \bar{z})$, not the full correlator $G(z, \bar{z})$. However, we can express $G_{L/R}(z, \bar{z})$ in terms of the logarithmic part of the full correlator $G_{\log}(z, \bar{z})$ or, equivalently, in terms of the discontinuity of the correlator across the logarithmic branch cut, since $G_{\text{disc}}(z, \bar{z}) = \pi G_{\log}(z, \bar{z})$. We have

$$G_{\log}(z, \bar{z}) = -\frac{\sin(\pi\sigma)}{\pi} \kappa(z, \bar{z}) F_{\sigma}(z) F_{\sigma}(\bar{z})$$

$$= -\frac{\sin(\pi\sigma)}{\pi} G_L(z, 1-\bar{z}). \quad (15)$$

Now, it is easy to see that the block expansion of the discontinuity only involves the $c_{\Delta, \ell}^{(2)}$ data.

$$G_{\log}(z, \bar{z}) = (z\bar{z})^{\sigma(1-\sigma)} \sum_{\Delta, \ell} \frac{c_{\Delta, \ell}^{(2)}}{2} g_{\Delta, \ell}(z, \bar{z}). \quad (16)$$

We will refer to the operators linked to the $c_{\Delta, \ell}^{(2)}$ data as log operators. Using $g_{\Delta, \ell}(z, \bar{z}) \sim_{z, \bar{z} \rightarrow 0} z^{(\Delta-\ell/2)} \bar{z}^{(\Delta+\ell/2)} + z^{(\Delta+\ell/2)} \bar{z}^{(\Delta-\ell/2)}$, we match the powers on both sides to extract the spectrum and the OPE coefficients. The spectrum is given as $(\Delta, \ell) = (\ell + 2n, \ell)$, $\ell, n \in \mathbb{Z}^{\geq 0}$. All OPE coefficients $c_{\Delta, \ell}^{(2)}$ hide a factor $[\sin(\pi\sigma)/\pi]$. Therefore, for clarity, we define $\tilde{c}_{\Delta, \ell}^{(2)} = -[\sin(\pi\sigma)/\pi] c_{\Delta, \ell}^{(2)}$. In this normalization, the coefficient $\tilde{c}_{0,0}^{(2)} = 1$.

Using (5) and (15), the conformal block decomposition of the CFT Legendre relation becomes

$$\frac{\sin(\pi\sigma)}{\pi} = |1-z|^{-2\sigma(1-\sigma)} \sum_{\Delta, \ell} \frac{\tilde{c}_{\Delta, \ell}^{(2)}}{2} \mathcal{L}[g_{\Delta, \ell}(z, 1-\bar{z})] \Big|_{\bar{z}=z}. \quad (17)$$

The above relation gives novel formulas for $1/\pi$ for any σ and z , among which the fastest, obtained when $\sigma = \frac{1}{2}$ and $z = \bar{z} = 1/2$, takes the following form:

$$\frac{1}{\pi} \approx \frac{a_0 + a_1 \log 2 + a_2 \log^2 2}{\sqrt{2}}, \quad (18)$$

where a_0, a_1, a_2 are rational numbers, and we have retained finitely many operators in the conformal block expansion. For instance, retaining the leading eight operators [28] in (18) leads to a six decimal place accuracy, which is in fact comparable to (12).

While novel and interesting in their own right, formulas resulting from (17) do not converge fast. This happens because (16) requires $z, \bar{z} \ll 1$ in $g_{\Delta, \ell}(z, \bar{z})$ for fast convergence, while the Legendre relation involves $g_{\Delta, \ell}(z, 1-z)$. This is essentially the same problem that Ramanujan faced and resolved by expressing the Legendre relation only in terms of $F_\sigma(z)$ in (10). One could just expand (10) in conformal blocks using (16) and get fast-converging series, especially at large n . For example, retaining only the log-identity operator, $n = 58$ gives new formulas for π with 18 decimal places accuracy [see Eq. (A11)] while $n = 1024$ (which can be worked out using a recursion relation [26]) gives 84 decimal places. We give more details in Supplemental Material [18]. However, our goal is to obtain a fast converging basis for the Legendre relation solely via CFT methods, and therefore, also provide a physics explanation for the existence of Ramanujan-like formulas parametrized by n . This is what we will discuss next.

