

Six-Loop Gravitational Interactions at the Sixth Post-Newtonian Order

Giacomo Brunello¹, Manoj K. Mandal^{2,3}, Pierpaolo Mastrolia^{2,3}, Raj Patil^{4,5}, Matteo Pegorin^{2,3,4}, Jonathan Ronca^{2,3}, Sid Smith^{2,3,6}, Jan Steinhoff⁴, and William J. Torres Bobadilla⁷

¹*Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126, Pisa, Italy and INFN Sezione di Pisa, Largo Pontecorvo 3, 56127 Pisa, Italy*

²*Dipartimento di Fisica e Astronomia, Università di Padova, Via Marzolo 8, 35131 Padova, Italy*

³*INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padova, Italy*

⁴*Max Planck Institute for Gravitational Physics (Albert Einstein Institute), D-14476 Potsdam, Germany*

⁵*Institut für Physik und IRIS Adlershof, Humboldt-Universität zu Berlin, Zum Großen Windkanal 2, D-12489 Berlin, Germany*

⁶*Higgs Centre for Theoretical Physics, University of Edinburgh, James Clerk Maxwell Building,*

Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom

⁷*Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, United Kingdom*



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We compute the gravitational interaction of two coalescing compact objects at sixth post-Newtonian order in the static limit, employing the diagrammatic approach within the effective field theory framework of general relativity. The calculation requires the evaluation of six-loop Feynman diagrams that are mapped onto two-point integrals with a gauge-theory-like structure, which are computed here for the first time. The resulting seventh-order contribution in Newton's constant is finite in three space dimensions. This result provides the most technically demanding missing ingredient for the determination of the conservative dynamics of the gravitational two-body system at sixth post-Newtonian order.

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Introduction—Progress in gravitational-wave (GW) astronomy is driven by observations of gravitational radiation emitted from coalescing compact-object binaries. To date, the LIGO-Virgo-KAGRA Collaboration has reported over 200 GW events from such mergers [1,2]. Future observatories—including next-generation ground-based detectors such as the Einstein Telescope [3] and Cosmic Explorer [4], and space missions like LISA [5]—will enhance our detection reach and sensitivity. Thus, they will offer unprecedented probes of gravity and the astrophysics of compact objects. Realizing the full scientific potential of these facilities, like the distributions of black hole masses and spins [6], constraints on the neutron-star equation of state [7], measurements of the Hubble-Lemaître parameter [8–10], and constraints on the theory of general relativity (GR) [11–14], requires highly accurate theoretical models for the GW signals. Limited-accuracy models may lead to systematic biases [15,16] that are already evident in the current detectors [17], a problem that will only escalate as the signal-to-noise ratio increases for future detectors.

Currently the modeling of the two-body system combines multiple theoretical and numerical frameworks. Numerical relativity [18–20] is used in the highly nonlinear

merger regime, where Einstein's equations are solved numerically on supercomputers. This method is highly accurate but computationally very intensive. On the other hand perturbative methods, particularly the post-Newtonian (PN) [21–71] methods are used to approximate the solution to Einstein's equations analytically in the inspiral regime, where the compact objects move with slow velocities and are widely separated. Other methods, such as the post-Minkowskian approximation [72–105] and the gravitational self-force (SF) [106–119] are also used for systems in the weak-field regime and for small mass ratios, respectively. The most commonly used approaches to build complete inspiral-merger-ringdown waveform models of compact binaries are the phenomenological methods [120–122] and effective-one-body formalism [123–128].

All binary systems observed thus far consist of two compact objects on bound trajectories that spiral toward each other until undergoing a nonlinear merger process. In the inspiral regime, the binary's components move at nonrelativistic velocities, and their orbital separation slowly decays. The nonrelativistic regime of motion at this stage allows for a perturbative treatment of the problem, which can be studied in powers of $v/c < 1$, where v is the orbital velocity of the compact binary and c is the speed of light. Consequently, physical quantities admit a series expansion, and the n th-order coefficient receives contributions from terms proportional to $G_N^k (v^2)^{n+1-k}$, with $0 \leq k \leq n+1$. Here, G_N is the Newton's constant and v^2 is the squared relative velocity, a parameter that is virial-related to G_N in

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the case of bound orbits ($v^2 \simeq G_N m/r$, where m and r are the typical mass and size of the system, respectively).

