

Complete Computation of All Three-Loop Five-Point Massless Planar Integrals

Dmitry Chicherin^{1,*}, Yu Wu^{2,†}, Zihao Wu^{3,4,5,‡}, Yongqun Xu^{6,§}, Shun-Qing Zhang^{6,||}, and Yang Zhang^{2,7,8,¶}

¹*LAPTh, Université Savoie Mont Blanc, CNRS, B.P. 110, F-74941 Annecy-le-Vieux, France*

²*Interdisciplinary Center for Theoretical Study, University of Science and Technology of China, Hefei, Anhui 230026, China*

³*State Key Laboratory of Nuclear Physics and Technology, Institute of Quantum Matter, South China Normal University, Guangzhou 510006, China*

⁴*Guangdong Basic Research Center of Excellence for Structure and Fundamental Interactions of Matter, Guangdong Provincial Key Laboratory of Nuclear Science, Guangzhou 510006, China*

⁵*Hangzhou Institute for Advanced Study, University of Chinese Academy of Sciences, Hangzhou 310024, China*

⁶*Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Boltzmannstraße 8, 85748 Garching, Germany*

⁷*Peng Huanwu Center for Fundamental Theory, Hefei, Anhui 230026, China*

⁸*Center for High Energy Physics, Peking University, Beijing 100871, China*



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We calculate all three-loop, five-point, massless planar Feynman integral families in the dimensional regularization scheme. This is a new milestone in Feynman integral computations. The analysis covers four distinct families of Feynman integrals for this configuration, for all of which we derive the canonical differential equations. Our results also confirm a prediction on the three-loop five-point alphabet. The boundary values are analytically determined. These differential equations offer a promising route toward efficient high-precision evaluation of the integrals. Our work establishes the foundation for next-to-next-to-next-to-leading-order calculations of the production of three massless final states, as well as corresponding bootstrap studies in gauge theories.

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Introduction—Precise computations in quantum field theory have become an important task in recent decades. They are essential for the physics project of current and future colliders, such as the high-luminosity large Hadron collider (HL-LHC) [1], future circular collider (FCC) [2], circular electron positron collider (CEPC) [3], and international linear collider [4].

The multiloop scattering amplitudes and corresponding multiloop Feynman integrals are key quantities in making precise predictions for experiments at these colliders. In recent years, with the help of the state-of-the-art computational techniques [5–16], we have witnessed substantial progress in these computations. They are represented by two-loop Feynman integrals and relevant next-to-next-to-leading-order amplitudes for phenomenology, most notably

in the computation of two-loop five-point integrals and amplitudes with up to several external masses [17–44], massless six-point integrals [45–48], and the associated six-point scattering amplitudes [49] and observables [50].

Despite the remarkable progress of the two-loop computations, many processes at colliders still require next-to-next-to-next-to-leading-order ($N^3\text{LO}$) amplitude computations, including a number of $2 \rightarrow 3$ processes, such as three-jet [27,30,51], isolated-photon-plus-dijet [52–54], diphoton-plus-jet [29,55–59], and triphoton production processes [32,60–63]. Taking the three-jet production as an example, observables involving three jets offer detailed information on the structure of hadronic final states, especially jet shape and radiation-pattern measurements, and can be used to extract the strong coupling constant α_s across a wide range of energy scales [30,64,65]. Current jet- and energy-correlator-based extractions of α_s are limited by theoretical uncertainties at the few-percent level, typically in the range of 2%–4% in recent analyses [30,66–69]. It is expected that $N^3\text{LO}$ predictions will further reduce the theoretical uncertainty of α_s . Moreover, $N^3\text{LO}$ calculations of diphoton-plus-jet and triphoton production are desirable for testing the expectation that perturbative corrections become milder and more consistent with convergence starting at this order [57,62,63]. We also expect that the demand for such $N^3\text{LO}$ computations will grow with the next generation of colliders.

