

Worksheet for Generalized Veneziano Amplitudes

Shota Komatsu^{1,*} and Pronobesh Maity^{2,†}

¹*Department of Theoretical Physics, CERN, 1211 Meyrin, Switzerland*

²*Laboratory for Theoretical Fundamental Physics, EPFL, Rte de la Sorge, Lausanne, Switzerland*



(Received 8 February 2026; accepted 18 May 2026; published 7 July 2026)

We present a worldsheet action that reproduces a class of dual resonance amplitudes discussed in the literature, which generalize the Veneziano amplitude for open strings. Our proposal builds on the chiral composite linear dilaton introduced recently. We further compute higher-point extensions and closed-string analogs, which exhibit partial crossing symmetry.

DOI: [10.1103/bplf-h1vx](https://doi.org/10.1103/bplf-h1vx)

Introduction—In 1968, Veneziano proposed [1] a mathematical expression for a $2 \rightarrow 2$ scattering amplitude of mesons that incorporates both the s -channel and t -channel resonances into a single analytic function, exhibiting the *dual resonance property*. This inspired numerous generalizations and variants [2–11], all sharing the same feature: simultaneous encoding of different scattering channels in one expression.

With the advent of quantum chromodynamics (QCD), these ideas lost prominence and came to be seen merely as historical precursors to string theory. However interest in these amplitudes has recently revived in the context of the S-matrix bootstrap [12–42], motivated by two main questions. First is the *uniqueness problem* [43–49]: do basic principles such as unitarity, crossing symmetry, and Regge behavior uniquely determine string theory amplitudes, or are there other consistent solutions? Second is *the search for amplitudes* that could describe scattering in large N QCD, capturing features such as confinement and Regge trajectories [50–53].

Despite extensive early works, a worldsheet description exists only for the original Veneziano amplitude. The absence of underlying worldsheet formulations has hindered progress in systematically studying higher-point generalizations and in understanding the origins of their physical and analytic properties, including their factorizability.

In this Letter, we propose a worldsheet action for a class of generalized Veneziano amplitudes introduced by Mandelstam [2], which depends on three parameters [54], b , λ , and δ :

*Contact author: shota.komatsu@cern.ch

†Contact author: pronobesh.maity@epfl.ch

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

$$T(s, t) = \int_0^1 dz z^{-s-b-1} (1-z)^{-t-b-1} [1-4\lambda z(1-z)]^\delta$$

$$= \frac{\Gamma(-s-b)\Gamma(-t-b)}{\Gamma(-s-t-2b)} {}_3F_2 \left[\begin{matrix} -s-b, -t-b, -\delta \\ -\frac{s+t+2b}{2}, \frac{1-(s+t+2b)}{2} \end{matrix}; \lambda \right]. \quad (1)$$

For $\lambda = b = 0$ or $\delta = b = 0$, this reduces to (the polarization-independent part of) the Veneziano amplitude, whereas this agrees with the open bosonic string amplitude upon setting $b = 1$ and $\delta = 0$. Our construction uses the *chiral composite linear dilaton theory* (CLD) introduced in our earlier work [55]. Using this framework, we also compute higher-point amplitudes and closed-string analogs. Unlike the open-string four-point case, these extensions are not fully crossing symmetric due to hidden quantum numbers carried by the strings. Nonetheless they can be made symmetric in several specific channels, retaining part of the dual resonance structure.

Generalized Veneziano amplitudes from worldsheet—Here we present a worldsheet theory reproducing the generalized Veneziano amplitudes (1). We first provide a minimal setup that corresponds to a specific choice of the parameters b , λ , and δ and then discuss generalizations.