The systematic PN expansion of the two-body problem began with the pioneering 1PN analysis in Refs. [21,22]. Since that foundational result, computing successive higher-order corrections has required sustained efforts and increasingly sophisticated analytical and computational techniques. The current state of the art is the 4PN conservative effective interaction between the two objects first derived in Refs. [29–33] and the 4.5PN radiation emitted by the binary [34,35]. However, to avoid systematic bias (like those seen in current observations [17]), it is urgent to complete the 5PN order or higher while current facilities reach their design sensitivity [15]. Next-generation detectors require further improvement in waveform accuracy of more than 2 orders of magnitude [129,130], hence the PN program should aim to reach 7PN accuracy within a decade.

Over the past decade, substantial progress has been made in pushing the conservative two-body dynamics [37,39–44,47–64] and emitted radiation [65–71] to increasingly high PN orders within the effective field theory (EFT) approach. In particular, this approach has enabled a systematic reformulation of the PN expansion in terms of classical Feynman diagrams [37,65]. When combined with modern multiloop techniques [38,41], this approach has led to the complete determination of the conservative dynamics at 4PN order [40–42], and partial determination at 5PN order [47–52]. The current state of the art is represented by the $\mathcal{O}(G_N^4)$ 6PN corrections [53], a subset of the complete 6PN corrections that requires the evaluation of the $\mathcal{O}(G_N^k)$ terms up to $k = 7$. On the other hand, the emitted radiation from a binary has been computed up to 3PN order [70].

As observed in Refs. [38,41], PN calculations are technically equivalent to evaluating multiloop massless two-point Feynman integrals. This brings classical gravity into direct contact with modern multiloop methods developed in quantum field theory (QFT). In the diagrammatic approach, the n th-order coefficient correction at order $\mathcal{O}[G_N^k(v^2)^{n+1-k}]$ receives contributions from multiloop diagrams, with the number of loops decreasing as the power of v^2 increases. Particularly, the static sector (independent of v^2 and corresponding to $k = n + 1$) requires evaluating diagrams with the maximal number of loops, namely n . In contrast, the $G_N(v^2)^n$ term (corresponding to $k = 1$) receives contributions from tree-level diagrams.

It is the main goal of this Letter to present the first evaluation of the static two-body effective interaction potential at the $\mathcal{O}(G_N^7)$ 6PN order.

Diagrammatic structure of gravitational two-body dynamics—The $\mathcal{O}(G_N^7)$ static term [131] is the most challenging component of 6PN conservative dynamics and includes, for the first time, a third-order SF term that is absent from lower PN orders. The corresponding

amplitude gets contributions from 1117 six-loop Feynman diagrams. Their integrands stem from a combinatorial construction of classical interaction graphs dressed by PN-expanded vertices, and involve thousands of distinct dimensionally regulated multiloop scalar integrals [132]. Their evaluation is made possible by the optimized use of state-of-the-art algorithms and software, partly developed to achieve this goal. The phase of the diagram generation is followed by the application of integration-by-parts (IBP) identities [133–135], implemented in a novel system solving strategy that combines the standard decomposition algorithm with modern concepts, including spanning cuts [136], syzygy-based methods [137–140], and improved seeding strategies [86,98,100,101,141–144], to reduce the several thousands of integrals contributing to the amplitude onto a set of 39 master integrals (MIs), out of which, after taking the Fourier transform and the $d = 3$ limit, only 21 turn out to give a nonvanishing contribution to the potential. While the evaluation of the static contribution at lower PN orders involved MIs associated to planar diagrams, at 6PN we observe the first contributions from two nonplanar graphs. The MIs are evaluated using both analytic methods and numerical techniques. For those cases where a direct analytic result cannot be obtained, we reconstruct the analytic expression from their numerical evaluation. In these cases, we opt for a combination of the auxiliary mass flow method [145,146] with the integer-relation finding algorithm [147] for the analytic reconstruction of high-precision numerical results.