*Contact author: chicherin@lapth.cnrs.fr

†Contact author: wy626@mail.ustc.edu.cn

‡Contact author: wuzihao@mail.ustc.edu.cn

§Contact author: yongqunxu@mail.ustc.edu.cn

||Contact author: sqzhang@mpp.mpg.de

¶Contact author: yzhphy@ustc.edu.cn

The above $N^3\text{LO}$ computations require a substantial amount of computational work to obtain their building blocks. With the automation of tree and one-loop amplitudes, as well as the progress on massless six-point integrals, three-loop five-point Feynman integrals currently serve as key ingredients for these $N^3\text{LO}$ computations. Very recently, one of the three-loop five-point integral families was computed in Ref. [70], presenting a first step on the journey. However, the remaining families, which are significantly more complicated, still remain to be computed.

In this Letter, we present for the first time the complete results of all massless planar three-loop five-point Feynman integrals, arranged in the so-called pentagon-box-box (PBB), box-pentagon-box (BPB), box-hexagon-box (BHB), and pentagon-box-pentagon (PBP) families listed in Fig. 1. For each family, we construct a complete pure integral basis which makes the differential equations simple [5,6]. Thanks to the conjectured three-loop planar pentagon alphabet [38] and the development of the state-of-the-art integration-by-parts (IBP) program NEATIBP 1.1 [12], we analytically reconstruct the differential equations in canonical form (CDE). Furthermore, we determine all boundary values analytically up to weight six. Thus all these integrals can be calculated by solving the differential equations, and their symbols are determined as well.

This Letter is organized as follows: In the section on integral families, the massless five-particle kinematics and four planar three-loop five-point families are introduced. In the section on IBP reduction via computational algebraic geometry, the techniques for performing IBP reduction are discussed. The complete pure basis for each family is given in the section on pure integral basis, which leads to the differential equation and alphabet section. The boundary values are extracted from the section on boundary values. The complete symbol space is identified in the symbology section. In the section on numerical evaluation, we validate our computation by a numeric approach. Finally, the last section provides a summary and an outlook for the future.

Integral families—The complete set of planar three-loop five-point integral families is displayed in Fig. 1. In previous work [70], the CDE and functions in the Euclidean region of the PBB family shown in Fig. 1(a) were determined.

We consider five-particle scattering. The five external momenta k_i , $i \in \{1, 2, 3, 4, 5\}$ are on shell and satisfy the momentum conservation, i.e., $k_i^2 = 0$, $\sum_{i=1}^5 k_i = 0$. We adopt the standard parity-even cyclic variables:

$$X := \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}, \quad s_{ij} = 2k_i \cdot k_j, \quad (1)$$

and one parity-odd invariant:

$$\epsilon_5 = 4\sqrt{G(k_1, k_2, k_3, k_4)}. \quad (2)$$

Here $G(p_1, \dots, p_n) \equiv \det(p_i \cdot p_j)$ and $1 \leq i, j \leq n$ is the Gram determinant.

For planar scattering at three-loop order, there are only four integral families, which are listed in Fig. 1. It is useful to introduce dual coordinates x_i with $k_i = x_i - x_{i-1}$, where the indices take cyclic values $x_6 \equiv x_1$, and x_{ij}^2 denotes $(x_i - x_j)^2$. The dual coordinates for three-loop momenta ℓ_1, ℓ_2, ℓ_3 are denoted by y_1, y_2, y_3 with $y_i = x_{i+6} - x_5$. Thereby, the inverse propagators are defined as follows:

$$\{\mathcal{D}_i\} = \{x_{17}^2, x_{27}^2, x_{37}^2, x_{47}^2, x_{57}^2, x_{18}^2, x_{28}^2, x_{38}^2, x_{48}^2, x_{58}^2, x_{19}^2, x_{29}^2, x_{39}^2, x_{49}^2, x_{59}^2, x_{78}^2, x_{79}^2, x_{89}^2\}. \quad (3)$$

