

Simulating Quantum Instruments with Projective Measurements and Quantum Postprocessing

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Quantum instruments describe both the classical outcome and the updated state associated with a quantum measurement. We ask whether these processes can be simulated using only a natural subset of resources, namely projective measurements on the system and quantum processing of the postmeasurement states. We show that the simulability of instruments can be connected to an entanglement classification problem. This leads to a computationally efficient necessary condition for simulation of generic instruments and to a complete characterisation for qubits. We use this to address relevant quantum information tasks, namely (i) the noise tolerance of standard qubit unsharp measurements, (ii) nonprojective advantages in information-disturbance trade-offs, and (iii) increased sequential Bell inequality violations under projective measurements. Moreover, we consider also d -dimensional Lüders instruments that correspond to weak versions of standard basis measurements and show that for large d these can permit scalable noise advantages over projective implementations.

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Introduction—It is often relevant to ask if complex quantum resources can be simulated by simpler quantum resources [1]. An important example of this concerns quantum measurements. Measurements are commonly associated with orthogonal projections, but their full scope corresponds to the more general notion of positive operator-valued measures (POVMs). Nonprojective measurements play an important role in quantum information protocols, but their physical implementation is more complex than that of projective measurements. It is therefore relevant to ask which POVMs can be simulated using projective measurements and which cannot [2]. In recent years, many experiments have realized POVMs that defy simulation with projective measurements [3–8].

While POVMs describe the classical output of a measurement, they do not describe the update of the quantum state. A complete description of the measurement process must address both the classical and quantum output. This is known as a quantum instrument [9] [see Fig. 1(a)]. Quantum instruments are not only an essential component of quantum theory, but they are also responsible for the fundamental trade-off between extracting information from a system and disturbing its state. Experiments have observed these trade-offs and demonstrated their applications [10–15].

Here, we ask whether quantum instruments can be simulated by an experimenter who only has access to standard projective measurements on the system, classical postprocessing of the measurement outcomes, quantum processing of the postmeasurement state and classical randomness; see Fig. 1(b). Thus, these simulations do not have access to the coherent nonprojective capabilities

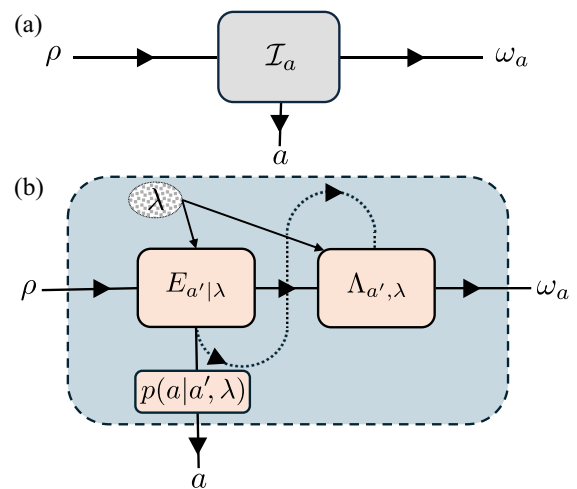


FIG. 1. Quantum instruments. (a) A quantum instrument transforms an incoming state ρ into a classical outcome a and an updated quantum state ω_a . (b) A projective simulable instrument uses a classical random variable, λ , to select a projective measurement, $\{E_{a'|\lambda}\}$, then passes the projected state through a quantum channel, $\Lambda_{a',\lambda}$, and finally uses a rule $p(a|a',\lambda)$ to decide the classical output.

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associated with generic instruments. This can be seen as a natural generalisation from projective simulation of POVMs to now account for the full measurement process. We show that the characterisation of simulable instruments can be relaxed to a type of entanglement classification problem. Using semidefinite programming (SDP) techniques (see the review [16]), this allows us to efficiently determine proofs of genuinely nonprojective instruments. Furthermore, for qubit systems, we prove that this characterization is both necessary and sufficient. This makes it a handy tool, which we showcase by considering three quantum information problems. First, we consider noisy qubit instruments, corresponding to unsharp measurements, and determine the visibilities needed to make a projective simulation possible. Second, we show how nonprojectivity implies advantages in information-disturbance trade-offs. Third, we show that sequential Bell inequality tests, that use only projective measurements and local randomness [17], can admit significantly larger violations than previously known. Moreover, we also address the role of Hilbert space dimension. Specifically, we consider instruments that weakly implement high-dimensional basis measurements and ask whether the large dimensionality implies scalable noise advantages over projectively simulable instruments. We find that for natural noise forms, scalable advantages are possible, but also that there exist noise forms for which the same is not true.