Post-Newtonian action: We consider the Einstein-Hilbert action for the dynamics of the gravitational degrees of freedom given by

$$S_{\text{EH}} = -\frac{c^4}{16\pi G_N} \int d^4x \sqrt{|g|} \left(R - \frac{1}{2} g_{\mu\nu} \Gamma^\mu \Gamma^\nu \right), \quad (1)$$

where G_N is Newton’s gravitational constant [148], $g_{\mu\nu}$ is the spacetime metric and g its determinant, R is the Ricci scalar, and $\Gamma_{\rho\sigma}^\mu$ is the metric compatible Christoffel connection. Here, the second term enforces harmonic gauge, $\Gamma^\mu = \Gamma_{\rho\sigma}^\mu g^{\rho\sigma} = 0$. The dynamics of each compact object in the binary is described by a point-particle worldline action given as

$$S_{\text{pp}} = \sum_{a=1,2} \int d\tau \left(-m_{(a)} c \sqrt{u_{(a)}^2} \right), \quad (2)$$

where $u_{(a)}^\mu = dx_{(a)}^\mu/d\tau$ is the four-velocity, such that $u_{(a)}^2 = u_{(a)}^\mu u_{(a)}^\nu g_{\mu\nu}$, $m_{(a)}$ denotes the mass of each object, the proper time τ is related to the coordinate time t as $d\tau = c dt$, and we work with mostly negative metric signature.

In the inspiral regime, the dynamics are characterized by three well-separated length scales: the Schwarzschild

radius R_s of each compact object, the orbital separation r , and the wavelength of gravitational radiation λ , with $\lambda \gg r \gg R_s$. In this regime, due to small velocities and weak fields, we expand the metric around flat spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the gravitational interaction is mediated by the gravitons $h_{\mu\nu}$. The separation of scales allows us to decompose the graviton field in short-distance instantaneous potential modes $H_{\mu\nu}$ scaling as $(k_0, \mathbf{k}) \sim (v/r, 1/r)$, and long-distance radiation modes $\bar{h}_{\mu\nu}$, scaling as $(k_0, \mathbf{k}) \sim (v/r, v/r)$ [37]. Since here we are only interested in the conservative binding potential, we discard the radiation modes and decompose the potential modes using a Kaluza-Klein (KK) decomposition [149,150] [151]. The metric components $g_{\mu\nu} (= \eta_{\mu\nu} + H_{\mu\nu})$ are encoded in a scalar field ϕ , a three-dimensional vector A_i , and a three-dimensional symmetric tensor σ_{ij} as

$$g_{\mu\nu} = e^{2\phi/c^2} \begin{pmatrix} 1 & -A_j/c^2 \\ -A_i/c^2 & -e^{-c_d\phi/c^2} \gamma_{ij} + A_i A_j / c^4 \end{pmatrix}, \quad (3)$$

with $\gamma_{ij} = \delta_{ij} + \sigma_{ij}/c^2$ and $c_d = 2(d-1)/(d-2)$. Then, integrating out the gravitational fields gives the two-body effective action,

$$\exp \left[\mathbf{i} \int dt \mathcal{L}_{\text{eff}} \right] = \int D\phi DA_i D\sigma_{ij} e^{\mathbf{i}(S_{\text{EH}} + S_{\text{pp}})}, \quad (4)$$

where the effective Lagrangian $\mathcal{L}_{\text{eff}}$ is decomposed as $\mathcal{L}_{\text{eff}} = \mathcal{K}_{\text{eff}} - \mathcal{V}_{\text{eff}}$ with $\mathcal{K}_{\text{eff}}$ the kinetic term and $\mathcal{V}_{\text{eff}}$ the effective interaction potential. The kinetic term is independent of graviton exchange and, at any PN order, follows from expanding $\sqrt{1 - v^2/c^2}$. By contrast, the potential encodes the gravitational interaction between the two bodies and is the primary quantity computed in this Letter.