In dimensional regularization, the loop momenta ℓ_i 's are $D = 4 - 2\epsilon$ -dimensional. With these propagators, all integral families under consideration can be written as:

$$I_{\nu_1, \dots, \nu_{18}}(\mathcal{N}) = e^{3\epsilon\gamma_E} \int \prod_{j=1}^3 \frac{d^{4-2\epsilon} \ell_j}{i\pi^{2-\epsilon}} \frac{\mathcal{N}(\ell, k)}{\prod_{i=1}^{18} \mathcal{D}_i^{\nu_i}}, \quad (4)$$

where γ_E is the Euler-Mascheroni constant, ν_k are the propagator powers, and $\mathcal{N}$ is a possible numerator. One can specify a particular family by restricting some of the indices ν_i to be nonpositive, e.g., $\nu_i \leq 0$, $i \in \{4, 6, 7, 9, 11, 12, 17\}$ holds for the PBB family in Fig. 1(a).

IBP reduction via computational algebraic geometry—IBP reductions constitute a major bottleneck in most state-of-the-art multiloop Feynman integral computations. Consequently, typical three-loop five-point integral family problems remain challenging using IBP reduction tools based on traditional methods [8,10,13,14]. We found that the public program NEATIBP [11,71], which employs computational algebraic geometry techniques [72–78] to generate IBP equations, together with finite-field techniques [7,9,15,79], helps to address the challenges.

Using this package, we find that the number of master integrals (MIs) for the four families is 316, 367, 431,

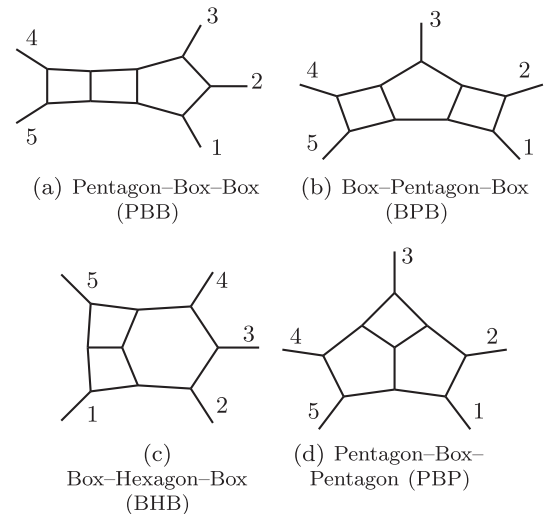


FIG. 1. Genuine three-loop five-point integral families.

TABLE I. Overview of the integral families.

	PBB	BPB	BHB	PBP
No. master integrals	316	367	431	734
No. genuine five-point integrals	120	164	169	342
No. independent five-point topologies	17	19	23	22
No. 10^3 CPU hours for IBP	~ 0.5	~ 1	~ 3	~ 7

and 734. The number of genuine five-point integrals and inequivalent five-point topologies are also listed in Table I. Hence, the tennis-court topologies (BHB and PBP) turn out to be considerably more formidable than the ladder-type topologies (PBB and BPB). Thanks to the recent development in the function SPANNINGCUT [77,78] implemented in NEATIBP 1.1 [12], we are able to make the reduction of the tennis-court families feasible. The central processing unit (CPU) times required for each family are listed in Table I. We remark that this reduction appears to be beyond the practical reach of other currently available public packages.

Pure integral basis—Building and solving differential equations (DE) for the integral basis after IBP reduction is one of the main methods to analytically compute such a basis. With a good choice of basis, called a pure integral basis, the DE takes a canonical form:

$$d\mathbf{I} = \varepsilon d\tilde{\mathbf{A}}\mathbf{I}, \quad d\tilde{\mathbf{A}} = \sum_{W_i \in \mathbb{A}} a_i d \log W_i, \quad (5)$$

and can be solved by iterative integration [5]. The CDE also tells us about the singularities of the integral families, represented by a set of functions W_i which are symbol alphabets [80].

In this Letter, we find the pure integral basis for three-loop five-point integrals, so that they can be calculated by the canonical differential equation method [5].

