

Thermoelectric Conduction in General Relativity: A Causal, Stable, and Well-Posed Theory

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We present a covariantly stable first-order framework for describing charge and heat transport in isotropic rigid media embedded in curved spacetime. Working in the Lorenz gauge, we show that the associated initial value problem is both causal and locally well posed in the fully nonlinear regime. We then apply such framework to explore a range of gravitothermoelectric effects in metals undergoing relativistic acceleration. These include (1) the separation of charge through acceleration, (2) the nonuniformity of Joule heating across accelerating circuits due to time dilation, and (3) the effect of redshift on magnetic diffusion. As an astrophysical application, we derive a relativistic Thomas-Fermi equation governing the charge distribution inside a compact object, also accounting for Seebeck charge displacements driven by cooling.

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Introduction—Suppose we place an electric circuit just above the event horizon of a black hole, or on the floor of a rocket undergoing extremely large uniform acceleration. How do gravity and acceleration modify the flow of electric current?

This simple question is the starting point of a long-standing line of research on *gravitoelectric effects* [1–10], namely phenomena that cause an electric circuit to effectively function as a gravimeter-accelerometer. The most elementary example is the well-known result that, in stationary conditions, it is the redshifted current $I\sqrt{-g_{tt}}$ (rather than I itself) that remains uniform along a wire [3]; see Fig. 1 for a quick proof. Another classic instance is the Schiff-Barnhill effect [11], according to which a conductor in a gravitational field develops an equilibrium charge separation (the Stewart-Tolman effect [1,10,12] is the same phenomenon with accelerations). While such effects are expected to be relevant primarily in locally charged compact object [12–27], some are sufficiently pronounced to be detectable even in Earth’s weak gravitational field, and have been exploited as precision tests of Einstein’s equivalence principle [28].

Despite the broad interest, a fully relativistic generalization of the standard solid-state theory of thermoelectricity (see, e.g., Ref. [29], Sec. 26, or [30], Sec. 21.C) is still missing. Certain relativistic fluid theories do contain the relevant transport mechanisms [31–35], and we may just

take their rigid limit. However, the resulting frameworks are either acausal (and therefore unstable [36–39]) or introduce a large number of additional “second order” transport coefficients whose values are generally unknown for relativistic materials. More importantly, none of the available models is known to admit a well-posed initial value formulation. In other words, it remains unclear whether their equations are mathematically solvable at all, particularly for objects undergoing relativistic rotation [40].

Here, we adopt the Benfica-Disconzi-Noronha-Kovtun strategy [41–43] to construct, for the first time to our knowledge, a causal, stable, and well-posed theory for thermoelectricity in relativistic media, which reduces to the standard “textbook” framework (see [29], Sec. 26 and [30], Sec. 21.C) in the Newtonian limit. We then illustrate its

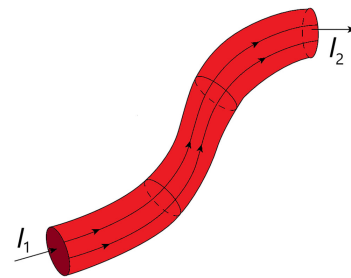


FIG. 1. Continuity of the redshifted current. Under stationary conditions, the Lie derivative $\mathfrak{L}_{\partial_t} J$ vanishes, where J^ν is the current density, and ∂_t is a Killing vector. Combining this with the conservation law $\nabla_\mu J^\mu = 0$ and the Killing property $\nabla_\mu (\partial_t)^\mu = 0$, we find that $\nabla_\mu [(\partial_t)^\mu J^\nu - J^\mu (\partial_t)^\nu] = 0$. Integrating over the portion of wire in the figure, and applying Stokes’ theorem, we obtain $\int_{\text{area}_1} J^\mu (\partial_t)^\nu dS_{\mu\nu} = \int_{\text{area}_2} J^\mu (\partial_t)^\nu dS_{\mu\nu}$, or, equivalently, $I_1 ||\partial_t||_1 = I_2 ||\partial_t||_2$.

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predictive power by discussing several applications in simplified geometries.

Conventions—The metric is treated as a fixed background, and has signature $(-, +, +, +)$. We work in natural units, with $c = \hbar = k_B = 1$, and $e^2 = 4\pi/137$ [44], Sec. 58.

