Minimum Energy and Photon Content in *PT*-Symmetric Metamaterials

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(Received 17 April 2026; revised 9 September 2026; accepted 18 September 2026; published 14 October 2026)

In the context of waves in space-time modulated materials, we ask two questions: (1) how much energy does it cost to break time reversal symmetry (T-symmetry) and transition to a PT-symmetric state? and (2) can a PT-symmetric system have a "ground state" in the sense that no photons are present? Our model system is a periodic metamaterial set in virtual motion with velocity c_g to become a space-time crystal. We find that the expectation of energy content is always increased on breaking symmetry. At the same time breaking T-symmetry introduces photon pairs even when we start from a T-symmetric ground state empty of photons, except in certain pathological examples that we describe. For a range of velocities, PT symmetry is broken so that energy must be continuously invested to preserve motion, creating a trail of photon pairs. Here, energy must be continuously invested to preserve motion. We make an analogy with acoustic radiation generated from breaking the sound barrier.

DOI: 10.1103/bc42-c2rg

Introduction—PT-symmetric systems are described by a Hamiltonian that is not Hermitian but nevertheless has real eigenvalues [1–3]. These do not correspond to energy but to some other conserved quantity the nature of which depends on details of the system. In a quantum system [4–6] there is an expectation value for the energy, $\langle U\rangle$, but actual measurements will vary from one identically prepared system to another. The question arises, but to our knowledge has not so far been answered, of what is the minimum value for $\langle U\rangle$. In the case of an optical system we can also ask whether states can be prepared having zero photon content, the PT-symmetric equivalent of a ground state, $\langle N\rangle = 0$. Since negative occupation numbers do not exist, this would imply that $N = 0$ is an eigenstate.

We explore these questions in the context of optical materials structured in space and time [7]. We consider the case of a structure set in uniform motion with a velocity c_g: the permittivity and permeability have the form $\varepsilon(x - c_g t), \mu(x - c_g t)$. By this we do not mean that the material itself moves, rather that ε, μ are locally modulated to simulate virtual motion: by analogy with a wave on the sea that appears to move but the water does not.

Such systems can have a Hamiltonian that is PT-symmetric with eigenvalues that are real but are not to be interpreted as energy content. In the system we shall examine, eigenvalues represent the total spin of all photons in the system and therefore spin is conserved [8].

We confirm that at lower velocities the system is PT-symmetric but there is a minimum energy required to start motion, increasing with c_g. Once this energy is invested the system continues without further input of energy. In contrast there is a regime of broken PT symmetry associated with systems where at some point the local velocity of light is the same as that of the grating. Here, we find constant emission of photon pairs with opposite spin (conserving total spin) and a constant supply of energy is needed to keep the system in motion: a sort of quantum friction [9,10]. As c_g approaches the symmetry-breaking transition, energy needed for motion diverges in our model, though including dispersion would give a finite but large number. An airplane breaking the sound barrier would be a good analogy, the "sonic boom" comprising a shower of photon pairs.

In the PT-symmetric region photon content is defined only as an expectation value that in general is nonzero as photon pairs fluctuate in and out of existence. However, there are special cases where the photon number commutes with the Hamiltonian and is therefore conserved and zero photon eigenstates can be defined.

PT symmetry in a moving grating—Our model is an elementary case of a metamaterial in uniform motion,

$$nc_0^{-1} = \varepsilon = \mu = c_1^{-1}/(1 + 2\alpha\cos(gx - \Omega t)), \quad (1)$$

where n is the refractive index, c_0 is the speed of light in vacuum. The velocity is given by $c_g = \Omega/g$ and the impedance $\sqrt{\mu/\varepsilon}$ is constant, eliminating backscattering

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and simplifying calculations. This profile is invariant under a simultaneous inversion in space $x \to -x$ and time reversal $t \to -t$. The model is a much used template for study of virtual motion in the context of classical waves [7,11–15]. We shall mention its limitations in our conclusions.