A new dispersive representation for LCFTs—Unlike the full correlator, the discontinuity of the correlator across the logarithmic cut given in (15) only involves $F_\sigma(z)$ and $F_\sigma(\bar{z})$ and, therefore, has a fast-converging block expansion. This directly points us toward the well-established dispersion relations for CFT correlators, which allow us to reconstruct the full CFT correlator just from its discontinuity. In particular, we consider the so-called stringy/parametric dispersion relation recently proposed and derived in [25, 29, 30], as it naturally leads to a one-parameter family of dispersive representations for CFT correlators.

The stringy dispersion relation—If a function $F(x, y)$ is $x \leftrightarrow y$ symmetric and satisfies $\lim_{x \rightarrow \infty} |F(x, y)| \rightarrow 0$ for $y \in \mathcal{D}$, then for all $-\lambda \in \mathcal{D}$ [29–35],

$$F(x, y) = \frac{1}{\pi} \int_{\mathcal{C}} d\xi H^{(\lambda)}(\xi, x, y) F_{\text{disc}}^{(x)}(\xi, \eta^{(\lambda)}(\xi, x, y)), \quad (19)$$

where the integral is over all the branch cuts $F(x, y)$ may have in x , and $F_{\text{disc}}^{(x)}(x, y)$ is the discontinuity of the function across the branch cuts. When the cuts are along the real line, it is simply $\lim_{\epsilon \rightarrow 0} [F(x + i\epsilon, y) - F(x - i\epsilon, y)]/2i$, which for real analytic functions equals $\text{Im}F(x, y)$. We have also defined

$$H^{(\lambda)}(\xi, x, y) = \left(\frac{1}{\xi - x} + \frac{1}{\xi - y} - \frac{1}{\xi + \lambda} \right),$$

$$\eta^{(\lambda)}(\xi, x, y) = \frac{(x + \lambda)(y + \lambda)}{\xi + \lambda} - \lambda. \quad (20)$$

The representation is independent of the parameter λ —see Ref. [25] for a neat proof and [18] for numerical checks.

Now, let us apply the above dispersion relation to the LCFT correlator. We first define $F_L(z, \bar{z}) = \kappa(z, \bar{z})^{-1} G_L(z, \bar{z}) = F_\sigma(z) F_\sigma(1 - \bar{z})$. Now, $F_L(z, \bar{z})$ is symmetric under $z \leftrightarrow 1 - \bar{z}$ and goes to zero at large z for any σ [36]. So, it can be expressed using the above dispersion relation as

$$F_L(z, \bar{z}) = \int_1^\infty d\xi H^{(\lambda)}(\xi, z, 1 - \bar{z}) \times F_{\log}(1 - \xi, \eta^{(\lambda)}(\xi, z, 1 - \bar{z})). \quad (21)$$

This converges for $\text{Re}(\lambda) > -1$. We have used the fact that the discontinuity of $F_L(z, \bar{z})$ across the branch cut along $z \in (1, \infty)$ is given by $\pi F_{\log}(z, \bar{z}) = \sin(\pi\sigma) F_\sigma(z) F_\sigma(\bar{z})$ using (15).

Now we can express the Legendre relation solely in terms of the discontinuity of the correlator as

$$\frac{\sin(\pi\sigma)}{\pi} = \int_1^\infty d\xi \mathcal{L} \left[H^{(\lambda)}(\xi, z, 1 - \bar{z}) \times F_{\log}(1 - \xi, \eta^{(\lambda)}(\xi, z, 1 - \bar{z})) \right]_{\bar{z}=z}. \quad (22)$$

Performing this integral analytically is hard in general, but a simple analytical verification is possible in the large λ limit, since the integrand can be shown to be a total derivative as follows:

$$\begin{aligned} \frac{\sin \pi \sigma}{\pi} \mathcal{L} \int_1^\infty d\xi H^{(\infty)}(\xi, z, w_{\bar{z}}) F_\sigma(1 - \xi) F_\sigma(z + w_{\bar{z}} - \xi) \Big|_{\bar{z}=z} \\ = -z(1 - z) \frac{\sin \pi \sigma}{\pi} \int_1^\infty d\xi \partial_\xi \left[H^{(\infty)}(\xi, z, w_{\bar{z}}) F_\sigma(1 - \xi)^2 \right], \\ = \frac{\sin \pi \sigma}{\pi}, \end{aligned} \quad (23)$$

where $w_{\bar{z}} = 1 - \bar{z}$ and in the last line follows because the entire contribution comes from the lower limit.