Preliminaries—We denote Hilbert spaces by $\mathcal{H}$, the space of density matrices as $\mathcal{D}(\mathcal{H})$, and the space of positive semidefinite linear operators as $\mathcal{L}_+(\mathcal{H})$. A quantum instrument, $\mathcal{I}$, transforms a state, $\rho \in \mathcal{D}(\mathcal{H}_A)$, into a classical outcome, a , and a corresponding postmeasurement state, $\omega_a \in \mathcal{D}(\mathcal{H}_{A'})$. It is represented as a set $\mathcal{I} = \{\mathcal{I}_a\}_{a=1}^N$, where each $\mathcal{I}_a: \mathcal{H}_A \rightarrow \mathcal{H}_{A'}$ is a completely positive and trace nonincreasing map. The postmeasurement state is given by $\omega_a = [\mathcal{I}_a(\rho)/\text{tr}(\mathcal{I}_a(\rho))]$ where the normalization is the probability of outcome a , namely, $p(a) = \text{tr}(\mathcal{I}_a(\rho))$. Normalization of the probability distribution means that the map $\sum_a \mathcal{I}_a$ is trace preserving. Using state-channel duality [9], instruments can be represented using Choi operators. Each $\mathcal{I}_a$ is then associated with a bipartite operator $\eta_a \in \mathcal{L}_+(\mathcal{H}_{A'} \otimes \mathcal{H}_A)$ defined as $\eta_a = (\mathcal{I}_a \otimes \mathbb{1})[\phi^+]$, where $\phi^+ = |\phi^+\rangle\langle\phi^+|$ and $|\phi^+\rangle = (1/\sqrt{d}) \sum_{i=0}^{d-1} |ii\rangle$ is the maximally entangled state of dimension $d = \dim(\mathcal{H}_A)$. Complete positivity and normalization are equivalent to $\eta_a \geq 0$ and $\text{tr}_{A'} \sum_a \eta_a = (\mathbb{1}/d)$, respectively. The (subnormalized) quantum output of the instrument can be written as $\mathcal{I}_a(\rho) = d \text{tr}_A((\mathbb{1}_{A'} \otimes \rho_A^T) \eta_a)$. The POVM, $\{M_a\}_a$, that is realized by the instrument is obtained from its reduced Choi operator as $M_a = d \text{tr}_{A'}(\eta_a)^T$.

Consider now an experimenter that has access only to the following three resources: (i) classical randomness and classical postprocessing, (ii) projective measurements on $\mathcal{H}_A$, and (iii) quantum postprocessing. Thus, the

experimenter draws a classical variable λ from some distribution $\{q_\lambda\}_\lambda$, selects a projective measurement $\{E_{a'|\lambda}\}$ with outcome a' ($E_{a'|\lambda}^2 = E_{a'|\lambda}$ and $\sum_{a'} E_{a'|\lambda} = \mathbb{1}$), and then implements a quantum channel $\Lambda_{a',\lambda}: \mathcal{H}_A \rightarrow \mathcal{H}_{A'}$ on the (subnormalized) postprojection state $E_{a'|\lambda} \rho E_{a'|\lambda}$. A postprocessing rule $p(a|a', \lambda)$ is used to determine the final classical output. This is illustrated in Fig. 1(b). Instruments of this type are written as

$$\mathcal{I}_a(\rho) = \sum_\lambda q_\lambda \sum_{a'} p(a|a', \lambda) \Lambda_{a',\lambda} [E_{a'|\lambda} \rho E_{a'|\lambda}]. \quad (1)$$

We refer to these as *projective instruments* (PIs) and denote their set as $\mathcal{P}$. In Supplemental Material available at [18], we prove that the postprocessing can be discarded w.l.g., i.e., that one may restrict to $p(a|a', \lambda) = \delta_{a,a'}$, and that one can w.l.g. also restrict to $\Lambda_{a',\lambda} = \Lambda_\lambda$.