Action and stress tensor: Consider the following action for bosonic open string:

$$S = S_{\mathbb{R}^{d-1,1}} + S_{\beta\gamma} + S_\chi + S_{bc}, \quad (2)$$

where $S_{\mathbb{R}^{d-1,1}}$ is an action of $\mathbb{R}^{d-1,1}$ free bosons,

$$S_{\mathbb{R}^{d-1,1}} = \int_\Sigma \frac{d^2z}{2\pi} \partial X^\mu \bar{\partial} X_\mu, \quad (3)$$

and S_{bc} is the action for bc ghosts. Throughout this Letter, we set $\alpha' = 1$ and $\partial := \partial_z$ and $\bar{\partial} := \partial_{\bar{z}}$. S_χ is an arbitrary conformal field theory (CFT) with the central charge c_χ , which describes the dynamics on an internal manifold. The most important part of the action is $S_{\beta\gamma}$, the action for CLD [55,56] given by

$$S_{\beta\gamma} = \int_{\Sigma} \frac{d^2 z}{2\pi} \left[\bar{\beta} \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} + q \left(\partial \varphi \bar{\partial} \varphi + \frac{\hat{R}}{2} \varphi \right) \right] + \frac{q}{\pi} \int_{\partial \Sigma} ds \hat{k} \varphi, \quad (4)$$

where $\varphi \equiv \log(\partial \gamma \bar{\partial} \bar{\gamma} / R^2)$ and $\hat{R}$ and $\hat{k}$ are the Ricci scalar and the geodesic curvature on the worldsheet, respectively. q is a c -number parameter, which is fixed uniquely by requiring the conformal anomaly cancellation as shown below in (6). Unlike the $\beta\gamma$ -system that appears in superstring theory, β here is a $(1,0)$ -form while γ is a $(0,0)$ -form and should be regarded as a target space coordinate of an extra “internal” manifold. In what follows, we compactify γ on a torus, by identifying the real part $\gamma^1 \equiv (\gamma + \bar{\gamma})/2$ as $\gamma^1 \sim \gamma^1 + 2\pi R$ and the imaginary part $\gamma^2 \equiv (\gamma - \bar{\gamma})/2i$ as $\gamma^2 \sim \gamma^2 + 2\pi \tilde{R}$.

A novelty as compared to our previous work [55,56] is the second line of (4), which is required for the Weyl invariance in the presence of worldsheet boundaries $\partial \Sigma$. See Supplemental Material (SM) for details [57].

Despite high nonlinearity of the action, the path integral can be performed explicitly since integrating out β s localizes γ to holomorphic maps ($\bar{\partial} \gamma = \partial \bar{\gamma} = 0$) [55,56]. Using the path integral, one can also deduce the operator product expansion (OPE). Computing the OPE of the stress tensor of matter fields,

$$T(z) = -\partial X^\mu \partial X_\mu(z) - \beta \partial \gamma(z) + 2q\{\gamma, z\} + T_\chi(z), \quad (5)$$

one can read off the central charge of $S_{\beta\gamma}$ to be $c_{\beta\gamma} = 2 + 24q$ [55]. Here $\{\gamma, z\}$ is the Schwarzian derivative, $\{\gamma, z\} = [\partial^3 \gamma \partial \gamma - \frac{3}{2}(\partial^2 \gamma)^2]/(\partial \gamma)^2$. Thus, conformal anomaly cancellation requires

$$q = 1 - \frac{d + c_\chi}{24}. \quad (6)$$

As discussed in [55,56], for the CLD to be well defined, q needs to satisfy $q > 0$, which translates to the condition $d + c_\chi < 24$. In particular, requiring the internal CFT to be unitary ($c_\chi > 0$) leads to a bound $d < 24$.