In the absence of backscattering, time evolution factorizes into forward and backward waves. As discussed in Ref. [6] only waves traveling in the same direction as c_g are of interest as these interact most strongly with the moving structure. Making this assumption, the displacement field evolves in time according to [16]

$$\partial_x n^{-1} D = -\partial_t D. \tag{2}$$

Since n is spatially periodic in a comoving Galilean frame $X = x - c_g t$, the eigenstates of (2) are defined by a Bloch wave vector k, and frequency ω. In this frame (2) becomes a non–Hermitian eigenvalue problem for $\omega(k)$,

$$-i\partial_X \left(n^{-1} - c_g\right) D = \omega D, \tag{3}$$

which is symmetric under simultaneous inversion of the X axis and complex conjugation [we assume n is an even function, as in Eq. (1)]. If $(n^{-1} - c_g)$ has the same sign throughout space, (3) can be transformed to a Hermitian problem in terms of a new field $D' = (n^{-1} - c_g)^{1/2} D$, showing that, in the PT-symmetric phase, the eigenvalue ω is real.

We can show this general behavior explicitly for the profile (1). Using the solution to (2) [given in [16] and reproduced in Eq. (A6) of the End Matter] and demanding the wave accumulate a phase $\exp(2\pi i k/g)$ over a single unit cell, the dispersion relation is found to be

$$\omega_m = c(k + mg) = \left(c_g + \sqrt{(c_1 - c_g)^2 - (2c_1\alpha)^2}\right)(k + mg), \tag{4}$$

where m is an integer.

We express ω in the observer rather than Galilean frame. Note the branch points at $c_g = c_1(1 \pm 2\alpha)$. These branch points arise from the existence of locations in the moving index profile (1) where the local wave velocity equals c_g, exactly the condition that prevents us from transforming (3) to a Hermitian problem. In these cases there is an unstable accumulation of energy over time [6,16,17]. For a grating velocity c_g between these values, the evolution of the D field breaks the PT symmetry of the system.

In the observer's frame many frequencies are found differing by multiples of Ω. Nevertheless, the eigenstates have the simple form [again see Ref. [16] and Eq. (A6) of the End Matter] $\exp(ikf) \times \partial_X f(X)$, where

$$|D_k| = \partial_X f(X) = \left(c_0 n^{-1} - c_g\right)^{-1}, \tag{5}$$

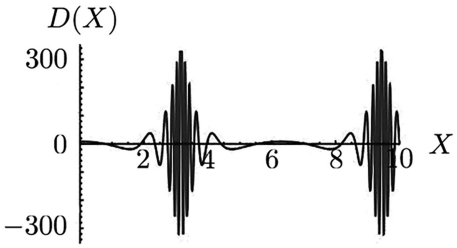


FIG. 1. Plot of $\mathrm{Re}D_k(X)$ for, $\alpha = 0.025, g = 1, c_1 = 1$, $c_g = 0.947, k = 10$. A space-time varying refractive index profile such as that given in Eq. (1) causes the electromagnetic field to concentrate around regions where the wave speed is closest to the synthetic velocity c_g. Note: c_g is close to the singularity at $1 - 2\alpha = 0.95$.

showing that lines of $|D_k|$ are most compressed where the local velocity, $c_0 n^{-1}$, comes close to c_g and that $|D_k|$ is conserved. Note that $|D_k|$ is independent of the wave vector. A typical eigenstate is plotted in Fig. 1.

Ground state energy and photon content—We now calculate the properties of the quantum ground state of the electromagnetic field in this space-time varying medium. To find the lowest expectation value of the energy we start from $\alpha = 0$ in Eq. (1), initially eliminating time variation from the medium. For a state with wave vector k we trivially identify the zero photons' ground state energy as $U_{0k} = \hbar c_1 k/2$. To access the moving state, we slowly increase α. This process conserves the wave vector. If the system is large but finite and the process is slow compared to frequency spacing between states, each state will continuously evolve into an eigenstate for the current value of α and minimize introduction of photons.