Integrand generation: The effective two-body potential $\mathcal{V}_{\text{eff}}$ can be expressed in terms of the connected, classical, one-particle-irreducible Feynman diagram as

$$\mathcal{V}_{\text{eff}} = \mathbf{i} \lim_{d \rightarrow 3} \int \frac{d^d \mathbf{p}}{(2\pi)^d} e^{\mathbf{i} \mathbf{p} \cdot (\mathbf{x}_{(1)} - \mathbf{x}_{(2)})} \times \text{[Diagram]} \quad (5)$$

where $\mathbf{p}$ is the momentum transferred between the two massive objects. The four-point amplitude consists of a sum of 1117 six-loop diagrams [152], as depicted in Fig. 1, grouped according to the power of $m_1^{7-n} m_2^{1+n}$, respectively, indicating the n th-order self-force contribution: the 0SF component corresponds to the simplest set of diagrams, encoding the test-body motion; whereas the 1SF, 2SF, and 3SF include contributions with an increasing level of complexity.

The code PNTHR. The Post-Newtonian Toolkit for Hamiltonian and Radiation [132] was extended [153] to generate the required six-loop graphs and the

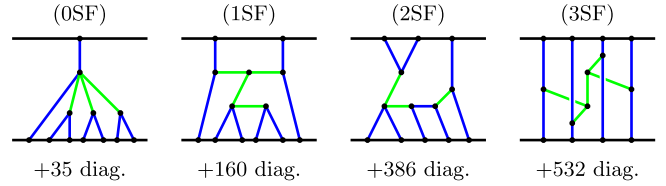


FIG. 1. Representative Feynman diagrams at six loop contributing to the effective potential at 6PN, in the static limit $\mathcal{O}(G_N^7)$: massive compact objects and the ϕ and σ fields are represented by black, blue, and green lines, respectively. The counting does not include diagrams that can be obtained by the exchange of the massive objects.

corresponding integrands. PNTHR uses QGRAF [156] to generate the skeleton of the diagrams, and after dressing them with the KK field and Feynman rules derived from the actions of PN expansion of GR given in Eqs. (1) and (2), prepares the integrands by carrying out the tensor algebra using xTensor [157].

As observed in [38,41], the static integrands of the 6PN four-point diagrams within the EFT approach to GR can be remarkably mapped onto those of six-loop two-point functions with massless internal lines, typical of higher order corrections in QFT, whose external momentum is the momentum $\mathbf{p}$ transferred of the massive bodies as

$$\text{[EFT Diagram]} \longleftrightarrow \text{[QFT Diagram]} \quad (6)$$

We employ an in-house topology-mapping algorithm to group the contributing diagrams into 97 equivalence classes identified by their graphs (Symanzik) polynomials.

Each topology represents a scalar integral, whose generic form reads as

$$I_{n_1, \dots, n_{27}} = \int_k \prod_{i=1}^{27} D_i^{-n_i}, \quad \text{with} \quad \int_k := \int \prod_{i=1}^6 \frac{d^d k_i}{(2\pi)^d} \quad (7)$$

in terms of 27 generalized denominators D_i , of which 13 are genuine propagators' denominators ($n_i \geq 1$, $i = 1, \dots, 13$), while the remaining 14 correspond to irreducible scalar products ($n_i \leq 0$ for $i = 14, \dots, 27$).

TABLE I. Contributions to different SF sectors at 6PN. Multiple MIs are common to different SF orders.

SF order	Diagrams	MIs
0	36	1
1	161	3
2	387	13
3	533	16
Total	1117	21

Integral decomposition: Dimensionally regulated integrals $I_{n_1, \dots, n_{27}}$ are not independent, and IBP identities [133–135] can be leveraged to decompose them, and consequently the amplitude in terms of a finite set of independent scalar integrals, dubbed MIs. They can be considered as generators of a vector space [158,159]. The number of MIs can be determined using methods inspired by differential geometry, for instance by counting the critical points associated with the Symanzik polynomials of the corresponding Feynman diagrams [160,161]. Symanzik polynomials are also known as the $\mathcal{U}$ - and $\mathcal{F}$ -polynomials, and arise from considering the Feynman integrals in the Feynman parameter or Lee-Pomeransky representations.