Boundary condition: To discuss amplitudes of open strings, we need to specify boundary conditions (bc) at $\partial \Sigma$. We choose the one corresponding to “D0-branes” in the γ plane, i.e., branes localized in γ^1 and extended in γ^2 . See Fig. 1. Consequently, we impose the Dirichlet bc for γ^1 and the Neumann bc for γ^2 ,

$$\gamma^1|_{\partial \Sigma} = \text{const}, \quad \partial_n \gamma^2|_{\partial \Sigma} = 0, \quad (7)$$

where ∂_n is a derivative normal to the boundary $\partial \Sigma$. Taking the boundary to be $\text{Im } z = 0$ and using the holomorphicity of γ s, (7) can be rewritten as

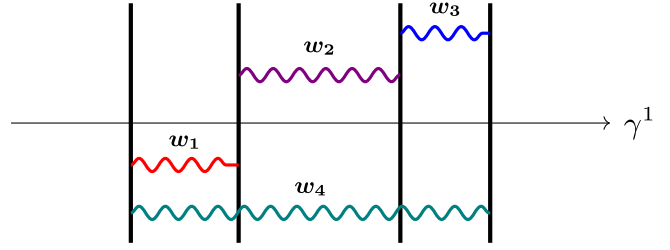


FIG. 1. Open strings between D0-branes placed along the γ^1 direction separated by $\pi w_a R$, where $w_a, a = 1, \dots, 4$ are fractional winding numbers.

$$\partial \gamma = -\bar{\partial} \bar{\gamma}|_{\partial \Sigma}. \quad (8)$$

Acting repeatedly the derivative parallel to the boundary $\partial + \bar{\partial}$ on (8), one can show $\partial^n \gamma = -\bar{\partial}^n \bar{\gamma}|_{\partial \Sigma}$ for any positive integer n . This implies $(\partial - \bar{\partial})\varphi|_{\partial \Sigma} = i\partial_y \varphi|_{\partial \Sigma} = 0$, where we used the real coordinates $z = x + iy$. The bc for β s can be determined by requiring the boundary contribution from the variation of γ^2 to vanish:

$$0 = \frac{1}{2\pi} \int_{y=0} dx \left[(\beta + \bar{\beta}) + iq\partial_x \left(\frac{\partial_y \varphi}{\partial \gamma} \right) \right] \delta \gamma^2. \quad (9)$$

Since the second term in the square bracket vanishes due to $\partial_y \varphi = 0$, we get the following bc for β s:

$$\beta + \bar{\beta}|_{\partial \Sigma} = 0. \quad (10)$$

Vertex operators and four-point amplitudes: We consider open-string vertex operators

$$\mathcal{V}_a(x_a) = e^{ik_a \cdot X(x_a) + iw_a R \int^{x_a} (\beta dz' - \bar{\beta} d\bar{z}')} \quad (11)$$

These vertex operators create open strings stretched between two D0-branes separated by $\pi w_a R$, as can be seen in the OPE with γ^1 :

$$\gamma^1(z, \bar{z}) \mathcal{V}_a(x) \sim -\frac{iw_a R}{2} \log \left(\frac{z-x}{\bar{z}-x} \right) \mathcal{V}_a(x). \quad (12)$$

By computing correlation functions using a path integral, one can read off the conformal dimension of $\mathcal{V}_a$, $h = \bar{h} = k_a^2 + q$. Thus, the on-shell condition reads as

$$M_a^2 = -k_a^2 = q - 1. \quad (13)$$

In the presence of vertex operators (11), the path integral of $\gamma(z)$ localizes to the “Mandelstam map” [58], $\gamma(z) = R\rho(z)$ with

$$\rho(z) = \sum_k w_k \log(z - x_k). \quad (14)$$

As described in Supplemental Material, this localization allows us to evaluate the worldsheet correlation function of matter fields as

$$G_4^{\text{matter}} = \mathcal{N}_4 \prod_{i < j}^4 x_{ij}^{2k_i k_j} \delta_{\sum_a w_a} \delta^d \left(\sum_i k_i \right) \times \left[\frac{(w_1 w_2 w_3 w_4)(w_1 + w_3)^2}{(x_{12} x_{14} x_{23} x_{34})^2} (x_c - a_+)(x_c - a_-) \right]^{\frac{q}{2}}, \quad (15)$$