Derivation of the theory—Following [3], consider a conductive medium, whose positive ions are constrained to move with a macroscopic velocity field u^μ that is Born rigid, i.e., proportional to a timelike Killing vector $\mathcal{K}^\mu = \mathcal{K}u^\mu$ [45–47]. In thermodynamic equilibrium, all fields, including the Faraday tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, are stationary, i.e., $\mathfrak{L}_{\mathcal{K}}\text{fields} = 0$, and we have that [31]

$$\nabla_\mu(\mathcal{K}T) = 0, \quad \nabla_\mu(\mathcal{K}\mu) - \mathcal{K}\mathcal{E}_\mu = 0, \quad (1)$$

where T is the temperature, μ is the electric charge chemical potential, and $\mathcal{E}_\mu = F_{\mu\nu}u^\nu$ the electric field in the conductor's local rest frame [48], Sec. 17.2.3. Moreover, despite the possible presence of charge separation, there is no current or heat flow in equilibrium [10,35], i.e.,

$$J^\mu = \rho_{\text{lte}}(T, \mu; \text{ion configuration})u^\mu, \\ W^\mu = \varepsilon_{\text{lte}}(T, \mu; \text{ion configuration})\mathcal{K}^\mu, \quad (2)$$

where “lte” stands for local thermodynamic equilibrium. Here, J^μ is the electric current, and $W^\mu = -T_{\text{mat}}^{\mu\nu}\mathcal{K}_\nu$, with $T_{\text{mat}}^{\mu\nu}$ is the matter stress-energy tensor [40].

Now, suppose the medium is *not* in thermodynamic equilibrium (but the ions still move rigidly). Then, we can model corrections to J^μ and W^μ as linear responses to the left sides of (1) not being zero, with susceptibilities $\sigma_i, \kappa_i = \mathcal{O}(\tau)$ (τ = electron mean free path). Then, the most general isotropic first-order theory is

$$\mathcal{K}J^\mu = \rho_{\text{lte}}\mathcal{K}^\mu - (\sigma_1 g^{\mu\nu} + \sigma_2 u^\mu u^\nu)[\nabla_\nu(\mathcal{K}\mu) - \mathcal{K}\mathcal{E}_\nu] \\ - (\sigma_3 g^{\mu\nu} + \sigma_4 u^\mu u^\nu)\nabla_\nu(\mathcal{K}T) + \mathcal{O}(\tau^2), \\ W^\mu = \varepsilon_{\text{lte}}\mathcal{K}^\mu - (\kappa_1 g^{\mu\nu} + \kappa_2 u^\mu u^\nu)[\nabla_\nu(\mathcal{K}\mu) - \mathcal{K}\mathcal{E}_\nu] \\ - (\kappa_3 g^{\mu\nu} + \kappa_4 u^\mu u^\nu)\nabla_\nu(\mathcal{K}T) + \mathcal{O}(\tau^2). \quad (3)$$

But let us recall that, out of equilibrium, T and μ are not uniquely defined, as the electrons' statistical distribution receives $\mathcal{O}(\tau)$ corrections, and there are many ways to approximate its shape with a Maxwellian fit [49–51]. This allows us to introduce field redefinitions of the form [40]

$$T = \tilde{T} + \tau_1 u^\nu \nabla_\nu \tilde{T} + \tau_2 u^\nu \nabla_\nu \tilde{\mu} + \mathcal{O}(\tau^2), \\ \mu = \tilde{\mu} + \tau_3 u^\nu \nabla_\nu \tilde{T} + \tau_4 u^\nu \nabla_\nu \tilde{\mu} + \mathcal{O}(\tau^2), \quad (4)$$

for some $\tau_i = \mathcal{O}(\tau)$. With a field redefinition of this kind, we can always set $\sigma_2 = \sigma_4 = \kappa_2 = \kappa_4 = 0$, as there are four fixable τ_i 's in (4). Neglecting the $\mathcal{O}(\tau^2)$, we finally obtain

$$\mathcal{K}J^\mu = \rho_{\text{lte}}\mathcal{K}^\mu - \sigma_1[\nabla^\mu(\mathcal{K}\mu) - \mathcal{K}\mathcal{E}^\mu] - \sigma_3\nabla^\mu(\mathcal{K}T), \\ W^\mu = \varepsilon_{\text{lte}}\mathcal{K}^\mu - \kappa_1[\nabla^\mu(\mathcal{K}\mu) - \mathcal{K}\mathcal{E}^\mu] - \kappa_3\nabla^\mu(\mathcal{K}T). \quad (5)$$

In the stationary limit, (5) reduces to the theory of [3], which allows one to assign physical interpretations to the transport coefficients (e.g., σ_1 = electric conductivity).