For the integrals with four-point legs and one off shell external leg, their pure basis has been extensively investigated in [81,82]. For genuine five-point integrals, the number of independent topologies in each family is listed in Table I. There exist partial overlaps between different families—once the three families PBB, BPB, and BHB are completed, only three topologies are new in PBP: the PBP top topology with 19 master integrals shown in Fig. 1(d) and the other two shown in Fig. 2. These topologies contain an exceptionally large number of integrals within a single sector, posing additional challenges for constructing pure bases.

To construct a pure basis for these genuine five-point integrals, we follow the loop-by-loop leading singularity analysis in momentum or Baikov representation [5,83,84], as well as the chiral numerator from local integrand [85] and $d \log$ integrals based on Dlogbasis algorithm [86]. To systematically construct numerators with unit leading

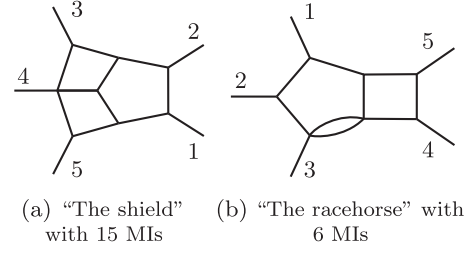


FIG. 2. New subtopologies in the PBP family.

singularities, we employ the algebraic geometric method of module lift in Baikov representation.

Here we list representative pure integrals on top topology of PBP shown in Fig. 1(d):

$$\mathcal{N}_4 = \frac{\varepsilon(s_{12} - s_{45})}{(1 + 2\varepsilon)\varepsilon_5} G(\ell_2, \ell_3, k_1, k_2, k_3, k_4), \quad (6)$$

$$\mathcal{N}_{13} = s_{12}s_{34}s_{45}(\ell_2 - k_1)^2(\ell_3 - k_{123})^2, \quad (7)$$

$$\mathcal{N}_{17} = s_{12}s_{45}(\ell_3 + k_5)^2 \left((\ell_2 - k_{123})^2(\ell_3 - k_{12})^2 - (\ell_3 - k_{123})^2(\ell_2 - k_{12})^2 \right), \quad (8)$$

$$\mathcal{N}_{19} = s_{12}s_{15}s_{45} \left((\ell_2 - k_{123})^2(\ell_3 - k_{12})^2 - (\ell_3 - k_{123})^2(\ell_2 - k_{12})^2 \right). \quad (9)$$

We abbreviate $\sum_{i=1}^j k_i$ as $k_{1..j}$. The complete pure basis for each family can be found in the auxiliary file Canonical_Basis.m (see Appendix).

Differential equation and alphabet—By taking derivatives of our pure integral basis and performing IBP reduction, we can check semianalytically that the resulting differential equation is proportional to the regulator ε . Furthermore, with the conjectured three-loop planar pentagon alphabet in [38], serving primarily as an auxiliary input, we can reconstruct the analytical CDE for each integral family in a particularly efficient manner. As a result, we confirm that the letter alphabet indeed consists of the following 56 letters:

$$\mathbb{A}_p^{3\text{-loop}} = \mathbb{A}_p^{2\text{-loop}} \cup \{\hat{W}_i\}_{i=1}^5 \cup \{\tilde{W}_i\}_{i=16}^{20} \cup \{\tilde{W}_i\}_{i=41}^{55} \cup \{\tilde{W}_i\}_{i=76}^{80}, \quad (10)$$

where $\mathbb{A}_p^{2\text{-loop}}$ denotes the well-known two-loop planar pentagon alphabet [18,87]. Each subset $\{W_i\}_{i=1}^5$ forms a cyclic orbit $W_i = \tau^{(i-1)}(W_1)$. Note that the alphabet is closed under the dihedral group, D_5 .