where $x_c = (x_{12} x_{34}) / (x_{13} x_{24})$ is the cross ratio and $a_{\pm}$ are given by

$$a_{\pm} = -\frac{w_1 w_4 + w_2 w_3 \pm 2\sqrt{w_1 w_2 w_3 w_4}}{(w_1 + w_3)^2}, \quad (16)$$

while $\mathcal{N}_4$ is an overall factor independent of worldsheet and target-space kinematics. Combined with the correlation functions of bc ghosts, this leads to the following expression for the four-point amplitudes

$$\mathcal{A}_4 = \tilde{\mathcal{N}}_4 \int_0^1 dx x^{-s+q-2} (1-x)^{-t+q-2} (x-a_+)^{\frac{q}{2}} (x-a_-)^{\frac{q}{2}}, \quad (17)$$

where $s = -(k_1 + k_2)^2$ and $t = -(k_2 + k_3)^2$ are Mandelstam variables and $\tilde{\mathcal{N}}_4$ is an overall factor independent of Mandelstam variables. Note that we used the $\text{PSL}(2, \mathbb{R})$ invariance to fix $x_{1,2,4}$ to $(0, 1, \infty)$. Compared with (1), (17) has a few differences: (1) Equation (17) is not s, t -crossing symmetric in general because of the extra factor $(x-a_+)^{(q/2)}(x-a_-)^{(q/2)}$. (2) The parameters b and δ are both fixed by q and are not independent in (17):

$$\delta = \frac{q}{2}, \quad b = 1 - q. \quad (18)$$

The first issue can be resolved by tuning w_a s. A particularly convenient choice is $w_1 = w_3$, under which $a_{\pm}$ satisfy $a_+ = 1 - a_-$ and are given by

$$a_{\pm} = \frac{1}{2} \pm \frac{i}{2} \sqrt{r(2+r)}, \quad (19)$$

with $r = w_2/w_1$. With this choice, the parameter λ in (1) is given by

$$\lambda = \frac{1}{(1+r)^2}. \quad (20)$$

Note that, for $q = 0$, the amplitude reduces to the standard open bosonic amplitude as can be seen in (18).

Generalization by dressing: The second issue can be overcome by making use of the internal CFT S_{χ} . For

concreteness, let us assume that S_{χ} contains an extra free boson $y(z, \bar{z})$ with an action

$$S_y = \int \frac{d^2 z}{2\pi} \partial y \bar{\partial} y. \quad (21)$$

We then dress the vertex operators $\mathcal{V}_{2,4}$ as

$$\tilde{\mathcal{V}}_2 = \mathcal{V}_2 e^{ipy}, \quad \tilde{\mathcal{V}}_4 = \mathcal{V}_4 e^{-ipy}. \quad (22)$$

This modifies the on-shell condition to $M_{2,4}^2 = -k_{2,4}^2 = q + p^2 - 1$ and (17) to

$$\begin{aligned} \mathcal{A}_4 / \tilde{\mathcal{N}}_4 &= \int_0^1 dx x^{-s+q+p^2-2} (1-x)^{-t+q+p^2-2} (x-a_+)^{\frac{q}{2}} (x-a_-)^{\frac{q}{2}}, \end{aligned} \quad (23)$$

changing (18) to

$$\delta = \frac{q}{2}, \quad b = 1 - q - p^2. \quad (24)$$

We can thus reproduce (1) with general parameters from the worldsheet. In particular, by setting $p^2 = 1 - q$, the lightest exchanged particle can be made massless as is the case with the original Veneziano amplitude.

Extensions—An advantages of having a worldsheet formulation is that it is straightforward to generalize the analysis beyond open-string four-point amplitudes. Here we present two generalizations: higher-point open-string amplitudes and closed-string four-point amplitudes.