IBP relations for the integrals of the considered problems stem from a system of symbolic equations reading as

$$\int_k \frac{\partial}{\partial k_\epsilon^\mu} \left(q^\mu \prod_{i=1}^{27} D_i^{-n_i} \right) = 0, \quad (8)$$

where $q \in \{k_1, k_2, k_3, k_4, k_5, k_6, p\}$. By *seeding* the equations, namely by inserting different values of the indices n_i and including symmetry relations (linear relations originating from shifting the loop momenta), we can generate a large system of linear equations for the integrals $I_{n_1, \dots, n_{27}}$, whose solutions gives the decomposition relations, exactly in d , in terms of 39 MIs.

Let us remark that the decomposition of the integrals arising in this problem constitutes a highly demanding computational task. The relevant integrals are scalar six-loop integrals of rank up to six, corresponding to at most six powers of scalar products in the numerator, or equivalently to numerators with mass dimension not exceeding 12. Their reduction to MIs cannot be achieved with commonly used public codes and instead required the development of a novel algorithm implemented within the code PRISM [162]: prime-field reduction of integrals using syzygy methods. This algorithm combines multiple optimization strategies to efficiently generate and solve a minimal IBP system.

Spanning cuts. Starting from the Symanzik polynomials of each topology, graph-theoretic considerations allow one to identify minimal edge cuts whose removal separates the diagram into two connected subgraphs, each containing at least one external leg. Loosely speaking, a cut can be considered by replacing propagators with Dirac delta functions $D_i^{-1} \rightarrow \delta(D_i)$ [136]. For higher denominator powers it is not so simple, however, due to the application of syzygies we do encounter integrals with these higher powers. Applying IBP reduction to integrals associated with spanning cuts decomposes the original IBP system into smaller cut subsystems that are significantly easier to solve. The solution of the full system is then obtained by consistently combining the results from all spanning cuts that collectively cover the original graph [163,164].

Syzygies. Recent progress in syzygy-based approaches [137–140] has significantly enhanced the algorithmic efficiency of IBP reductions. When integrated with spanning cuts, these methods substantially reduce the size and complexity of the IBP systems to be generated and solved. In PRISM, syzygy relations are computed using Singular [165] and are embedded directly into the IBP generation step, thereby constraining the system at the outset and improving the overall performance of the reduction algorithm.

Seeding strategy. PRISM employs an improved seeding strategy, inspired by techniques used in Kira and NeatIBP [138,139,141].

In the top sector, integrals are seeded up to rank six, with the maximal rank reduced according to the degree of the corresponding syzygy solution. As the algorithm descends to lower sectors, the maximum seed rank is decreased by one at each step, while integrals with squared denominators' exponents are excluded. Additional constraints inspired by rectangular seeding are imposed and tailored to the specific integrals appearing for each topology. In several instances, these combined strategies lead to a substantial reduction in both the size of the IBP systems and the overall computational cost.

After generating the IBP systems for each spanning cut and topology, the resulting linear systems are solved using FiniteFlow [166]. The solver first applies a mark-and-sweep procedure by solving the system at a random numerical phase-space point modulo some large prime number (sample point). This allows for the elimination of redundant equations and identification of the relevant MIs from the predefined basis obtained through counting. Furthermore, the row reduction operations are remembered and can be applied much faster for subsequent runs. The reduced systems are then solved over finite fields at many more sample points, and the rational coefficients of the MIs are reconstructed from these evaluations.

The IBP reduction of these integrals constituted the most computationally demanding part of the calculation. In particular, the generation of the IBP system, which required solving syzygy equations, together with the initial mark-and-sweep procedure for each spanning cut, represented the most time-consuming steps.

Master integrals: The IBP reduction yields 39 MIs, out of which only 21 contribute to the conservative potential. They are shown in Fig. 2. Let us observe that the 0SF sector gets contribution from just one MI, namely $\mathcal{I}_3$, as expected [162,163], whereas the 1SF, 2SF, and 3SF sectors get contributions from 3, 13, and 16 MIs, respectively, as given in Table I.