Causality and well-posedness—The vanishing of $\{\sigma_{2i}, \kappa_{2i}\}$ allows us to establish several self-consistency results.

Theorem 1—Suppose that (5) holds, where $\sigma_i(T, \mu, x^\alpha)$ and $\kappa_i(T, \mu, x^\alpha)$ are such that $\sigma_1\kappa_3 - \sigma_3\kappa_1 \neq 0$ everywhere. Then, in the Lorenz gauge ($\nabla_\mu A^\mu = 0$), Maxwell's system

$$\nabla_\mu F^{\nu\mu} = J^\nu, \\ \nabla_\mu J^\mu = 0, \\ \nabla_\mu W^\mu = \mathcal{K}\mathcal{E}_\mu J^\mu, \quad (6)$$

defines a well-posed Cauchy problem for the variables $\Psi = \{A^\nu, \mu, T\}$, given initial data on spacelike surfaces. Furthermore, the resulting dynamics propagate signals precisely at the speed of light.

Proof—It will be enough to show that the system (6) takes the form $\nabla_\mu \nabla^\mu \Psi = \mathcal{F}(\Psi, \nabla_\alpha \Psi, x^\alpha)$, as then we can invoke Theorem 10.1.3 of [52]. To this end, we recall that, in the Lorenz gauge, the first line of (6) reads $\nabla_\mu \nabla^\mu A^\nu = R^\nu_\mu A^\mu - J^\nu$, where $R_{\mu\nu}$ is the Ricci curvature [53], Sec. 22.4. This already has the form we want, since J^μ contains only first derivatives. The second and third lines can be rearranged, with a bit of algebra, in the form

$$\begin{bmatrix} \sigma_1 & \sigma_3 \\ \kappa_1 & \kappa_3 \end{bmatrix} \begin{pmatrix} \nabla_\mu \nabla^\mu \mu \\ \nabla_\mu \nabla^\mu T \end{pmatrix} = \text{terms with at most } \nabla \Psi. \quad (7)$$

To arrive here, one needs to use the product rule, and the identity $\nabla_\mu \mathcal{E}^\mu = -u_\nu J^\nu + F^{\mu\nu}\nabla_\mu u_\nu$. Now, the condition $\sigma_1\kappa_3 - \sigma_3\kappa_1 \neq 0$ guarantees that the matrix in (7) is invertible everywhere, allowing us to isolate $\nabla_\mu \nabla^\mu \Psi$. ■

Entropy—To first order in the mean free path, the entropy current of the conductor reads [54]

$$\mathcal{K}s^\mu = s_{\text{lte}}\mathcal{K}^\mu - \frac{\kappa_1 - \mu\sigma_1}{T}[\nabla^\mu(\mathcal{K}\mu) - \mathcal{K}\mathcal{E}^\mu] \\ - \frac{\kappa_3 - \mu\sigma_3}{T}\nabla^\mu(\mathcal{K}T) + \mathcal{O}(\tau^2). \quad (8)$$

Its four divergence can be computed with the aid of (6). Defined $V^\mu = [\mathcal{K}^{-1}\nabla^\mu(\mathcal{K}\mu) - \mathcal{E}^\mu, \mathcal{K}^{-1}\nabla^\mu(\mathcal{K}T)]$, we have

$$T\nabla_\mu s^\mu = V^\mu \begin{bmatrix} \sigma_1 & \sigma_3 \\ \frac{\kappa_1 - \mu\sigma_1}{T} & \frac{\kappa_3 - \mu\sigma_3}{T} \end{bmatrix} V_\mu^T + \mathcal{O}(\tau^2). \quad (9)$$

Then, Onsager's principle [55], Sec. 14.5 immediately tells us that $T\sigma_3 = \kappa_1 - \mu\sigma_1$, in agreement with [56]. Moreover, the second law of thermodynamics requires that the 2×2

(symmetric) matrix above be positive definite. In fact, the conservation laws in (6) imply that $\mathcal{K}_\mu V^\mu = \mathcal{O}(\tau)$ on shell, so V^μ is effectively spacelike in (9), up to an error that scales like $\mathcal{O}(\tau^2)$. The resulting inequalities are

$$\sigma_1 > 0, \quad \kappa_3 - \mu\sigma_3 - T\sigma_3^2/\sigma_1 > 0, \quad (10)$$

the latter being interpretable as the positivity of the heat conductivity [29], Sec. 26.