Simulability of instruments—How to determine whether an instrument is PI simulable? We begin with giving a general necessary condition. To this end, let us associate every N -outcome projective measurement, $\{E_a\}_{a=1}^N$, with a rank vector $\vec{r} = (r_1, \dots, r_N)$, where $r_a = \text{rank}(E_a)$. Thus, every N -tuple of nonnegative integers such that $\sum_a r_a = d$ is a valid rank vector. We can separately consider the measurements associated with each such $\vec{r}$. Therefore, we write $\lambda = (\chi, \vec{r})$, where χ is a random variable for selecting projective measurements with rank vector $\vec{r}$. Consider now the Choi representation of the generic PI defined in Eq. (1) with $p(a|a', \lambda) = \delta_{a,a'}$ and $\Lambda_{a',\lambda} = \Lambda_\lambda$. It reads,

$$\eta_a = \sum_{\chi, \vec{r}} q_{\chi, \vec{r}} (\Lambda_{\chi, \vec{r}} \otimes \mathbb{1}) [\nu_{a|\chi, \vec{r}}], \quad (2)$$

where $\nu_{a|\chi, \vec{r}} = (E_{a|\chi, \vec{r}} \otimes \mathbb{1}) \phi^+ (E_{a|\chi, \vec{r}} \otimes \mathbb{1})$. Observe that (i) the local projection $E_{a|\chi, \vec{r}}$ of ϕ^+ succeeds with probability (r_a/d) , and (ii) the operator $\nu_{a|\chi, \vec{r}}$ is locally confined to an r_a -dimensional subspace. Consequently, the entanglement dimension (a.k.a its Schmidt number, SN [19] [20]) of $\nu_{a|\chi, \vec{r}}$ is at most r_a . This provides the intuition for our first result (see Supplemental Material [18] for details).

Theorem 1: Instrument simulation—Consider any N -outcome projective instrument and let $\{\eta_a\}_{a=1}^N$ be its Choi representation. There exists a decomposition

$$\eta_a = \sum_{\vec{r}} \sigma_{a|\vec{r}}, \quad \text{where } \sigma_{a|\vec{r}} \in \mathcal{L}_+(\mathcal{H}_{A'} \otimes \mathcal{H}_A) \\ \text{tr}(\sigma_{a|\vec{r}}) = q_{\vec{r}} \frac{r_a}{d}, \quad \sum_a \text{tr}_{A'}(\sigma_{a|\vec{r}}) = q_{\vec{r}} \frac{\mathbb{1}}{d}, \quad \text{SN}(\sigma_{a|\vec{r}}) \leq r_a, \quad (3)$$

where $\vec{r}$ runs over all rank vectors and SN denotes the Schmidt number.

We can interpret the operators $\{\sigma_{a|\vec{r}}\}_a$ as corresponding to a subnormalized Choi representation of a PI that uses only measurements with rank vector $\vec{r}$. The subnormalization, $q_{\vec{r}}$,

corresponds to the probability of selecting $\vec{r}$ and it is given by $q_{\vec{r}} = \sum_a \text{tr}(\sigma_{a|\vec{r}})$. Thus, Theorem 1 states that if the instrument $\mathcal{I}$ is not a convex combination over the Choi representations associated with the different rank vectors, then $\mathcal{I} \notin \mathcal{P}$ and hence it is genuinely nonprojective.

Theorem 1 is not a sufficient condition for simulability because not all Choi representations with a Schmidt number as in Eq. (3) can be associated with PIs. Importantly, however, for the practically most relevant case, namely qubits ($d = 2$), it turns out to be an exact characterisation.

Theorem 2: Qubit instrument simulation—For instruments with qubit input, Theorem 1 is both necessary and sufficient.