The MIs $\mathcal{I}_1$ through $\mathcal{I}_{10}$ are directly solvable as iterated massless one-loop two-point integrals, and are evaluated analytically in generic dimension d using the standard identity

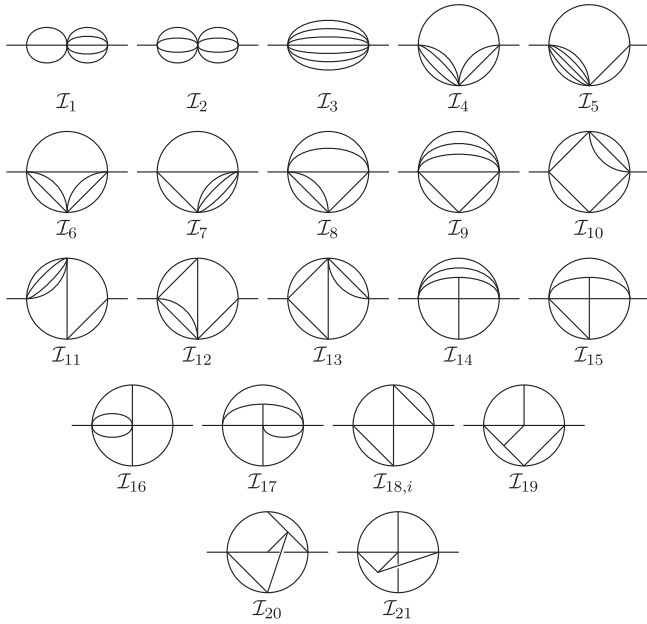


FIG. 2. The 21 MIs contributing to the 6PN static potential. The topology $\mathcal{I}_{18,i}$ has two MIs, $i \in \{1, 2\}$, but only $\mathcal{I}_{18,2}$ appears in the final result.

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^{\nu_1} [(k-p)^2]^{\nu_2}} = \frac{(p^2)^{\frac{d}{2} - (\nu_1 + \nu_2)}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2} - \nu_1) \Gamma(\frac{d}{2} - \nu_2) \Gamma(\nu_1 + \nu_2 - \frac{d}{2})}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(d - \nu_1 - \nu_2)}. \quad (9)$$

Remarkably, this subset is sufficient for the determination of the 0SF and 1SF terms of the complete result. Using the identity in Eq. (9), MIs $\mathcal{I}_{11}$, $\mathcal{I}_{12}$, $\mathcal{I}_{13}$ are solvable as two-loop kite integrals with d -dependent denominator exponents, and are evaluated analytically using the techniques proposed in Refs. [169–173]. The MIs $\mathcal{I}_{14}$ through $\mathcal{I}_{21}$ are evaluated numerically around $d = 3 + \epsilon$ dimensions, using the auxiliary mass flow method implemented in AMFlow [145,146]. Their analytic Laurent coefficients are reconstructed via the PSLQ algorithm [147], employing a tailored ansatz for the expected transcendental constants, built from harmonic polylogarithm constants [174] evaluated at unit argument with transcendental weight up to three [175]. We note that this is the first PN order at which the effective potential receives contributions from nonplanar master integrals, namely, $\mathcal{I}_{20}$ and $\mathcal{I}_{21}$.

The definitions and analytic expressions of the 21 MIs of the problem can be found in Supplemental Material [176].

Checks. The 13 MIs that have been obtained analytically in d dimensions have been checked by verifying that they satisfy dimensional recurrence relations [177–180], and via an independent numerical evaluation using AMFlow. Independent numerical checks of all the six-loop MIs were performed using FEYNTRIP [181], which evaluates the integrals via Monte Carlo integration

within a tropical-geometry framework. Such a comparison requires the determination of a quasifinite basis of MIs in $d = 5$, the construction of dimensional recurrence relations, and additional IBP reductions to relate the two bases. The resulting agreement between these computational strategies can be considered a stringent validation test on the MIs as well as on the IBP decomposition.

Results—Following Eq. (5), the effective two-body potential is obtained by Fourier transforming the amplitude using expression given in Supplemental Material [176], expressed as linear combination of MIs, and performing the Laurent expansion in $\epsilon = d - 3$. Finally, the resulting static two-body effective interaction potential at 6PN order reads

$$\mathcal{V}_{6\text{PN}}^{G_N^7} = -\frac{G_N^7}{r^7} \left(\frac{5}{16} m_1^7 m_2 + \frac{190}{9} m_1^6 m_2^2 + \frac{37651}{144} m_1^5 m_2^3 + \frac{5852}{9} \frac{m_1^4 m_2^4}{2} + (1 \leftrightarrow 2) \right). \quad (10)$$

This remarkably simple expression constitutes the main result of this Letter. The robustness of this result has been assessed in several ways.