The next step is to plug the following conformal block decomposition of the discontinuity [see Eq. (16)] in (21). This leads to a new family of bases for the expansion of the LCFT correlator, parametrized by λ [37]:

$$F_{\log}(z, \bar{z}) = |1 - z|^{-\sigma(1-\sigma)} \sum_{\Delta, \ell} \frac{c_{\Delta, \ell}^{(2)}}{2} g_{\Delta, \ell}(z, 1 - \bar{z}). \quad (24)$$

Although the expansion is independent of λ , convergence improves significantly when λ is taken to be large. In fact,

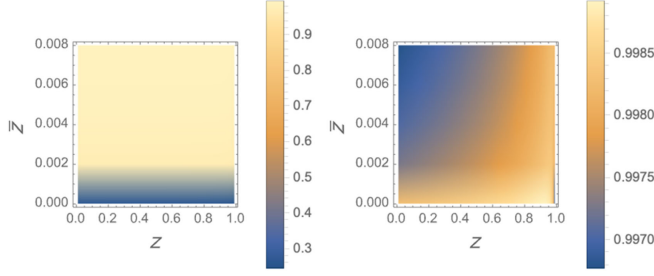


FIG. 2. Heat plot showing ratio of sum of blocks ($\Delta = 2n + \ell$, $n \leq 2, \ell \leq 4$) to exact answer. Left: CBD; right: dispersive representation ($\lambda = 500$).

as we demonstrate in the heat plot in Fig. 2, it is significantly better than the usual conformal block decomposition, especially in the small $\bar{z}$ region.

Finally, plugging (24) in (22) leads us to a family of expansions of the Legendre relation in terms of the conformal data, parametrized by λ . These expansions are fast converging when λ is taken large. To see this, note that they involve conformal blocks in the form $g_{\Delta, \ell}(1 - \xi, \eta^{(\lambda)}(\xi, z, 1 - \bar{z}))$. For fast convergence, we need both the arguments of the blocks to be small. In the large λ limit, $\eta^{(\lambda)}(\xi, z, 1 - \bar{z}) \sim 1 + z - \bar{z} - \xi$, and since the Legendre relation requires $\bar{z} = z$, both the arguments become $1 - \xi$. Now we just need the integral to be dominated by $\xi = 1$, which happens because the kernel $H^{(\lambda)}(\xi, z, 1 - \bar{z})$ peaks near $\xi \sim 1$ and more so when $\bar{z} \sim 0$. Therefore, via purely CFT methods, we have obtained a family of fast-converging expansions of the Legendre relation, just like the derivation of Ramanujan's $1/\pi$ formulas that we reviewed before.

In fact, quite strikingly, the improvement in convergence is so drastic, that in the strict $\lambda \rightarrow \infty$ limit, *the entire contribution to the Legendre relation comes solely from the log-identity operator* [38]. This can be shown analytically. Plugging the block expansion from (16) in the second line of (23), we get

$$z(1-z) \int_1^\infty d\xi \partial_\xi \left[H^{(\infty)}(\xi, z, w_{\bar{z}}) \left((1-\xi)^{-2\sigma(1-\sigma)} \times \sum_{\Delta, \ell} \frac{c_{\Delta, \ell}^{(2)}}{2} g_{\Delta, \ell}(1-\xi, 1-\xi) \right) \right]. \quad (25)$$

This is a total derivative, and only the lower limit $\xi \rightarrow 1$ contributes. Since $g_{\Delta, \ell}(1-\xi, 1-\xi) \sim (1-\xi)^{\Delta+\ell}$, the entire contribution comes from $\Delta = \ell = 0$. We also demonstrate this numerically in Fig. 3 for $\sigma = 1/2$.