The novel three-loop letters take the following representative forms:

$$\hat{W}_1 = s_{23}s_{34} - s_{34}s_{45} + s_{45}s_{15}, \quad (11)$$

$$\tilde{W}_{16} = \Delta_4^{(1)}, \quad (12)$$

$$\tilde{W}_i = \frac{q_i - \sqrt{\Delta_4^{(1)}}}{q_i + \sqrt{\Delta_4^{(1)}}}, \quad i \in \{41, 46, 51\}, \quad (13)$$

$$\tilde{W}_{76} = \frac{q_{76} - \epsilon_5 \sqrt{\Delta_4^{(1)}}}{q_{76} + \epsilon_5 \sqrt{\Delta_4^{(1)}}}, \quad (14)$$

where q_i are polynomials in the Mandelstam variables, and $\Delta_4^{(1)}$ is a quartic polynomial given by

$$\begin{aligned} \Delta_4^{(1)} = & s_{15}^2 s_{12}^2 + s_{23}^2 s_{12}^2 - 2s_{15} s_{23} s_{12}^2 - 2s_{23}^2 s_{34} s_{12} \\ & + 2s_{15} s_{23} s_{34} s_{12} + 2s_{15} s_{34} s_{45} s_{12} + 2s_{23} s_{34} s_{45} s_{12} \\ & + s_{23}^2 s_{34}^2 + s_{34}^2 s_{45}^2 - 2s_{23} s_{34} s_{45}, \end{aligned} \quad (15)$$

and $\Delta_4^{(i)} = \tau^{(i-1)}(\Delta_4^{(1)})$ for $i = 1, \dots, 5$. The symbol letters can also be generated using the code [88]. We collect their explicit expressions in the ancillary files described in the Appendix.

Consequently, the CDE for each integral family can be written as

$$d\mathbf{I}_f = \epsilon d\tilde{\mathbf{A}}_f \mathbf{I}_f, \quad d\tilde{\mathbf{A}}_f = \sum_{W_i \in \mathbb{A}_f} a_{f,i} d\log W_i, \quad (16)$$

where $f \in \{\text{PBB}, \text{BPB}, \text{BHB}, \text{PBP}\}$ labels the integral family, $a_{f,i}$ are $\mathbb{Q}$ -valued constant matrices, and $\mathbb{A}_f$ denotes the subset of letters relevant to each family. Explicitly, we find

$$\begin{aligned} \mathbb{A}_{\text{PBB}} &= \mathbb{A}_{\text{P}}^{2\text{-loop}}, & \mathbb{A}_{\text{BPB}} &= \mathbb{A}_{\text{P}}^{2\text{-loop}} \cup \{\hat{W}_i\}_{i=1}^5 \\ \mathbb{A}_{\text{BHB}} &= \mathbb{A}_{\text{PBP}} = \mathbb{A}_{\text{P}}^{3\text{-loop}}, \end{aligned} \quad (17)$$

where $\mathbb{A}_{\text{P}}^{3\text{-loop}}$ is given in Eq. (10). When fixing a specific orientation of the external legs, additional letters may be omitted. Further details are provided in the auxiliary file `Atilde.m`.

Our explicit computation provides a proof that the conjectured three-loop planar pentagon alphabet [38] is sufficient for planar three-loop five-point Feynman integrals, as no additional letters are required in the reconstructed CDE.

Boundary values—The boundary values are necessary for solving these CDEs and themselves have interesting properties. The boundary values can be fixed by imposing the absence of spurious singularities [18,46,89,90]. Following the strategy applied to the PBB family (Sec. VII in [70]), we determine the boundary constants for the remaining families at the symmetric Euclidean point $x_0 = \{-1, -1, -1, -1, -1\}$. Exploiting the physical consistency conditions, the boundary values can be written formally as

$$\mathbf{I}_i^{(n)}(x_0) = c_i \mathbf{I}_1^{(n)}(x_0) + \sum_{j,k,l} d_{ijk} \int_{\gamma_k} d\tilde{\mathbf{A}}_{jl} \times \mathbf{I}_l^{(n-1)}(x), \quad (18)$$

where $\mathbf{I}_1^{(n)}(x)$ denotes a trivial integral computable by Feynman parametrization, and the rational constants c_i and d_{ijk} are fixed by the requirement that all spurious poles cancel.