Finite and rational potential. The final potential (10) is finite in $d = 3$ spatial dimensions and purely rational due to the exact cancellation of the ϵ poles, appearing at 2SF and 3SF orders, as well as the cancellation of the transcendental constant π^2 [36], which is present in the contributions of the individual 1SF-, 2SF-, and 3SF-order terms. The purely rational expression of the 6PN potential at $\mathcal{O}(G_N^7)$ is in line with the known result of the static contribution at the lower PN orders, and the cancellation of the divergent terms along with the transcendental constant appearing in the intermediate expressions can be considered a nontrivial validation of our result.

The $m_1^7 m_2$ term. We further verified the result against the known test body limit. The static contribution to the interaction potential, linear in m_2 , is given by

$$\mathcal{V}^{m_2 \ll m_1} = m_2 \sqrt{-g_{00}} = m_2 \sqrt{\frac{1 - \frac{G_N m_1}{r}}{1 + \frac{G_N m_1}{r}}}, \quad (11)$$

where the metric component g_{00} is expressed in harmonic coordinates. The Taylor expansion in G_N yields the numerical coefficient $-5/16$ at $\mathcal{O}(G_N^7)$, which exactly agrees with the coefficient of the $m_1^7 m_2$ term in Eq. (10).

The static contribution at 5PN. Let us remark that our complete computational pipeline, from the automatic diagrams and integrands generation to the master integral decomposition, and the Fourier transform of the amplitude from momentum to position space in $d = 3$ dimensions has been validated by computing the 5PN static contribution, reproducing the result of Refs. [47,48].

Conclusion—In this Letter, we computed the static contribution to the conservative two-body potential at

6PN order within the EFT framework. This term constitutes the most technically demanding component of the 6PN conservative dynamics and, for the first time, incorporates a third-order SF contribution. The calculation required the evaluation of six-loop massless two-point Feynman integrals, never addressed before. We achieved this by combining state-of-the-art multiloop techniques, including automatic diagrams generation, IBP identities supplemented by spanning cuts, syzygy-based methods, and optimized seeding strategies. The analytic expressions of the MIs are obtained through a combination of analytic and numerical evaluations, together with reconstruction algorithms. Our result, together with the methods developed to obtain it, has significant implications for several future research directions. On the one hand, it represents a major milestone toward the completion of the full 6PN conservative dynamics, which is essential for achieving accurate theoretical predictions of the inspiral phase of compact binary coalescences for current and future gravitational-wave observatories. In particular, velocity-dependent contributions at 6PN order—requiring the evaluation of lower-loop integrals with higher-rank numerators—as well as radiation-reaction effects at this order, can be addressed using the automated framework developed in this Letter. This approach naturally extends and complements existing partial results obtained up to three-loop order using EFT methods [53], as well as complementary information on local-in-time 6PN contributions derived from alternative approaches [182–184]. Our result thus fills a crucial missing gap and completes the currently known local contributions to the 6PN dynamics, and can be considered the state-of-the-art for post-Newtonian correction in general relativity.

On the other hand, important missing ingredients remain in the form of hereditary, or nonlocal-in-time, effects. These contributions have been partially determined at 5PN and 6PN order in Refs. [52,185–187]. A dedicated and systematic analysis will be required to complete these terms at 6PN. With regard to higher-order PN corrections, the static contribution at 6PN order provides a key input for the application of the factorization theorem [47] to the derivation of the 7PN static term, which follows from purely combinatorial properties together with all lower-order static contributions.

Beyond its implications for general relativity, this Letter illustrates a fruitful cross-fertilization between gravitational and gauge-theory computations. The diagrammatic techniques, IBP strategies, and reduction algorithms developed for high-order post-Newtonian gravity closely mirror—and in several respects extend—those employed in multiloop gauge-theory calculations. As such, they pave the way for previously inaccessible high-precision results in gauge theories, including the six-loop β functions in quantum electrodynamics and quantum chromodynamics and related observables, while at the same time benefiting from

conceptual and technical advances originally developed in the context of perturbative QFT.

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Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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