This remarkable result also enables us to get an approximate (which becomes increasingly accurate as n increases) LCFT handle on the modular equations as discussed in the Appendix. Furthermore, the feature that allowed us to express the discontinuity of the correlator itself in terms of conformal blocks was just the logarithmic nature of the

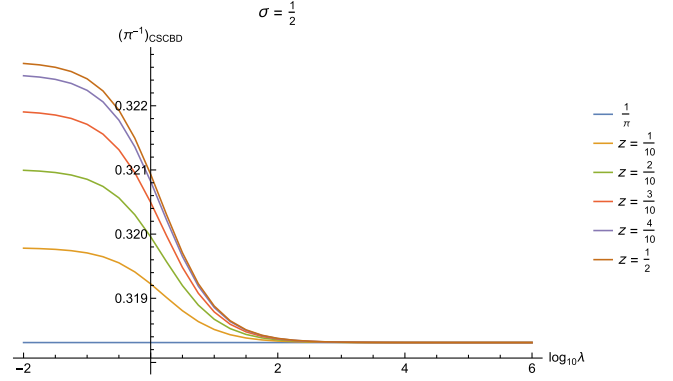


FIG. 3. Plot showing the contribution of the log-identity operator in the dispersive block expansion of $1/\pi$ as λ is increased for $\sigma = 1/2$ and various z . We observe that at large λ , the log-identity operator captures the full Legendre relation, giving $1/\pi$.

branch cut. Therefore, the above result is likely a general feature of LCFT correlators. We emphasize that it only arises when the correlator is expressed via the stringy dispersion relation and then acted upon by the $\mathcal{L}$ operator—operations that were motivated by insights from Ramanujan's formulas.

Discussion—We have seen how CFT considerations shed new light on Ramanujan's formulas and, in turn, how we are led to new, fast converging expansions of LCFT correlators in terms of the CFT data, using lessons learned from Ramanujan's formulas. It will be interesting to exploit this connection to set up the bootstrap program for LCFTs. In fact, given that the log-identity operator reproduces the full CFT Legendre relation in the large λ limit, we expect the LCFT bootstrap to completely fix the full LCFT correlator.

Additionally, in the Appendix, we show that the dispersive representation with the log-identity operator gives a very good approximation to the modular equations Fig. 4. Do the modular equations and singular values reviewed above carry a deeper physical significance beyond their mathematical roles? We would like to point out that in Saleur's work [3], G_L and G_R can be thought of as configurations of polymers stretching horizontally and vertically. The modular equation can be reinterpreted as the probability of the vertical configurations being a factor of n larger than the horizontal ones. Hence, in the $n \gg 1$ limit, we have a chiral polymer configuration. In the Appendix, we also give a holographic interpretation of the $\sigma = 1/2$ Legendre relation in terms of the membrane paradigm. We hope these observations will motivate further research into these remarkable mathematics-physics connections.

Acknowledgments—We thank B. Ananthanarayan, Agnese Bissi, Justin David, Debhashis Ghoshal, Rajesh Gopakumar, Srijan Kumar, Rob Myers, Sridip Pal, Prashanth Raman, Arnab Saha, Chaoming Song, Spenta Wadia, Masaaki Yoshida, and Xinan Zhou for useful discussions. We also thank Gaurav Bhatnagar and

Kaushal Verma for raising related questions which spurred this research. A. S. thanks Perimeter Institute for support and hospitality. A. S. acknowledges support from SERB core grant CRG/2021/000873 and from a Quantum Horizons Alberta chair professorship.

Data availability—The data that support the findings of this article are openly available [13].