This approach allows us to calculate the eigenstate at each stage of the process but not to normalize it. Conservation laws [16] for the lines of force solve the problem. The integral

$$\frac{g}{2\pi}\int_0^{2\pi/g} |D_k(\alpha = 0, X)| dX = \sqrt{\frac{n(\alpha = 0) U_{0k}}{c_1}} \tag{6}$$

is conserved as α is increased. The fields in Eq. (6) have been normalized to start with the static zero-point energy content, U_{0k},

$$U_{0k} = \frac{c_1}{n(\alpha = 0)} |D_k(\alpha = 0)|^2. \tag{7}$$

We may now calculate enhancement of the zero point energy by evaluating Eq. (6) for nonzero α,

$$U_k = \frac{g}{2\pi}\int_0^{2\pi/g} |D_{n,k}(\alpha, X)|^2 \frac{c_1}{n(\alpha, X)} dX. \tag{8}$$

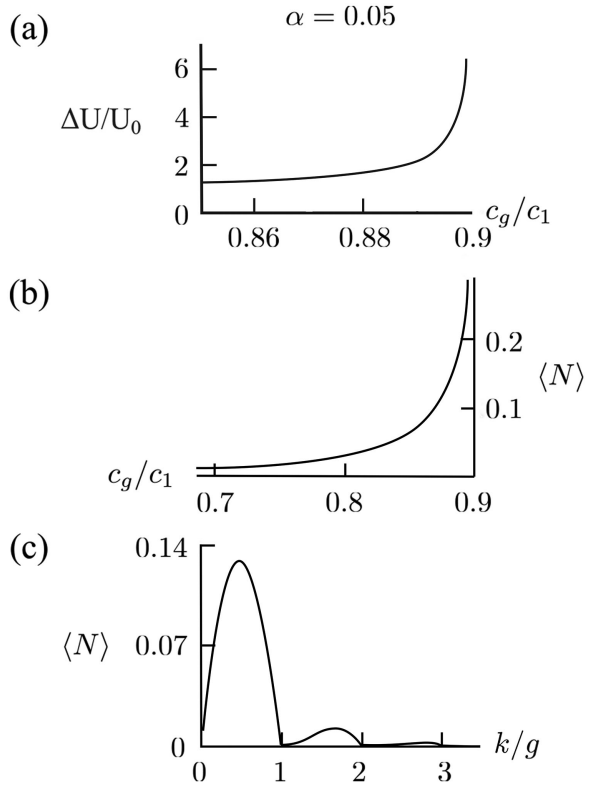


FIG. 2. (a) Plot of $\Delta U/U_0$ versus c_g/c_1. For these parameters ($\alpha = 0.05, g = 1.0$) the system enters the PT symmetry-broken regime at $c_g/c_1 = 0.9$. (b) Average number of photons added to the ground state versus grating velocity for $k = 0.7g$. We sum Eq. (11) over m to get $\langle N \rangle$ shown in the plots. (c) Average number of photons added versus wave vector, $c_g = 0.89c_1$. Whenever the wave vector is an integer multiple of g, no photons are created. Note that in this model, which neglects dispersion, enhancement is the same for all states irrespective of frequency.

Using Eqs. (5)–(8) the energy cost of motion is calculated and plotted against c_g in the top panel in Fig. 2. Here, we neglect the contribution of left-going modes, which although important, does not diverge at the PT-symmetry breaking point. A more detailed calculation based on a quantization of the field is given in the End Matter, leading to the same result and showing that, as demonstrated in Fig. 2(a), the ground state energy U_k of each mode diverges as we approach the symmetry broken phase where $\omega_m(k)$ in Eq. (4) becomes complex.

We estimate the change in total energy, ΔU, by integrating over all the fluctuations up to a cutoff frequency typically $\hbar\omega_{\text{cut}} \approx 4$ eV ,

$$\Delta U \approx \frac{1}{(2\pi)^3} \int_0^{k(\omega_{\text{cut}})} \frac{\hbar\omega}{2} 4\pi k^2 dk - U_0 = \frac{\hbar\omega_{\text{cut}}^4 (n^3 - 1)}{16\pi^2 c_0^3} \approx 10 \text{ Jm}^{-3}, \tag{9}$$

where we choose $n \approx 3$. This is not a large number but the point we make is one of principle: that in PT-symmetric systems it is nonzero.

Even though there is no physical motion of the medium, the field energy (8) increases from the stationary system to the singularity at $c_g/c_1 = 1 - 2\alpha = 0.9$ where PT symmetry is broken. Equation (4) shows that ω becomes complex for $c_1(1 - 2\alpha) < c_g < c_1(1 + 2\alpha)$, indicating a continual increase in energy. In this PT-broken phase, input of energy is required to keep the system in motion, a sort of quantum friction. It was shown elsewhere [6] that photon pair production is responsible for the increase in energy.

Next we calculate the number of photons present.