Proof—We give the main idea here and present details in Supplemental Material [18]. For qubits, there are only two rank vectors; $\vec{s} = (2, 0, \dots, 0)$ and $\vec{t} = (1, 1, 0, \dots, 0)$, up to permutations. Thus, $\nu_{a|\vec{s}} = \phi^+$ for $a = 1$, yielding arbitrary Choi operators in (2), and $\eta_a = 0$ for $a \neq 1$. This corresponds to (3), since $\text{SN} \leq 2$ trivially holds for any density matrix over $\mathcal{D}(\mathcal{H}_{A'} \otimes \mathbb{C}^2)$. $\vec{t}$ leads to $\nu_{a|\vec{t}} = \frac{1}{2} E_{a|\vec{t}} \otimes E_{a|\vec{t}}^T$ for $a \in \{1, 2\}$ and via (2) to Choi operators $\frac{1}{2} \sum_{\chi} q_{\chi, \vec{t}} \varphi_{a, \chi, \vec{t}} \otimes E_{a|\vec{t}}^T$, for some arbitrary $\varphi_{a, \chi, \vec{t}}$. Any subnormalized separable Choi operator can be written on this form and it is equivalent to the characterization in (3) for $\vec{t}$. ■

Theorem 2 reduces the full characterization of PIs acting on qubits to a type of separability problem. For qubit-qubit systems, separability is equivalent to positive partial transpose [21]. This means that for instruments whose input and output are qubits, we can replace in Eq. (3) the only nontrivial Schmidt number constraint ($\text{SN} \leq 1$) with $\sigma_{a|\vec{t}}^T \geq 0$. This is crucial since it makes Eq. (3) an SDP. This allows us to efficiently decide the membership problem to $\mathcal{P}$.

Building on Theorem 1, SDPs can also be used to falsify simulability for generic instruments. Note that the only condition in (3) that is not SDP compatible is that concerning the Schmidt number. While there exists no computable necessary and sufficient condition for Schmidt number characterization [22], many partial criteria are known (see, e.g., [20, 23–27]). We employ one such partial Schmidt number criterion which admits a semidefinite formulation [28], as this results in an SDP relaxation of $\mathcal{P}$. A practical choice is based on the reduction map $\Theta_s(X) = \text{tr}(X)\mathbb{1} - (1/s)X$. It has the property that $(\Theta_s \otimes \mathbb{1})[\sigma] \geq 0$ for all σ with $\text{SN} \leq s$ [20, 29]. Thus, if the SDP is infeasible, no simulation exists. Our implementation is available at [30].

We now proceed to demonstrate the practical relevance of these methods via three well-known quantum information applications. These concern, respectively, unsharp measurements, information-disturbance relations, and nonlocality.

Noisy unsharp observables—Consider an instrument realizing an unsharp Pauli observable with eigenbasis $\{|0\rangle, |1\rangle\}$. The associated Lüders instrument has Kraus operators $K_a = \sqrt{[1 - (-1)^a \gamma/2]}|0\rangle\langle 0| + \sqrt{[1 + (-1)^a \gamma/2]}|1\rangle\langle 1|$, for outcomes $a \in \{1, 2\}$, and sharpness parameter $\gamma \in [0, 1]$. Thus, $\mathcal{I}_a(\rho) = K_a \rho K_a^\dagger$ and its Choi representation is $\eta_a = (K_a \otimes \mathbb{1})[\phi^+](K_a^\dagger \otimes \mathbb{1})$. This nonprojective qubit instrument is extremal [31] and frequently used in quantum information [15, 32–34]. The associated POVM is $M_a = K_a^\dagger K_a$ and it is always projective simulable, but the instrument is PI only when $\gamma \in \{0, 1\}$.

We determine the instrument's best approximation by a PI. To this end, we find the precise amount of noise that these instruments must be exposed to in order to make them PI simulable for any given γ . Thus, we consider mixtures

$$\eta_a^v \equiv v\eta_a + (1-v)\eta_a^{\text{noise}}, \quad (4)$$

where $\{\eta_a^{\text{noise}}\}_a$ represents a noise instrument and $v \in [0, 1]$ is the visibility. We focus on three standard noise models: dephasing noise, white noise, and worst-case noise. Respectively, these correspond to choosing $\eta_a^{\text{noise}} = (|00\rangle\langle 00| + |11\rangle\langle 11|/4)$, $\eta_a^{\text{noise}} = \frac{1}{8}$, and optimizing over η_a^{noise} . The former corresponds to outputting a random a and then emitting the state $|a\rangle$. The second also outputs a random a but emits the maximally mixed state. In contrast, since the final noise form makes the simulation as powerful as possible, we refer to it as worst-case noise.