Higher-point amplitudes: Let us extend the computation to $n(> 4)$ -point amplitudes. Using the vertex operators (11) and evaluating the path integral, we obtain

$$G_n^{\text{matter}} = \mathcal{N}_n \prod_{1 \leq i < j \leq n} x_{ij}^{2k_i k_j - q} \delta_{\sum_a w_a} \delta^d \left(\sum_i k_i \right) \times \left| \Delta(P_n) \prod_{k=1}^n w_k \right|^{\frac{q}{2}}. \quad (25)$$

Here $\Delta(P_n)$ is a discriminant of a $(n-2)$ degree polynomial associated with the Mandelstam map $\rho(z)$ (14),

$$\begin{aligned} P_n(z) &\equiv \left(\prod_{k=1}^n (z - x_k) \right) \partial \rho(z) \\ &= \sum_{k=1}^n w_k \prod_{j \neq k} (z - x_j). \end{aligned} \quad (26)$$

Written explicitly, it is given by

$$\Delta(P_n) = \left(\sum_{k=1}^n w_k x_k \right)^{2n-6} \prod_{I < J}^{n-2} (Z_I - Z_J)^2, \quad (27)$$

where Z_I s are zeros of $\partial\rho(z)$, $\partial\rho(Z_I) = 0$. As shown in SM, $\Delta(P_n)$ transforms under $\text{PSL}(2, \mathbb{R})$, $x_k \rightarrow [(ax_k + b)/(cx_k + d)]$ ($ad - bc = 1$), as

$$\Delta(P_n) \rightarrow \frac{\Delta(P_n)}{\prod_k (cx_k + d)^{2n-6}}, \quad (28)$$

guaranteeing that (25) behaves as a correlation function of (quasi)primary operators of dimension 1.

As usual, scattering amplitudes can be computed by fixing three points using $\text{PSL}(2, \mathbb{R})$, including the ghost contribution and integrating over the remaining moduli. For instance, setting $x_{1,4,5}$ to $(0, 1, \infty)$, the five-point amplitude reads as

$$\begin{aligned} \frac{A_5}{\tilde{\mathcal{N}}_5} &= \int_{0 \leq \chi_1 < \chi_2 \leq 1} d\chi_1 d\chi_2 \chi_1^{-s_{12}+q-2} (1-\chi_1)^{-s_{24}+q-2} \\ &\times \chi_2^{-s_{13}+q-2} (1-\chi_2)^{-s_{34}+q-2} (\chi_2-\chi_1)^{-s_{23}+q-2} \Delta(Q_5)^{\frac{q}{2}}, \end{aligned} \quad (29)$$

where $\tilde{\mathcal{N}}_5$ is an overall factor independent of the Mandelstam variables, $s_{ij} = -(k_i + k_j)^2$, and $\chi_{1,2}$ are two conformal cross ratios defined by

$$\chi_1 = \frac{x_{12}x_{45}}{x_{14}x_{25}}, \quad \chi_2 = \frac{x_{13}x_{45}}{x_{14}x_{35}}. \quad (30)$$

$\Delta(Q_5)$ is a discriminant of a polynomial

$$\begin{aligned} Q_5(z) &= \lim_{x_{1,4,5} \rightarrow (0,1,\infty)} \frac{P_5(z)}{x_5} \Big|_{x_2 \rightarrow \chi_1, x_3 \rightarrow \chi_2} \\ &= a_3 z^3 + a_2 z^2 + a_1 z + a_0, \end{aligned} \quad (31)$$

with

$$\begin{aligned} a_3 &= w_5, & a_0 &= w_1 \chi_1 \chi_2, \\ a_2 &= -[w_4 + w_5 + (w_2 + w_5)\chi_1 + (w_5 + w_3)\chi_2], \\ a_1 &= -[(w_1 + w_3)\chi_1 + (w_1 + w_2)\chi_2 + (w_1 + w_4)\chi_1 \chi_2]. \end{aligned} \quad (32)$$

Analogous to the four-point case, the amplitude (29) is not crossing symmetric in general but can be made partially crossing symmetric by tuning w_i s. For instance, by setting $w_1 = w_4$ and $w_2 = w_3$, the amplitude becomes symmetric with respect to two independent exchanges, $k_1 \leftrightarrow k_4$ and $k_2 \leftrightarrow k_3$. One can also make the lightest exchanged particles in some of the channels massless by dressing some of the vertex operators by the free boson (21) (for

instance, by $\tilde{\mathcal{V}}_2 = \mathcal{V}_2 e^{ipy}$, $\tilde{\mathcal{V}}_3 = \mathcal{V}_3 e^{-ipy}$) and tuning the momentum p .