As shown in the End Matter (see also Ref. [16]), in the rest frame of the index profile, the modes of our system have a conserved frequency ω. The Hamiltonian in this frame can therefore be written as $\hat{H} = (4\pi)^{-1} \int d\omega\, \hbar\omega [\hat{a}_\pm^\dagger(\omega)\hat{a}_\pm(\omega) + \hat{a}_\pm(\omega)\hat{a}_\pm^\dagger(\omega)]$, where the "$\hat{a}_\pm$" are the annihilation operators for right- and left-going modes. This indicates that the number of photons is conserved in the rest frame of the medium, and, assuming the index profile becomes uniform at infinity, the vacuum state can be defined as usual as $\hat{a}_\pm|0\rangle = 0$. This all holds, provided that PT symmetry is not broken, at which point the modes can no longer be defined over the whole of space.

Meanwhile, in the frame where the index profile is in motion, the Hamiltonian must give rise to the transformed Heisenberg equations, e.g., $\partial_t \hat{A} + c_g \partial_x \hat{A} = i\hbar^{-1}[\hat{H}, \hat{A}]$. The Hamiltonian in this frame $\hat{H}'$ thus has an additional term proportional to the Minkowski momentum,

$$\hat{H}' = \hat{H} + \frac{c_g}{2} \int dx [\hat{D}\partial_x \hat{A} + \partial_x \hat{A}\, \hat{D}] = \varepsilon_0^{-1} \int dx\, n^{-1} \hat{D}^2. \tag{10}$$

The additional term on the first line of (10) is not proportional to the number operator and therefore the photon number in the $|0\rangle$ state is nonzero in the x, t coordinate system. This is an example of the known reference frame dependence of particle number found in, e.g., the Fulling-Davies-Unruh effect [18,19].

To count the photons in the $|0\rangle$ state we move to Fourier space in order to identify each frequency component of the eigenstates $D_{m,k}(x,t)$ (k the Bloch vector and m the band index). These Fourier components determine the Bogoliubov transformation between the creation and annihilation operators and are defined as $d_{m,k,m'} = (g/2\pi) \int_0^{2\pi/g} dx\, D_{m,k}^* e^{-i(k+m'g)x}$, where $D_{m,k}$ is normalized in the same way as in Eq. (6). As our model system is impedance matched, the displacement and magnetic fields are related, as they are in free space, by $B = \pm(c_0\varepsilon_0)^{-1} D_{m,k}$. We can thus imagine instantaneously "turning off" the index profile and, without this leading to

any additional excitation, measuring the photon content in terms of these Fourier components,

$$\langle N_{m,k}\rangle = \langle 0|\hat{a}^{\dagger}_{m,k}(\omega)\hat{a}_{m,k}(\omega)|0\rangle = \frac{2}{\hbar|\omega_{m,k}|\varepsilon_0}\sum_{m'}\left|d_{m,k,m'}\right|^2, \tag{11}$$

where $\hat{a}_{m,k}$ is an annihilation operator for the free space mode with wave vector $k+mg$ and frequency $\omega_{m,k}$. This tells us that we can always count the number of photons in our system by dividing the square of the Fourier coefficients by $\hbar|\omega+m\Omega|$, and summing over all Fourier components. This approach is described further in [20,21]. Note that if we make the same calculation except using $\hbar(\omega+n\Omega)$, i.e., without taking the modulus, we calculate the spin. It has been shown [8] that spin is conserved in this system. It follows that any increase in photon number is associated with transitions between positive and negative frequencies during the increase of α.

The middle panel in Fig. 2 shows the total number of photons produced as the grating velocity approaches c_1, the lower critical velocity. Comparing the middle and upper panels we note that photon production makes a minimal contribution to energy that is mainly vested in modifying the original ground state.

Dependence of photon production on wave vector is presented in the bottom panel of Fig. 2. Photon production is heavily biased toward low values of k and is strictly zero when k is an integer multiple of g. To understand this result we note that photons can only be created in $\pm$ frequency pairs, which in the absence of backscattering is the only way to conserve spin. Matrix elements of the Hamiltonian coupling frequencies are proportional to frequency [21] and frequency changes in single steps of Ω in consequence of Eq. (1). If $\omega=m\Omega$ the route to negative frequency passes through $\omega=0$, and is therefore barred: no photons can be created. Furthermore, starting from a large positive value of ω means a longer journey to $\omega=0$ and matrix elements, weighted by frequency as they are, favor migration to even larger ω.