For all three noise types, we have determined analytically the critical visibility for PI simulation (see Supplemental Material [18]). The optimality of these results is verified by evaluating the SDP. The three critical visibilities are illustrated as solid lines in Fig. 2. We see that

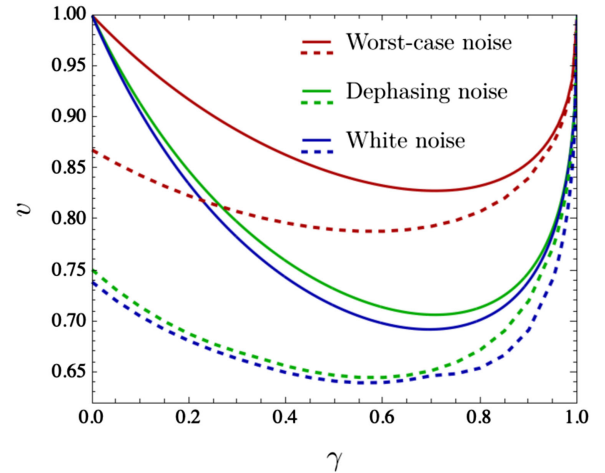


FIG. 2. Critical visibilities for simulation. Solid lines are the critical visibilities for PI simulation of Lüders instruments for qubit observables with sharpness γ . Dashed lines are upper bounds on the critical visibility for Lüders instruments for qutrit measurements.

sizeable amounts of noise are tolerated before simulation is possible, even in the worst-case setting. This attests to robust nonprojective advantages. For dephasing and worst-case noise, the most noise-tolerant instrument corresponds to $\gamma = (1/\sqrt{2})$, while for white noise it is somewhat smaller. In particular, the shape of the curves highlights that instruments that are close to sharp measurements ($\gamma = 1$) are harder to simulate than instruments that are close to noninteracting measurements ($\gamma = 0$).

Information-disturbance relations—Extracting information from a state induces a disturbance in it [32,35]. However, one may expect that nonprojective instruments induce smaller disturbances than PIs. By examining an appropriate information-disturbance trade-off, we show how to check whether an instrument, $\mathcal{I}$, defies PI simulation.

Let us consider an example of such a situation, which pertains to all pure qubit states in quantum theory. For clarity, we explicitly separate the information extraction part from the state-disturbance part. Let ψ be an arbitrary pure qubit state. We want to determine if it belongs to the northern (N) or southern (S) Bloch hemisphere. The success probability becomes $p_{\text{win}} = \frac{1}{2} \int_N d\psi \text{tr}(\mathcal{I}_N(\psi)) + \frac{1}{2} \int_S d\psi \text{tr}(\mathcal{I}_S(\psi))$, where the outcomes are labeled $a \in \{N, S\}$. Also, we want to extract this information while disturbing the state as little as possible. We quantify this through the fidelity of the postmeasurement state, which corresponds to measuring $\{N_{b|\psi}\}_b = \{|\psi\rangle\langle\psi|, \mathbb{1} - |\psi\rangle\langle\psi|\}$. The average fidelity is $F = \int d\psi \langle\psi| \sum_a \mathcal{I}_a(\psi) |\psi\rangle$. The optimal trade-off between F and p_{win} depends on whether we use generic quantum instruments (Q) or PIs. In Supplemental Material [18], we show that the two trade-offs are

$$F_{\text{PI}} = \frac{5-4p_{\text{win}}}{3}, \quad F_Q = \frac{2 + \sqrt{16p_{\text{win}}(1-p_{\text{win}})} - 3}{3}, \quad (5)$$

where $p_{\text{win}} \in [1/2, 3/4]$ ranges from its trivial value (random guess) to the maximal value possible. We see that nonprojective instruments can better preserve the fidelity than can the PIs; the latter are restricted to a linear trade-off (see Fig. 3). Hence, a better-than-linear trade-off implies that the instrument is genuinely nonprojective.