Importantly, for higher-point amplitudes, one cannot make them fully crossing-symmetric or render the lightest exchanged particles massless in every channel. This is because the winding conservation, $\sum_i w_i = 0$, forces external particles into unequal roles; some w_i must be positive while others are negative.

Closed-string amplitudes: To compute closed-string amplitudes, we simply use (4) without the boundary term. As shown in [55], the resulting action is the Weyl invariant.

Inserting vertex operators in the bulk of the worldsheet, $\mathcal{V}_a = e^{ik_a \cdot X + i w_a R \int^{z_a} (\beta dz' - \bar{\beta} d\bar{z}')}$, dressing them by the free boson as $\tilde{\mathcal{V}}_2 = \mathcal{V}_2 e^{ipy}$ and $\tilde{\mathcal{V}}_4 = \mathcal{V}_4 e^{-ipy}$, and performing the path integral, we obtain

$$\frac{A_4^{\text{closed}}}{\tilde{\mathcal{N}}_4^{\text{closed}}} = \int_{\mathbb{C}} d^2 z |z|^{-\frac{s}{2}-2b-2} |1-z|^{-\frac{t}{2}-2b-2} |(z-a_+)(z-a_-)|^q, \quad (33)$$

with $b = 1 - q - (p^2/4)$ and $a_{\pm}$ given by (16).

Similarly to the open-string case, (33) does not exhibit any crossing symmetry for general w_i s but can be made s - t symmetric by setting $w_1 = w_3$, i.e., symmetric with respect to $z \leftrightarrow 1 - z$. However, it cannot be made s - t - u symmetric, i.e., $z \leftrightarrow 1 - z \leftrightarrow [z/(z-1)]$. The reason for this limitation is again the winding conservation, which puts external particles on unequal footings.

Relatedly, although the complex integral (33) can be evaluated explicitly using the tricks by Kawai-Lewellen-Tye [59], the result does not factorize simply into two open-string amplitudes (1). Instead, it is given by a sum of product of Appell's hypergeometric function $\mathbb{F}_1(a, b_1, b_2, c|x, y)$. See SM for details.

It would be interesting to find a worldsheet action that gives rise to a generalized Virasoro-Shapiro amplitude with full crossing symmetry. To achieve it, one needs to relax the winding conservation, which forces the light-cone-gauge-like map. One possibility is to find a mechanism that localizes the worldsheet to a map with the ‘‘Witten-type’’ interaction vertices [60].

Conclusion—We presented a worldsheet action reproducing the generalized Veneziano amplitudes, originally introduced by Mandelstam. Using the same action, we also provided generalization to open-string higher point amplitudes and closed-string analogs. Being the first worldsheet description of generalized Veneziano amplitudes, our construction motivates further studies of the subject. We list a few here: (1) Unitarity of the generalized Veneziano amplitudes for $d = 4$ was analyzed in [28] by studying the positivity of their partial-wave coefficients. It would be useful to revisit this question from a worldsheet perspective, extending the result in [28] to general dimensions and establishing a no-ghost theorem from the worldsheet [61].

(2) It would be interesting to look for the worldsheet actions reproducing other generalized Veneziano amplitudes, for instance, the hypergeometric amplitudes discussed in [23]. (3) Our construction gave closed-string amplitudes with only partial crossing symmetry. As mentioned, developing a worldsheet model that produces a fully crossing-symmetric generalized closed-string amplitude would be highly illuminating. (4) The generalized Veneziano amplitude (1) bears resemblance to off-shell string amplitudes computed from string field theory [62]. It would be interesting to understand possible connections, if any.

Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

-
- [1] G. Veneziano, *Nuovo Cimento A* **57**, 190 (1968).
 - [2] S. Mandelstam, *Phys. Rev. Lett.* **21**, 1724 (1968).
 - [3] K. Bardakci and H. Ruegg, *Phys. Lett.* **28B**, 342 (1968).
 - [4] D. D. Coon, *Phys. Lett.* **29B**, 669 (1969).
 - [5] S. Matsuda, Model for infinite Veneziano series, Report No. UCRL-19266, Lawrence Radiation Laboratory, University of California, Berkeley, California, 1969.
 - [6] H.-M. Chan and S. T. Tsou, *Phys. Lett.* **28B**, 485 (1969).
 - [7] M. A. Virasoro, *Phys. Rev.* **177**, 2309 (1969).
 - [8] D. J. Gross, *Nucl. Phys.* **B13**, 467 (1969).
 - [9] J. A. Shapiro, *Phys. Lett.* **33B**, 361 (1970).
 - [10] L. J. Romans, *Conf. Proc. C* **8903131**, 57 (1989).
 - [11] D. B. Fairlie and J. Nuyts, *Nucl. Phys.* **B433**, 26 (1995).
 - [12] Y.-t. Huang, J.-Y. Liu, L. Rodina, and Y. Wang, *J. High Energy Phys.* **04** (2021) 195.
 - [13] I. Ridkokasha, *Mod. Phys. Lett. A* **36**, 2150031 (2021).
 - [14] P. Maity, *J. High Energy Phys.* **04** (2022) 064.
 - [15] N. Arkani-Hamed, L. Eberhardt, Y.-t. Huang, and S. Mizera, *J. High Energy Phys.* **02** (2022) 197.
 - [16] F. Figueroa and P. Tourkine, *Phys. Rev. Lett.* **129**, 121602 (2022).
 - [17] Y.-t. Huang and G. N. Remmen, *Phys. Rev. D* **106**, L021902 (2022).
 - [18] N. Geiser and L. W. Lindwasser, *J. High Energy Phys.* **12** (2022) 112.
 - [19] J. Chakravarty, P. Maity, and A. Mishra, *J. High Energy Phys.* **10** (2022) 043.
 - [20] N. Geiser and L. W. Lindwasser, *J. High Energy Phys.* **04** (2023) 031.
 - [21] R. Bhardwaj, S. De, M. Spradlin, and A. Volovich, *J. High Energy Phys.* **08** (2023) 082.
 - [22] A. S.-K. Chen, H. Elvang, and A. Herderschee, *Phys. Rev. Lett.* **133**, 091601 (2024).
 - [23] C. Cheung and G. N. Remmen, *Phys. Rev. D* **108**, 026011 (2023).
 - [24] A. S.-K. Chen, H. Elvang, and A. Herderschee, *Phys. Rev. Lett.* **131**, 031602 (2023).
 - [25] C. B. Jepsen, *J. High Energy Phys.* **06** (2023) 114.
 - [26] C. Cheung and G. N. Remmen, *Phys. Rev. D* **108**, 086009 (2023).
 - [27] L.-Y. Chiang, Y.-t. Huang, and H.-C. Weng, *J. High Energy Phys.* **05** (2024) 289.
 - [28] K. Häring and A. Zhiboedov, *J. High Energy Phys.* **10** (2024) 075.
 - [29] R. Bhardwaj and S. De, *arXiv:2309.07214*.
 - [30] K. C. Rigatos, *Phys. Rev. D* **109**, 086008 (2024).
 - [31] J. Berman, H. Elvang, and A. Herderschee, *J. High Energy Phys.* **03** (2024) 021.
 - [32] N. Geiser, *J. High Energy Phys.* **10** (2024) 010.
 - [33] C. Eckner, F. Figueroa, and P. Tourkine, *Phys. Rev. D* **111**, 126005 (2025).
 - [34] K. C. Rigatos and B. Wang, *Phys. Rev. D* **110**, 126024 (2024).