In practice, fixing these constants requires integrating the CDE from the base point x_0 to the spurious singularities x_k along a path γ_k . To render the integration expressible in terms of Goncharov polylogarithms [91–93], we choose paths along which the alphabet can be rationalized. For the ladder-type families PBB and BPB, which involve a single square root ϵ_5 , we use the path $\gamma: s_{i,i+1} = -x/(1-x)^2$ [18].

For the tennis-court families BHB and PBP, the additional square roots $\{\sqrt{\Delta_4^{(i)}}\}_{i=1}^5$ introduce a $\sqrt{x}$ dependence, which can be rationalized by introducing $x \rightarrow t^2$:

$$\gamma: s_{i,i+1} = \frac{-t^2}{(1-t^2)^2}. \quad (19)$$

Using this parametrization, the three-loop planar pentagon alphabet $\mathbb{A}_{\text{P}}^{3\text{-loop}}$ reduces to 17 effective letters.

Performing the integration explicitly, we obtain all boundary values up to weight six analytically, which involve over $\mathcal{O}(10^6)$ Goncharov polylogarithms. The corresponding numerical values can be obtained to high precision using NUMPOLYLOG [94]. These numerical boundary constants were further validated against independent PYSECDEC calculations; see the section on numerical evaluation.

Symbology—It is important to study the analytic structure of intrinsic multiloop Feynman integrals through symbology. In this Letter, symbols of all three-loop five-point planar massless integrals are derived.

The CDE for each family and each permutation in D_5 is obtained from the transformation rules of the alphabet. Then, with the weight-zero boundary values, the symbols of the pure basis are calculated up to the weight six.

We count independent symbols to study the function space. It is difficult to obtain independent symbols directly from all D_5 permutations and integral families, since the number of terms in these symbols is huge. We use a divide-and-conquer strategy to determine the symbol space from each family under all D_5 permutation and then merge to get independent symbols. The number of letters and independent symbols appearing in each weight order are listed in Table II. The symbol manipulation is powered by finite-field computational tools [9,79].

Up to weight three, the symbols only consist of two-loop pentagon letters, but starting from weight four, new letters appear. For three-loop five-point integrals, the symbols' first entries only allow $W_1, \dots, W_5$, while first two entries allow 35 combinations of letters. Furthermore, 810 letter pairs never appear in these three-loop Feynman integrals.

Numerical evaluation—Using the CDE and boundary values in the sections on the differential equation and alphabet and on boundary values as input, we can evaluate each integral family at a generic Euclidean point by applying

TABLE II. Function space of three-loop five-point massless Feynman integrals. For each weight, the number of letters and the number of independent symbols are given.

Weight	1	2	3	4	5	6
Number of letters	5	10	20	51	56	56
Number of symbols	5	20	76	285	1000	2220

a newly developed, parallelizable Chebyshev pseudospectral transport (CHESS) method to solve the CDE numerically [95–97]. For example, using a 2024 MacBook Pro with a 14-core M4 Pro, we obtain the values with 10 significant digits of the 734 pure integrals in the PBP family, in about 39 seconds at the following generic point:

$$X: \left\{ -\frac{36}{29}, -\frac{32}{23}, -\frac{45}{31}, -\frac{33}{19}, -\frac{25}{18} \right\}. \quad (20)$$

We also evaluate the same system by numerically solving the CDE with DIFFEXP [98]. The corresponding run takes about 10 minutes on a single core.

As a further comparison, we evaluate one of the top-sector integrals—with $\mathcal{N}_{13}$ in Eq. (7), by direct Monte Carlo simulation with the help of the public program PYSECDEC [99–101], which takes more than $\mathcal{O}(10^3)$ CPU hours. The results are listed in Table III, which shows the agreement within the margin of error. For the subfamilies accessible to the current implementation of AMFLOW [102,103], we performed additional cross-checks, finding complete agreement.