Sequential Bell nonlocality—There has recently been much interest in sequential violations of Bell inequalities [36]. In this scenario, Alice and Bob share a two-qubit entangled state Ψ and aim to violate the CHSH inequality, which reads $\mathcal{S}_{AB} \equiv \langle A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \rangle_\Psi \leq 2$, where A_x and B_y are Alice's and Bob's observables. The average state after Bob's measurement is $\Psi_{\text{post}} = \frac{1}{2} \sum_{b,y} (\mathbb{1} \otimes K_{b|y}) \Psi (\mathbb{1} \otimes K_{b|y}^\dagger)$, where $K_{b|y}$ are the Kraus operators of Bob's y th instrument. Bob's share of Ψ_{post} is relayed to Charlie who measures it in another CHSH test with Alice; $\mathcal{S}_{AC} \equiv \langle A_0 C_0 + A_0 C_1 + A_1 C_0 - A_1 C_1 \rangle_{\Psi_{\text{post}}}$, where C_z is Charlie's observable. The goal is to achieve

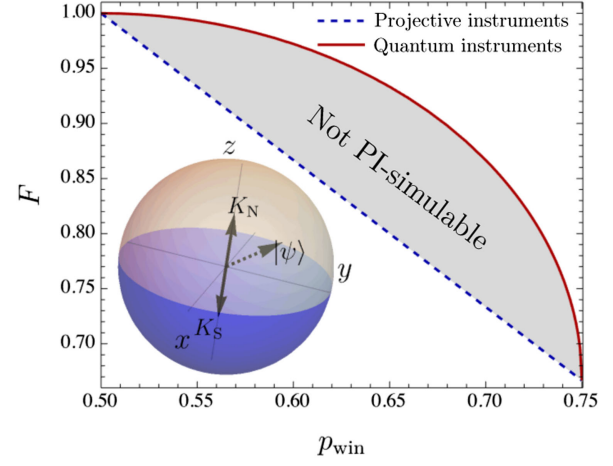


FIG. 3. Information-disturbance trade-off. Fidelity between pre- and postmeasurement state versus success probability of hemisphere discrimination. Inset: illustration of the hemisphere discrimination problem.

a double violation, namely $\mathcal{S}_{AB} > 2$ and $\mathcal{S}_{AC} > 2$. Thus, Bob's instrument must measure strongly enough to violate the inequality, but weakly enough to make a violation possible also for Charlie. Nonprojective instruments are natural for this task [33] but it was recently found that PIs suffice; they achieve $\mathcal{S}_{AB} = \mathcal{S}_{AC} \approx 2.046$ [17].

We use our SDP characterization of qubit PIs to show that larger double violations are possible than previously known, albeit still weaker than what is possible with nonprojective operations. For this, we have employed an alternating convex search routine [16] in which we iteratively optimise over (i) the qubit assemblage prepared by Alice for Bob, (ii) the PI instruments of Bob, and (iii) the qubit measurements of Charlie. See Supplemental Material [18] for details and [30] for our implementation. This procedure returns an explicit quantum model for the double violations. The results are illustrated in Fig. 4 together with

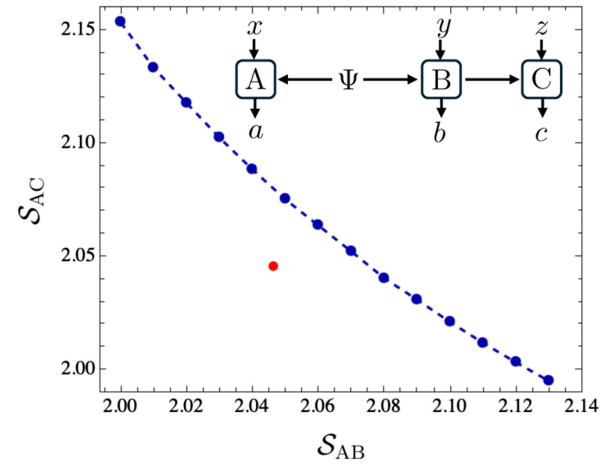


FIG. 4. Sequential nonlocality. Numerical trade-off between CHSH parameters $\mathcal{S}_{AB}$ and $\mathcal{S}_{AC}$ for PIs. Red dot is reported in [17]. Inset: an illustration of the sequential CHSH scenario.

the previously known double violation (red dot) from Ref. [17]. We find significantly enhanced violations, and a trade-off between the CHSH parameters that is convex. Notably, we find that the best two-qubit entangled state is not maximally entangled. This is an important reason for improving on Ref. [17], which restricts its analysis to maximally entangled states. This restriction may appear reasonable since this state indeed is optimal when considering generic quantum instruments [37], but as we have shown it is unjustified for PIs.