- [1] S. Ramanujan, Modular equations and approximations to π , *Q. J. Math.* **45**, 350 (1914).
- [2] J. Guillera, History of the formulas and algorithms for π , *La Gaceta RSME* **10**, 159 (2007).
- [3] H. Saleur, Polymers and percolation in two-dimensions and twisted $N = 2$ supersymmetry, *Nucl. Phys.* **B382**, 486 (1992).
- [4] V. Gurarie, Logarithmic operators in conformal field theory, *Nucl. Phys.* **B410**, 535 (1993).
- [5] M. Flohr, Bits and pieces in logarithmic conformal field theory, *Int. J. Mod. Phys. A* **18**, 4497 (2003).
- [6] J. L. Cardy, Critical percolation in finite geometries, *J. Phys. A* **25**, L201 (1992).
- [7] M. A. I. Flohr and A. Muller-Lohmann, Conformal field theory properties of two-dimensional percolation, *J. Stat. Mech.* (2005) P12004.
- [8] R. Langlands, P. Pouliot, and Y. Saint-Aubin, Conformal invariance in two-dimensional percolation, *Bull. Am. Math. Soc.* **30**, 1 (1994).
- [9] V. Gurarie, M. Flohr, and C. Nayak, The Haldane-Rezayi quantum Hall state and conformal field theory, *Nucl. Phys.* **B498**, 513 (1997).
- [10] A. Bissi, L. Donnay, and B. Valsesia, Logarithmic doublets in CCFT, *J. High Energy Phys.* **12** (2024) 031.
- [11] M. Grinberg and J. Maldacena, Proper time to the black hole singularity from thermal one-point functions, *J. High Energy Phys.* **03** (2021) 131.
- [12] J. Guillera, A method for proving Ramanujan series for $1/\pi$, [arXiv:1807.07394](https://arxiv.org/abs/1807.07394).
- [13] F. Bhat and A. Sinha, *Mathematica* notebook for this paper, Wolfram Community, STAFF PICKS, 2025, <https://community.wolfram.com/groups/-/m/t/3438221>.
- [14] M. R. Gaberdiel and H. G. Kausch, A rational logarithmic conformal field theory, *Phys. Lett. B* **386**, 131 (1996).
- [15] M. R. Gaberdiel and H. G. Kausch, A local logarithmic conformal field theory, *Nucl. Phys.* **B538**, 631 (1999).
- [16] M. R. Gaberdiel, An Algebraic approach to logarithmic conformal field theory, *Int. J. Mod. Phys. A* **18**, 4593 (2003).
- [17] H. G. Kausch, Curiosities at $c = -2$, [arXiv:hep-th/9510149](https://arxiv.org/abs/hep-th/9510149).
- [18] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/c38g-fd2v> for more details, which includes Refs. [19–24].
- [19] M. S. Mahadeva Naika, Modular equations in the spirit of Ramanujan, in *Talk in IIT, Bangalore* (2012), https://www.iitb.ac.in/csl/events/conference/ramanujan/rmit2012talks/IIITB_MSM.pdf.
- [20] H. Rosengren, String theory amplitudes and partial fractions, *Ramanujan J.* **67**, 26 (2025).
- [21] J. G. Wan, Series for $1/\pi$ using Legendre's relation, [arXiv:1302.5984](https://arxiv.org/abs/1302.5984).
- [22] Y. He, Logarithmic operators in $c = 0$ bulk CFTs, *SciPost Phys.* **19**, 008 (2025).
- [23] P. A. Pearce, J. Rasmussen, and J. B. Zuber, Logarithmic minimal models, *J. Stat. Mech.* (2006) P11017.
- [24] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nucl. Phys.* **B241**, 333 (1984).
- [25] F. Bhat, A. P. Saha, and A. Sinha, A stringy dispersion relation for field theory, [arXiv:2506.03862](https://arxiv.org/abs/2506.03862).
- [26] J. Borwein and P. J. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, Canadian Mathematical Society Series of Monographs and Advanced Texts (John Wiley & Son, New York, 1987).
- [27] M. Hogervorst, M. Paulos, and A. Vichi, The ABC (in any D) of logarithmic CFT, *J. High Energy Phys.* **10** (2017) 201.
- [28] Explicitly $a_0 = (21825486901/2147483648)$, $a_1 = -(66594751779/2147483648)$, $a_2 = (52661096730/2147483648)$.
- [29] A. P. Saha and A. Sinha, Field theory expansions of string theory amplitudes, *Phys. Rev. Lett.* **132**, 221601 (2024).
- [30] F. Bhat, D. Chowdhury, A. P. Saha, and A. Sinha, Bootstrapping string models with entanglement minimization and machine learning, *Phys. Rev. D* **111**, 066013 (2025).
- [31] A. Sinha and A. Zahed, Crossing symmetric dispersion relations in quantum field theories, *Phys. Rev. Lett.* **126**, 181601 (2021).
- [32] R. Gopakumar, A. Sinha, and A. Zahed, Crossing symmetric dispersion relations for Mellin amplitudes, *Phys. Rev. Lett.* **126**, 211602 (2021).
- [33] P. Raman and A. Sinha, QFT, EFT and GFT, *J. High Energy Phys.* **12** (2021) 203.
- [34] A. Bissi and A. Sinha, Positivity, low twist dominance and CSDR for CFTs, *SciPost Phys.* **14**, 083 (2023).
- [35] C. Song, Crossing-symmetric dispersion relations without spurious singularities, *Phys. Rev. Lett.* **131**, 161602 (2023).
- [36] More precisely, it behaves as $z^{-\sigma}$ for $0 \leq \sigma \leq 1$ and $z^{-1/2} \log(z)$ when $\sigma = 1/2$.
- [37] The validity of this can be checked numerically following, e.g., the analysis in [34].
- [38] The dispersive representation does not appear useful for getting series representations for $1/\pi$ as there is a $1/\pi$ explicitly appearing outside the integral. However, we can reuse the dispersion relation to replace the $F_\sigma F_\sigma$ factors in the integral, which could be used for such a purpose. This idea is briefly discussed in Supplemental Material [18].
- [39] H. G. Kausch, Symplectic fermions, *Nucl. Phys.* **B583**, 513 (2000).
- [40] This theory also plays a role in 2d topological gravity [41].
- [41] D. Gaiotto and L. Rastelli, A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model, *J. High Energy Phys.* **07** (2005) 053.
- [42] H. G. Kausch, Extended conformal algebras generated by a multiplet of primary fields, *Phys. Lett. B* **259**, 448 (1991).
- [43] S. A. Hartnoll, Lectures on holographic methods for condensed matter physics, *Classical Quantum Gravity* **26**, 224002 (2009).
- [44] A. Lewkowycz and J. Maldacena, Generalized gravitational entropy, *J. High Energy Phys.* **08** (2013) 090.