This result opens the possibility of systems which have PT zero point energy and no photons.

Consider a wave on a ring, in which the field obeys periodic boundary conditions requiring $k=2\pi n/L$. Then choosing $g=2\pi/L$ in Eq. (1), no state can have photons added.

With the exception of this pathological case we argue that even in the lowest energy state, photons will always be present. Suppose that instead of creating states of the moving system by slowly acting on the initial ground state, we suddenly start the motion. Immediately after the start there are still no photons present. This state is not an eigenstate of the time evolution operator but can be expanded in terms of the complete set of eigenfunctions. In reciprocal space this looks like

$$D_{m,k} = \sum_{m'} F_{m'm} e^{i(k+mg)x - i(\omega_{m'}+m\Omega)t}, \tag{12}$$

where subscript m' labels the eigenfunctions and m the Fourier component of that eigenvector. The eigenfunctions will be in phase only at points in time when they sit in that exact relationship to the grating. At other times they will represent a state with nonzero photon expectation value. On average the photon occupation will be the sum of the component eigenvalue contributions. All eigenstates have the same velocity, c, and have velocity $(c-c_g)$ relative to the grating. Therefore, photon content will oscillate in time with a periodicity $2\pi/(g(c-c_g))$, where $2\pi/g$ is the grating period.

Conclusions—Our simple model demonstrates the main features of energy and photon minimization in a system that transitions from separate T and P symmetries to a combined PT symmetry. Namely that it always costs energy, even if the starting point is the ground state. During the process, photon production cannot be avoided except for certain special cases, one of which we have demonstrated. This necessary investment of energy means that the systems we consider here cannot spontaneously crystalize into the PT-symmetric state.

In the velocity range $c_1(1-2\alpha) < c_g < c_1(1+2\alpha)$, PT symmetry is broken and a regime of friction ensues. In our model, to cross this threshold requires an infinite investment of energy, a consequence of neglecting dispersion. Nevertheless, even in a more sophisticated model crossing the threshold would require major investment of energy.

The main limitation of our model is the absence of frequency dispersion in ε,μ, which is always present in real systems. This will remove infinities from the solutions due to infinite compression of waves. This cannot happen when dispersion is present.

We neglect backscattering, which even outside the transluminal region can break PT symmetry by introducing band gaps in k. However, backscattering can be minimized by avoiding sharp interfaces, much as a camera lens can be "bloomed" to avoid reflections.

These limitations are discussed in the End Matter but neither backscattering nor dispersion affect the main qualitative conclusions of energy investment required for motion and a critical value of c_g beyond which PT symmetry is broken.

There is already literature on realizing artificial event horizons for surface water waves (among other kinds of waves, e.g., in Bose-Einstein condensates, or spin waves) [22–24]. Some further comments may be found in the End Matter.

Acknowledgments—J. B. P. and S. A. R. H. acknowledge support from the Engineering and Physical Sciences Research Council (EPSRC) under Grant No. EP/Y015673/1, and S. A. R. H. acknowledges additional

support through the EPSRC grant "Photon management in dynamic complex scattering media" (EP/Z535928/1).

Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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End Matter

Appendix—Here, we provide more details to show how the electromagnetic field can be quantized in this moving index profile and the calculation of the ground state energy. In the case where $\varepsilon = \mu = n$, Maxwell's equations reduce to

$$\frac{\partial E}{\partial x} = \frac{\partial B}{\partial t}\frac{\partial}{\partial x}\left(n^{-1}B\right) = c_0^{-2}\frac{\partial}{\partial t}(nE), \tag{A1}$$

where we assume propagation along the x axis and $\mathbf{E} = E\mathbf{e}_z$ and $\mathbf{B} = B\mathbf{e}_y$. In terms of the potentials, the first of the above equations is trivial, whereas the second reduces to

$$\left(\frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial t}n\right)\left(\frac{1}{n}\frac{\partial}{\partial x} + \frac{1}{c_0}\frac{\partial}{\partial t}\right)A = 0. \tag{A2}$$