High dimensionality—We now turn to instruments that operate on system of higher-than-qubit dimension and ask how the advantages of nonprojectivity depend on the dimension d . We focus on Lüders instruments for an unsharp measurement of the d -dimensional computational basis. The Kraus operators are

$$K_a = \sqrt{\frac{1+\gamma}{2}}|a\rangle\langle a| + \sqrt{\frac{1-\gamma}{2(d-1)}}(\mathbb{1} - |a\rangle\langle a|), \quad (6)$$

where $\gamma \in [0, 1]$ is the sharpness parameter. The coefficient in front of $\mathbb{1} - |a\rangle\langle a|$ is fixed by normalization.

We begin with the case of qutrits ($d = 3$). As previously, we consider the mixture of the instrument with noise of the dephasing, white and worst-case type, respectively; see Eq. (4). Although we do not have a complete characterization of $\mathcal{P}$ for qutrits, our SDP relaxation of this set is sufficient to reveal significant noise advantages over the qubit case. This is shown by the dashed lines in Fig. 2. These enhanced results motivate us to ask whether the advantage of non projective instruments over PIs is scalable for large d . To answer this, we consider the instrument (6) for arbitrary d and study its mixture with noise. We focus on two types of noise, namely dephasing and worst-case noise. Considering both turns out to be important because they yield sharply contrasting results. Let us start with the former.

Let $\eta_a = (K_a \otimes \mathbb{1})[\phi^+](K_a^\dagger \otimes \mathbb{1})$ be the Choi representation of the instrument (6), let $\eta_a^{\text{noise}} = (1/d^2) \sum_{i=0}^{d-1} |ii\rangle\langle ii|$ be the completely dephased instrument and consider their mixture, as in Eq. (4). In Supplemental Material [18] we prove the following upper bound on the critical dephasing visibility required for a PI simulation:

$$v_{\text{deph}}(d) \leq \frac{2(d-1)}{d(1+\gamma + \sqrt{(1-\gamma^2)(d-1)}) - 2}. \quad (7)$$

This bound is tight when $\gamma \geq 1 - (2/d)$ and appears to be near optimal otherwise. Our proof is based on using the strong duality theorem of SDP. By considering the dual form of our SDP relaxation of $\mathcal{P}$, we can find upper bounds on $v_{\text{deph}}(d)$ by constructing feasible points. We have explicitly provided such constructions. Importantly, the critical visibility (7) scales as $O(1/\sqrt{d})$ for any $\gamma \neq 1$. This

tends to zero for large d and shows that the nonprojective feature of the instruments becomes unboundedly noise robust for high dimensions. We refer the reader to Supplemental Material [18] and Refs. [38,39] for further details.

However, such a favorable scaling is not to be expected from arbitrary types of noise. In the Supplemental Material [18], we show that there exists a tailor-made type of noise which allows us to make a PI simulation for the corresponding noisy Lüders instrument (4) for any d and γ with at least 50% visibility. This shows that particularly detrimental forms of noise can lead to powerful projective simulations even in high-dimensional systems.

Conclusion—We have introduced projective quantum instruments. This class of instruments is motivated by the operational limitations of an experimenter who attempts to perform a measurement process using only projective measurements on the incoming system and quantum processing of the postprojection state. One may view this as a resource-theoretic setting in which experimenters cannot perform nonprojective measurements.

We have developed general and computationally efficient methods for characterizing the quantum instruments that admit a projective realization. For qubits, we found a complete characterization which allowed us solve several quantum information tasks. We also showed that instruments acting on high-dimensional quantum systems can under relevant noise conditions have scalable advantages over projective instruments. This indicates that dimensionality is a powerful resource for quantum instruments. Interestingly, the analogous has been shown not to hold for POVMs: Ref. [40] shows that there exists a nonzero isotropic-noise visibility at which a projective measurement simulation is possible for any POVM in any dimension. This highlights that considering the full measurement process leads to sizably stronger advantages from non-projectivity than considering only the classical outcome of the same process. A simple intuition for the enhanced scaling encountered for instruments stems from our reported connection between their simulability and an entanglement classification problem. It is well-known that entanglement under several standard noise conditions can uphold scalable robustness as the dimension increases. We have seen that this qualitative feature carries over to the advantages of nonprojective instruments over projective instruments. While we also showed that nonprojective advantages are not always scalable, this is to be expected due to the tailored selection of noise that is maximally detrimental for the considered instrument. Such noise is typically not physically natural.