End Matter

Appendix—

More about the $c = -2$ LCFT: The LCFT we focus on in this Letter is the well-studied $c = -2$ symplectic fermion model [3,39,40]. The model consists of a two-component fermionic field χ^α whose modes $\chi_n^\alpha, n \in \mathbb{Z}$ generate a chiral algebra. The Virasoro algebra with $c = -2$ is contained within this algebra. The algebra has a unique irreducible highest-weight representation, with the highest-weight state being the vacuum state Ω . This representation can be extended to obtain reducible but indecomposable representations, which contain, in addition to the vacuum state Ω , another state ω . The states ω and Ω lead to a two-dimensional Jordan block structure for L_0 ,

$$L_0\Omega = 0, \quad L_0\omega = \Omega. \quad (\text{A1})$$

It is this nondiagonalizability of the L_0 operator that leads to logarithmic correlation functions in LCFTs.

The symplectic fermion model further has a global $SL(2, \mathbb{C})$ symmetry which allows us to introduce twisted states by quotienting with respect to the cyclic subgroups $\mathcal{C}_N$. The twist fields μ_σ are defined by

$$\chi^\alpha(e^{2\pi i}x)\mu_\sigma(0) \sim e^{2\pi i\sigma}\chi^\alpha(x)\mu_\sigma(0), \quad (\text{A2})$$

where σ is the twist. For the $\mathcal{C}_N$ -twisted models, the highest-weight twist states have $\sigma = (k/N), k = 1, \dots, N-1$ and conformal weights $h = -[\sigma(1-\sigma)/2]$. Their OPE features the logarithmic doublet (Ω, ω)

$$x^{\sigma(1-\sigma)}\mu(x)\mu(0) \sim \omega + \Omega \log x + \dots \quad (\text{A3})$$

The $\mathcal{C}_2$ -twisted model has been shown to be equivalent to the well-known triplet model [39], which is an extension of the Virasoro (1,2) algebra by a triplet of fields $W^a(z)$, $a = 1, 2, 3$. In general, all Virasoro (1, q) models have been shown to be extended to triplet algebras and lead to rational LCFTs [14,42]. Although we consider the $q = 2, c = -2$ case in this Letter, the connection to Ramanujan formulas also exists for the other Virasoro (1, q) models. We explain the connection to LCFTs of other central charges in [18].

As is evident from (A3), the correlation functions of twist fields have logarithmic singularities. In particular, the correlator considered in (4) is the four-point correlator of two twist and two antitwist fields (with twists $\sigma^* = 1 - \sigma$),

$$G(z, \bar{z}) = \langle \mu_\sigma(0)\mu_{\sigma^*}(z, \bar{z})\mu_\sigma(1)\mu_{\sigma^*}(\infty) \rangle. \quad (\text{A4})$$

Note that while the twist and antitwist fields are distinct, they have the same conformal weights. $\sigma = \frac{1}{2}$ is the exception and leads to the correlator of identical scalar primaries with $h = -\frac{1}{8}$.