Considering only right-going waves, which propagate with the motion of the index profile, we can take the right-most part of this factorized wave equation,

$$\left(\frac{\partial}{\partial x} + \frac{n}{c_0}\frac{\partial}{\partial t}\right)A_+ = 0. \tag{A3}$$

As described in the main text, in the comoving frame $X = x - c_g t$, the refractive index is time-independent and the above wave equation can be written for a fixed frequency ω,

$$\frac{\partial A_+}{\partial x} = \frac{i\omega}{c_0}\frac{n}{1 - nc_g/c_0}A_+, \tag{A4}$$

with the solutions

$$A_+(X,t) = \exp\left[\frac{i\omega}{c_0}\int^X \frac{n(X')dX'}{1 - n(X')c_g/c_0}\right]. \tag{A5}$$

The corresponding displacement field D_+ is defined in terms of a time derivative of the vector potential in the observer frame. Thus, in terms of the comoving frame coordinates,

$$\begin{aligned} D_+(X,t) &= -\varepsilon_0 n\left(\frac{\partial A_+}{\partial t} + c_g\frac{\partial A_+}{\partial X}\right) \\ &= \frac{i\omega n\varepsilon_0}{1 - nc_g/c_0}\exp\left[\frac{i\omega}{c_0}\int^X \frac{n(X')dX'}{1 - n(X')c_g/c_0} - c_0 t\right]. \end{aligned} \tag{A6}$$

In terms of the comoving frame coordinates, we have thus found the modes of the system, with $-D_+$ being the conjugate momentum to the vector potential A_+. We write these operators as

$$\hat{A} = \hat{A}_+ + \hat{A}_-$$
$$\hat{D} = \hat{D}_+ + \hat{D}_-, \tag{A7}$$

(with "±" indicating right and left propagation). The quantum theory of light requires the following commutator between these operators:

$$\left[\hat{A}(X,t), -\hat{D}(X',t)\right] = i\hbar\delta(X - X'). \tag{A8}$$

We now look for the expansion of $\hat{A}_+$ and $\hat{D}_+$ such that this commutation relation is fulfilled. Without assuming any periodicity we can write

$$\hat{A}_+ = \int_0^\infty \frac{d\omega}{2\pi}\sqrt{\frac{\hbar}{2\varepsilon_0\omega c_0}}e^{\left[\frac{i\omega}{c_0}\int^X \frac{n(X')dX'}{1-n(X')c_g/c_0} - c_0 t\right]}\hat{a}_+ + \text{H.c.}, \tag{A9}$$

where the displacement field operator follows from Eq. (A6). Assuming $[\hat{a}_+(\omega), \hat{a}_+^\dagger(\omega')] = 2\pi\delta(\omega - \omega')$, the commutation relation between $\hat{A}_+$ and $\hat{D}_+$ is

$$\left[\hat{A}_+(X,t), -\hat{D}_+(X',t)\right] = \frac{i\hbar}{2}\delta(X - X') \tag{A10}$$

in agreement with the required commutation relation. The factor of $1/2$ arises because we are only considering right-going waves. Performing an identical analysis for left-going waves produces the same result, with the sum equaling (A8). Combining (A6) and (A9) shows that, as described in the main text, each mode in the expansion of the displacement field has a fixed normalization $\sqrt{\hbar/(2\omega\varepsilon_0 c_0)}$ that is independent of the grating velocity c_g.

The argument given in the main text uses the fact that the normalization of the modes (A6) necessary to fulfil the commutation relation (A8) requires a fixed amplitude independent of both the index profile and the grating velocity. This normalization leads to an energy for each mode that does depend on the grating profile and velocity. This means that there is an energetic cost for setting the grating into motion: for making the transition between a T-symmetric and a *PT*-symmetric vacuum state.