Summary and outlook—In this Letter, we completed the analytic computation of massless planar three-loop five-point Feynman integrals in dimensional regularization, which marks a new milestone in multiloop multileg Feynman integral computation. The complete pure bases for four planar integral families were explicitly provided. Combined with the recent development of the NEATIBP program, we obtained the complete canonical differential equations for the four families. The analytic boundary values up to weight six are also obtained by imposing the absence of spurious singularities. With these computations, fast numerical evaluation for these integrals can be implemented by solving the differential equation.

Our results provide the complete planar three-loop massless five-point integrals for the corresponding three-loop amplitudes with five massless external states. These integrals, as well as recent results for two-loop six-point integrals [45–48], are necessary ingredients for future $\mathcal{N}^3\text{LO}$ studies of $2 \rightarrow 3$ scattering processes, such as three-jet production. The complete phenomenological predictions require future work on scattering amplitudes, renormalization, infrared subtraction, and phase-space integration.

We believe that the state-of-the-art techniques presented in this Letter, such as the IBP reduction and pure integral basis determination via modern computational algebraic

geometry and the alphabet studies, would boost other three-loop and higher-loop precision computations in collider physics and gravitational wave physics.

The computations of the Feynman integrals in this Letter also have applications on theoretical studies of gauge theory. In the recent breakthrough [104], our results on three-loop five-point integrals enable the bootstrap of Wilson loops with a Lagrangian insertion at the three-loop level in $\mathcal{N} = 4$ super-Yang-Mills theory. Furthermore, our results constitute an important step for the direct calculation of three-loop five-point *all-plus* amplitudes in pure Yang-Mills theory, which can be used to test the conjectured dualities between maximally transcendental pieces and Wilson loops with a Lagrangian insertion [105,106].

Finally, the planar computation provides an essential stepping stone toward the nonplanar three-loop five-point problem.

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Data availability—The data that support the findings of this article are openly available [107], embargo periods may apply.

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End Matter

Appendix—To maximize impact and reproducibility, we provide a detailed description of the ancillary files:

`Letters.m`, the three-loop planar pentagon alphabet in Eq. (10).

`<family_name>/Atilde.m`, the CDE matrix $\tilde{A}$ defined in Eq. (16).

`<family_name>/BCnum.m`, numeric boundary values up to finite order with high precision.

`<family_name>/Canonical_Basis.m`, a pure integral basis with uniform transcendental weight which leads to the CDE.

`<family_name>/SBW.tar.xz`, a compressed file for the symbol of the pure integrals.

`ForbiddenPairs.txt`, the forbidden pairs of letters, each integer labels the letters in Eq. (10).

All the ancillary files can be downloaded from the website [107].

TABLE III. Value of integral (7) evaluated at the Euclidean point (20) by PYSECDEC, DIFFEXP, or Chebyshev method.

ϵ order	PYSECDEC	DIFFEXP	CHESS
ϵ^0	$(2.000000 \pm 4.91) \times 10^{-8}$	2	2.000000
ϵ^1	$(-2.339370 \pm 6.88) \times 10^{-5}$	-2.339426	-2.339426
ϵ^2	$(-16.19419 \pm 4.36) \times 10^{-4}$	-16.19414	-16.19414
ϵ^3	$(-17.36078 \pm 2.22) \times 10^{-3}$	-17.36262	-17.36262
ϵ^4	$(18.99960 \pm 1.04) \times 10^{-2}$	18.98939	18.98939
ϵ^5	$(79.57537 \pm 4.03) \times 10^{-2}$	79.52752	79.52752
ϵ^6	$(256.3690 \pm 1.61) \times 10^{-1}$	256.2476	256.2476

Moreover, we show the numerical value of one top-sector integral of PBP in Eq. (7), obtained by PYSECDEC and DIFFEXP in Table III.