 - [35] C. Eckner, F. Figueroa, and P. Tourkine, *J. High Energy Phys.* **02** (2025) 103.
 - [36] B. Wang, *J. High Energy Phys.* **04** (2024) 143.
 - [37] J. Berman and H. Elvang, *J. High Energy Phys.* **09** (2024) 076.
 - [38] N. Arkani-Hamed, C. Figueiredo, and G. N. Remmen, *J. High Energy Phys.* **04** (2025) 039.
 - [39] R. Bhardwaj, M. Spradlin, A. Volovich, and H.-C. Weng, *Phys. Rev. D* **110**, 106016 (2024).
 - [40] G. Mansfield and M. Spradlin, *J. High Energy Phys.* **02** (2025) 145.
 - [41] F. Bhat, D. Chowdhury, A. P. Saha, and A. Sinha, *Phys. Rev. D* **111**, 066013 (2025).
 - [42] A. Gadde and S. Jain, *J. High Energy Phys.* **05** (2026) 027.
 - [43] S. Caron-Huot, Z. Komargodski, A. Sever, and A. Zhiboedov, *J. High Energy Phys.* **10** (2017) 026.
 - [44] C. Cheung and G. N. Remmen, *J. High Energy Phys.* **01** (2023) 122.
 - [45] N. Arkani-Hamed, C. Cheung, C. Figueiredo, and G. N. Remmen, *Phys. Rev. Lett.* **132**, 091601 (2024).
 - [46] J. Albert, W. Knop, and L. Rastelli, *J. High Energy Phys.* **02** (2025) 157.
 - [47] C. Cheung, A. Hillman, and G. N. Remmen, *Phys. Rev. Lett.* **133**, 251601 (2024).
 - [48] C. Cheung, A. Hillman, and G. N. Remmen, *Phys. Rev. D* **111**, 086034 (2025).
 - [49] C. Cheung, G. N. Remmen, F. Sciotti, and M. Tarquini, *arXiv:2508.09246*.
 - [50] J. Albert and L. Rastelli, *J. High Energy Phys.* **08** (2022) 151.
 - [51] J. Albert and L. Rastelli, *J. High Energy Phys.* **09** (2024) 039.
 - [52] C. Fernandez, A. Pomarol, F. Riva, and F. Sciotti, *J. High Energy Phys.* **06** (2023) 094.
 - [53] J. Albert, J. Henriksson, L. Rastelli, and A. Vichi, *J. High Energy Phys.* **09** (2024) 172.
 - [54] We set the parameter a in [2] to 1 by rescaling s and t . We also introduced a new parameter λ following [28].
 - [55] S. Komatsu and P. Maity, *Phys. Rev. Lett.* **135**, 231603 (2025).
 - [56] S. Komatsu and P. Maity, *arXiv:2512.00861*.
 - [57] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/bplf-h1vx> for additional technical details.
 - [58] S. Mandelstam, in *Workshop on Unified String Theories* (1985).
 - [59] H. Kawai, D. C. Lewellen, and S. H. H. Tye, *Nucl. Phys.* **B269**, 1 (1986).

- [60] E. Witten, [Nucl. Phys. **B268**, 253 \(1986\)](#).
- [61] Our construction requires $q > 0$, i.e., $d + c_\chi < 24$. However, because of the unconventional structure of CLD, it is unclear if this condition is sufficient for the unitarity of spacetime amplitudes. To clarify this issue, one has to derive the no-ghost theorem for the CLD.
- [62] L. Rastelli and B. Zwiebach, [J. High Energy Phys. 01 \(2008\) 018](#).