A physical interpretation of the Legendre relation via holography: Consider a scalar φ in an asymptotically AdS_{d+1} Schwarzschild black brane background,

$$ds^2 = \frac{R^2}{z^2} \left[-(1-z^d)dt^2 + \frac{dz^2}{1-z^d} + d\vec{x}^2 \right], \quad (\text{A5})$$

where R is the AdS radius and z is the radial bulk coordinate [11]. We fix the mass of the scalar so that it satisfies the Breitenlohner-Freedman bound, i.e., $m^2 = -d^2/(4R^2)$. It turns out that, under these conditions, the bulk-bulk Green's function of the scalar has exactly the structure of an LCFT correlator. More precisely, consider the Fourier modes of the bulk-bulk Green's function in the $\vec{x}, t$ directions, denoted as $\hat{G}(z, z', \omega, \vec{k})$. The zero-mode $\hat{G}(z, z') = \hat{G}(z, z', \omega, \vec{k})|_{\omega=\vec{k}=0}$ is especially physically significant and satisfies

$$\partial_z \left(\frac{R^{d-1}}{z^{d-1}} (1-z^d) \partial_z G(z, z') \right) - \frac{R^{d+1}}{z^{d+1}} m^2 G(z, z') = \delta(z - z'). \quad (\text{A6})$$

The two linearly independent solutions of the homogeneous part of this differential equation are given by

$$g_{\text{inf}}(z) = z^{1/2} F_{\frac{1}{2}}(z), \quad g_{\text{hor}}(z) = z^{1/2} F_{\frac{1}{2}}(1-z) \quad (\text{A7})$$

and are called so because $g_{\text{inf}}(z)$ is regular at the AdS boundary at $z = 0$ (with a logarithmic singularity at $z = 1$), while $g_{\text{hor}}(z)$ is regular at the black brane horizon at $z = 1$ (with a logarithmic singularity at $z = 0$). From here, $G(z, z')$ can be computed and is given as

$$G(z, z') = \begin{cases} -\frac{\pi}{dR^{d-1}} g_{\text{inf}}(z) g_{\text{hor}}(z'), & z' > z, \\ -\frac{\pi}{dR^{d-1}} g_{\text{hor}}(z) g_{\text{inf}}(z') & z < z'. \end{cases} \quad (\text{A8})$$

Thus, we see that the bulk-bulk Green's function has precisely the structure of an LCFT correlator.

The Legendre relation in this context is simply the Wronskian of the homogeneous part of the differential equation:

$$W(z) = g_{\text{inf}}(z) \partial_z g_{\text{hor}}(z) - g_{\text{hor}}(z) \partial_z g_{\text{inf}}(z) = -\frac{1}{\pi} \frac{1}{1-z}. \quad (\text{A9})$$

One can check that the above equation is exactly the same as the Legendre relation in (2) or (5) with $\sigma = 1/2$.

Physical role of the Wronskian and the static susceptibility: Together with the Green's function [Eq. (A6)],

Eq. (A9) implies a radially conserved symplectic “flux” for the pair $\{g_{\text{inf}}, g_{\text{hor}}\}$. Integrating Eq. (A6) across $z = z'$ gives the usual jump condition on $\partial_z G$, and substituting the piecewise form (A8) with the Wronskian identity (A9) fixes the overall normalization of G independent of where it is evaluated. In boundary terms, this same conservation makes the *static susceptibility* of the dual operator at $\omega = k = 0$ purely IR-determined: expanding the IR-regular bulk solution near the AdS boundary as $\phi(z) = z^{d/2}(A \log(\mu z) + B + \dots)$ (where A is the source coefficient and B the response), horizon regularity plus the constant Wronskian fix the ratio B/A , so $\chi(0) = B/A$ is real and scheme-dependently constant along the radial direction. This is the same “radial Gauss law” logic that underlies horizon formulas for transport (membrane paradigm), specialized here to the scalar static limit: the conserved Wronskian turns horizon regularity directly into the boundary zero-frequency response (see, e.g., Ref. [43]).