From the above equation for A_+ (A3), we can see that, for right-going waves the magnetic field and the displacement field are related as follows:

$$B_+ = -\frac{1}{c_0\varepsilon_0}D_+ \tag{A11}$$

(an identical relation holds in free space). Therefore, the energy (in the observer's frame, where the grating is in motion) can be written as

$$\begin{aligned} U &= \frac{1}{2}\int dx\left(\varepsilon_0\varepsilon E_+^2 + \mu_0\mu H_+^2\right) \\ &= \frac{1}{2}\int dx\left[\frac{D_+^2}{n\varepsilon_0} + \frac{B_+^2}{n\mu_0}\right]. \\ &= \frac{1}{\varepsilon_0}\int dx\frac{D_+^2}{n} \end{aligned} \tag{A12}$$

Taking the case where the index profile is uniform at infinity, the modes in Eq. (A9) reduce to plane waves at large $|X|$, with the Doppler shifted dispersion relation $n(\omega + c_g k)/c_0 = k$ due to the use of a Galilean transformation. The vacuum state can thus be defined by $\hat{a}_+|0\rangle = 0$, and the ground state energy (A12) is

$$\langle 0|\frac{1}{\varepsilon_0}\int dx\frac{\hat{D}_+^2}{n}|0\rangle = \int dX\int_0^\infty \frac{d\omega}{2\pi}\frac{\hbar\omega}{2c_0}\frac{n}{(1 - nc_g/c_0)^2}, \tag{A13}$$

an energy that increases with c_g and diverges when the index profile contains points where $c_g = c_0/n$. Note that when the medium is homogeneous and its velocity is zero $c_g = 0$, the integral over frequency can be replaced with one over wave vector $dk = nd\omega/c_0$, and (A13) reduces to the usual result for the vacuum energy. When restricted to a single mode and single period of the moving index profile, this is expression (8) given in the main text.

Our model can be extended to 3D through numerical studies. The wave vector then becomes three dimensional, $\mathbf{k} = (k_x, k_y, k_z)$, where we assume that the grating travels in the z direction, and the frequency, $\omega = c\sqrt{k_x^2 + k_y^2 + k_z^2}$, is only asymptotically linear in k_z. Breaking of *PT* symmetry now occurs at values of c_g closer to c weakening the effect of motion. The same argument can be made when dispersion is present. This also removes the linearity of $\omega(k_z)$, weakening the effect of motion but not removing it.

Turning now to dissipation, *PT* symmetry is lost when dissipation is present and motion requires continuous input of energy at all values of c_g. Emission of photons will commence smoothly as a function of c_g rather than with a sharp transition.

Our model neglects backscattering eliminating all band gaps. The system remains *PT*-symmetric with real eigenvalues at low values of c_g. Weak backscattering removes the linearity of $\omega(k_z)$ but only substantially in the vicinity of the gaps, only slightly weakening the effect of c_g, which must come closer to c to break *PT* symmetry. However, when $c_g > c$ band gaps change their nature to gaps in k rather than ω, breaking *PT* symmetry. Since the gaps arise from hybridization of positive and negative frequencies

emission of photon pairs can occur without violating spin conservation.

The most likely candidates for experimental realization have to do with propagation of intense pulses in optical fibers. It has been shown that a modulation of the refractive index created by an intense pulse of light can propagate with the speed of light [25] and compress radiation through the "push broom" effect [26]. More recently pulse propagation in fibers has been used in the search for a laboratory analog of Hawking radiation [27], which is precisely the radiation we expect in the symmetry broken phase of our system. Observation of radiation would be evidence of PT-symmetry breakdown. Most recent progress is reported in [28] where details of this experimental approach can be found.

To give a specific example: imagine a train of pulses in a fiber propagating at the minimum dispersion wavelength of $\lambda = 1.3\mu$. Silica fiber has a damage threshold of around $10^3 W$ and assuming pulses approaching this limit would create a moving modulation in the refraction index of around $\delta n = \alpha \approx 0.005$ [29]. Modulation of n is modest but the advantage of a fiber is that it can be very long if need be. Vacuum modes accompanying the grating are slowly transformed into grating modes by the weak modulations, acquiring the extra energy we predict. Our theory neglects dispersion and in the PT broken symmetry case would predict exponential growth of energy with fiber length achieved by compressing the frequencies in a mode. Inclusion of dispersion would halt this process once those frequencies were outside the PT-broken regime.

Dispersion is very small indeed in a silica fiber so let us make the modest assumption that compression continues until it has doubled the amount of energy in the mode. This assumes a fiber long enough that saturation occurs. The rate of increase of energy is given by Eq. (4) of our manuscript, $\Im k = 4\alpha k$, so a doubling of the energy content can be expected after

$$e^{+4\alpha k \ell} = 2, \ \ell = 7.4 \times 10^{-6} m. \tag{A14}$$

Now calculate the number of extra photons emerging,

$$\dot{N}_0 = \frac{c_0}{2\pi} \int_{k_1}^{k_2} \frac{dk}{2} = 2.2 \times 10^{14} \ \mathrm{s}^{-1}, \tag{A15}$$

where $(k_2 - k_1) = 4.7 \times 10^6 \ \mathrm{m}^{-1}$. We have populated the original modes with half a photon and assumed that this number doubles for each mode within a range $(k_2 - k_1)$. It seems that this number is highly observable.