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Data availability—The data that support the findings of this Letter are openly available [30].

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End Matter

Witnesses of nonprojectivity—Beyond the information-disturbance trade-off discussed above, it is possible to compute arbitrary witnesses of instrument nonprojectivity. The experimenter prepares a set of states $\{\psi_x\}$, passes them through $\mathcal{I}$, and measures the quantum output with some POVMs $\{N_{b|y}\}_b$, where y indexes the choice of measurement. The statistics of this experiment becomes $p(a, b|x, y) = \text{tr}(\mathcal{I}_a(\psi_x)N_{b|y})$. To detect nonprojective behavior, consider a linear witness

$$W \equiv \sum_{a,b,x,y} c_{abxy} p(a, b|x, y) \leq \beta, \quad (\text{A1})$$

where c_{abxy} are real coefficients and β is a bound satisfied by all PIs. We can efficiently compute these bounds using our SDP relaxations for $\mathcal{P}$. This admits also further simplification: the linearity of W lets us separately consider the witness for each rank vector and then select the best result. If the experimenter observes a violation of the inequality (A1), then no PI simulation is possible. See Supplemental Material for further details.

Noisy SIC instruments—It is interesting to ask whether instruments that implement extremal measurements are more noise tolerant with respect to projective simulation than are the extremal measurements themselves. We show that the answer can be positive, by considering the qubit symmetric informationally complete POVM [41]. It reads $\{\frac{1}{2}|\varphi_a\rangle\langle\varphi_a|\}_{a=1}^4$, corresponding to Bloch vectors $[(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)]/\sqrt{3}$. When this POVM is mixed with the trivial POVM $\{\frac{1}{4}\}_{a=1}^4$, the critical visibility for projective simulation is $v = \sqrt{2/3} \approx 0.817$ [2]. We define the SIC instrument in terms of Kraus operators $K_a = (1/\sqrt{2})\varphi_a$ and mix it with a

noise instrument, as in (4). The noise instrument can be selected in any way compatible with the realized POVM being $\{\frac{1}{4}\}_{a=1}^4$. Two examples are white-noise and dephasing-noise instruments. Evaluating the critical visibility for PI simulation, we find that it is not the same. For dephasing, we find again $v = \sqrt{2/3}$ while for white noise we find $v \approx 0.773$. This showcases the relevance of considering the full measurement process also for extremal POVMs.

Projective simulation of POVMs—By simply ignoring the quantum output, our approach to projectively simulating quantum instruments reduces to a projective simulation of POVMs. Since this problem has received prior interest (see, e.g., [2,42–44]) we make explicit how our methods apply to it as a special case. Let $\{M_a\}$ be a POVM on $\mathcal{H}_A$. In a projective simulation, we wish to decompose it as $M_a = \sum_{\vec{\lambda}} q_{\vec{\lambda}} E_{a|\vec{\lambda}}$, where $\{E_{a|\vec{\lambda}}\}$ are projective measurements. As before, we partition the space of projective measurements by their rank vectors $\vec{r}$. This leads to the characterization

$$M_a = \sum_{\vec{r}} F_{a|\vec{r}}, \quad \text{where} \quad F_{a|\vec{r}} \succeq 0, \\ \text{tr}(F_{a|\vec{r}}) = q_{\vec{r}} r_a, \quad \sum_a F_{a|\vec{r}} = q_{\vec{r}} \mathbb{1}. \quad (\text{C1})$$

Here, $q_{\vec{r}} = (1/d) \sum_a \text{tr}(F_{a|\vec{r}})$ is the probability of selecting the rank vector $\vec{r}$. This can be viewed as an SDP relaxation of the set of projective simulable measurements and it is obtained directly from Theorem 1 by reducing the Choi operators to the space $\mathcal{H}_A$. This gives a computationally simple way to prove the failure of projective measurement simulability.