Order of the modular equation: Bulk interpretation: In AdS language, an order- n modular equation is most naturally read as passing to the n -fold Euclidean cover [44] of the black-brane thermal circle (the replica “cigar”). Along the modular slice used in (6), this cover renders the two radial bases g_{inf} and g_{hor} *equivalent up to fixed algebraic factors*, and at the singular value z_0 they *align*, so the mixed-channel expression collapses to a single channel. Operationally, this is precisely why the step from (6) to (10) is possible and why the resulting series converges parametrically faster: on the n -fold cover each term effectively accounts for n windings at once.

Holographic meaning of the dispersive formula: Eq. (21) is a spectral reconstruction of the static bulk Green’s function from the *horizon cut*. The integration variable $\xi \in [1, \infty)$ runs along the branch cut through $z = 1$ of the radial hypergeometric solutions. The $F(1 - \xi)$ factor is the *analytic continuation* g_{inf} across the horizon onto the second Riemann sheet; this is a reflection of the fact that the cut of the UV block is proportional to the IR block. The kernel factor propagates this cut (absorptive) data to the bulk points $(z, \bar{z})$. In this language the horizon cut supplies the spectral density (ingoing condition), while the conserved flux fixes the static normalization; together they reproduce the physical Green’s function.

The parameter λ labels a one-parameter family of dispersion representations of the same Green’s function: changing λ reshuffles the weight between the two radial channels inside the integrand but leaves G invariant. Differentiating Eq. (21) with respect to λ shows that the variation of the kernel and of its argument combine into a total ξ derivative plus the radial equation of motion for the hypergeometric block. The bulk term vanishes by the equation of motion, and the total ξ derivative contributes only boundary terms at the endpoints of the cut, which

vanish. Equivalently, in holographic language, changing λ amounts to adding a homogeneous solution whose coefficient is fixed to zero by the ingoing (horizon) condition and the Wronskian normalisation. Thus G is λ independent: λ is a dispersive “gauge” choice rather than new physics.

$1/\pi$ formulas from CFT: The explicit formula from the log-identity operator contributing to (10) in the main text for $n = 7$ is quite simple:

$$\frac{1}{\pi} \approx \frac{8\sqrt{7} - 11}{8\sqrt{3\sqrt{7} + 8}} \quad (\text{A10})$$

giving 5 decimal places. The $n = 58$ case gives

$$\frac{1}{\pi} \approx \frac{1}{2} \left(13233864246\sqrt{2} - 18715707421 + 87\sqrt{29}(28247341\sqrt{2} - 39947352) \right)^{\frac{1}{2}}. \quad (\text{A11})$$

This leads to 18 decimal places agreement. While easy to explain, the formula for $n = 1024$, which leads to 84 decimal places for the log-identity operator, is too cumbersome to display.

Modular equations from CFT: The dispersive representation gives an approximate handle on the modular equations. Using the dispersive representations of $F_{\frac{1}{2}}(z)F_{\frac{1}{2}}(1 - \bar{z})$ and $F_{\frac{1}{2}}(\bar{z})F_{\frac{1}{2}}(1 - z)$, retaining only the log-identity operator in the $\lambda \gg 1$ limit, we can make contour plots of the ratio.

As is evident, the log-identity operator on its own provides a very good approximation to the $z = f_n(\bar{z})$ slices obtained from the modular equations. As n increases, the CFT approximation provides a more accurate solution. For example, the percentage deviations at $z = 1/100$ from the known solutions at $n = 3, 7, 23$ are 10%, 0.02%, $10^{-7}\%$, respectively.

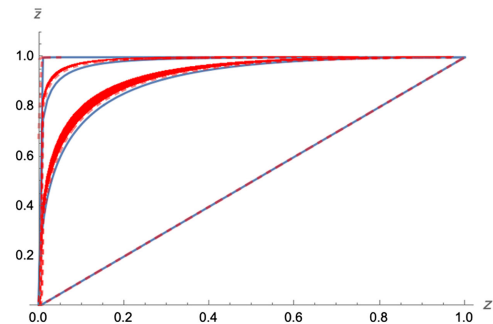


FIG. 4. Contour plot of the ratio of $G_L(z, \bar{z})/G_R(z, \bar{z})$. The blue solid lines and red dashed lines show the $n = 1, 2, 3, 7$ as in Fig. 1. The thickened red lines shade the regions between 1.9 (lower), 2 (upper) and 2.9 (lower), 3 (upper).