Other possible experimental realizations have been referenced in the main text. For example, the classical fluctuations of these water waves referenced in [22,23] can be used as an experimental investigation of the Hawking spectrum [24]. For shallow, stationary water, the dispersion relation is linear, $\omega = k\sqrt{gh}$, where g is the acceleration due to gravity, and h is the water depth. If the depth is a function of x, and the water is flowing at velocity c_g, this will look formally almost identical to our electromagnetic system, with $1/\sqrt{h}$ playing the role of, e.g., the index. Although this fluid system will also not be impedance matched, so long as we change $\sqrt{h}$ slowly enough in space, we can ignore reflections, except for the very lowest frequency parts of the wave. This assumption was already made in the above references, including in the experiment, which was well described by the theory.

We present a simulation comparing propagation through a nondispersive system with modulated profile,

$$n(x,t) = n_b + \frac{1}{2}\delta n\big(1 + \cos\big(\omega_g(t - x/c_g)\big)\big),$$
$$n_b = 2, \ c = 1, \ c_g = 0.5, \ \omega_g = 2/\pi, \ \delta n = 0.2, \tag{A16}$$

and through a dispersive profile,

$$n(x,\omega,t) = n_b + \frac{\alpha(\omega)}{2}\delta n\big(1 + \cos\big(\omega_g(t - x/c_g)\big)\big), \tag{A17}$$

where $\alpha(\omega)$ is given in Fig. 3, $\alpha(\omega)$ satisfies the Kramers-Kronig relations, with a real part that is approximately constant up to a frequency ω_c.

Dissipation and dispersion are intimately connected through Kramers-Kronig so we cannot have one without the other and their effects are entangled. Roughly speaking, dispersion will ultimately limit the degree of compression experienced by a pulse when velocities at higher frequencies disperse out of the PT symmetry-broken range.

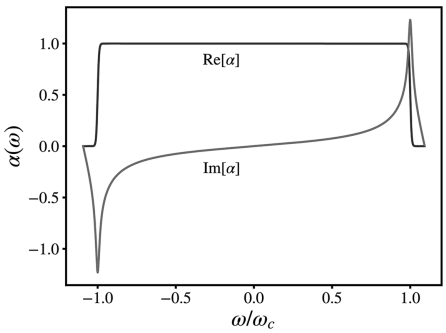


FIG. 3. Real and imaginary parts of $\alpha(\omega)$ the function used to modify the space-time-dependent index from the nondispersive form.

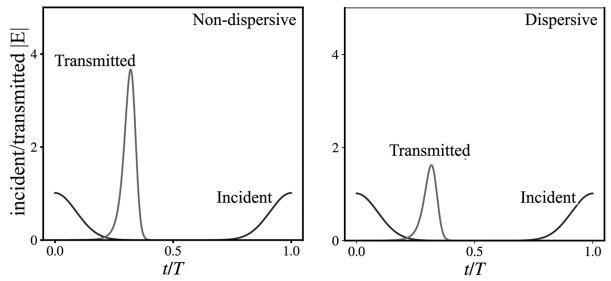


FIG. 4. Propagation through six periods of the grating defined in (A14) and (A15). Left: nondispersive; right: dispersive. Blue lines present the incident profile, red lines the transmitted profile of $|E|$. T is the period of the grating. The scale shows an incident pulse of maximum amplitude of unity.

Absorption on the other hand limits the increase in amplitude of a pulse by competing with the gain mechanism.

Figure 4 shows propagation through six periods of the "grating." The main effect of the dispersion is that localization saturates and the pulse amplitude stops growing, compared to the nondispersive situation. Where this starts to happen is entirely dependent on the frequency at which dispersion or loss becomes important. This phenomenon makes our divergent results finite: at the onset of symmetry breaking the fields no longer have a divergent amplitude with increasing propagation distance.