Covariant stability—Since the theory is causal, stability in the rest frame implies covariant stability [39,57]. Hence, we linearize (6) around a uniform charge-neutral state in Minkowski space, with $\mathcal{K} = 1$ and $u^\mu = (1, 0, 0, 0)$. By analyzing the “curl” Maxwell equations, we find that there are four transversal quasinormal modes, all of which obey a causal magnetic diffusion equation:

$$\partial_t \mathcal{B}_j = \frac{1}{\sigma_1} \partial_\mu \partial^\mu \mathcal{B}_j, \quad (11)$$

which is stable [57]. The remaining four quasinormal modes are longitudinal and have no magnetic field. Their evolution can be studied by linearizing $\nabla_\mu J^\mu = 0$ and (9). This results in the system

$$\left\{ \begin{bmatrix} \chi_{\mu\mu} & \chi_{\mu T} \\ \chi_{T\mu} & \chi_{TT} \end{bmatrix} \partial_t - \begin{bmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_5 \end{bmatrix} \partial_\mu \partial^\mu \right\} \begin{bmatrix} \delta\mu \\ \delta T \end{bmatrix} + \begin{bmatrix} \sigma_1 \\ \sigma_3 \end{bmatrix} \partial_\mu \mathcal{E}^\mu = 0, \quad (12)$$

where χ_{ij} is the susceptibility matrix $\partial(\rho, s)/\partial(\mu, T)$, and we defined $T\sigma_5 = \kappa_3 - \mu\sigma_3$. Let us note that the two matrices in (12) are both symmetric and positive definite, and thus can be rewritten as $N\Lambda N^T$ and NN^T , respectively, with N invertible and $\Lambda = \text{diag}(\lambda_1, \lambda_2)$, $\lambda_i > 0$. Hence, assuming a plane-wave form $\Psi \propto e^{\Gamma t + ikz}$ ($k \in \mathbb{R}$), and invoking the equation $\partial_\mu \mathcal{E}^\mu = \delta J^0$, (12) becomes

$$[\Gamma^2 + (\Pi + \Lambda)\Gamma + \Pi\Lambda + k^2]\Upsilon = 0, \\ \text{with } \Upsilon = N^T \begin{bmatrix} \delta\mu \\ \delta T \end{bmatrix}, \quad \Pi = N^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} N. \quad (13)$$

In Supplemental Material [58], we prove that, as long as Π and Λ are non-negative definite (which they are), the solutions of (13) always have $\Re\Gamma \leq 0$, i.e., are stable.

The regime of validity and physical content of each mode is discussed in Supplemental Material [58].

Setup for applications 1,2,3—In what follows, we apply our theory to investigate extreme gravitoelectric phenomena arising at high accelerations. To this end, we discuss the behavior of rigid pieces of equipment aboard a rocket undergoing hyperbolic (i.e., uniformly accelerated) motion in Minkowski space, see Fig. 2. As usual, we work in Rindler coordinates ($ds^2 = -g^2 z^2 dt^2 + dx^2 + dy^2 + dz^2$),

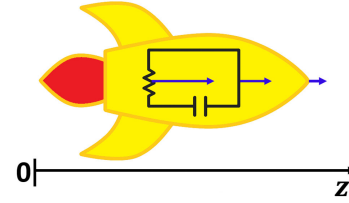


FIG. 2. A rocket containing a circuit undergoes rigid motion in Minkowski space ($ds^2 = -dT^2 + dx^2 + dy^2 + dZ^2$), with velocity u^μ proportional to the generator of boosts, $\mathcal{Z}\partial_T + T\partial_Z$. In Rindler coordinates [$T = z \sinh(gt)$, $Z = z \cosh(gt)$], we have $u^\mu \partial_\mu = (gz)^{-1} \partial_t$, and the proper acceleration (blue arrows) of a piece of equipment is $a = 1/z$, which increases toward the rear of the rocket, and diverges at Rindler’s horizon ($z = 0$).

so that the local velocity of the equipment aligns with the Killing vector ∂_t , and thus $\mathcal{K} = gz$.

Application 1: Inertia of the electron—We consider a piece of metal with uniform transport properties ($\sigma_1 = \text{const}$), which is externally kept in an isothermal state ($\mathcal{K}T = \text{const}$, $\nabla_\mu W^\mu = \mathcal{K}\mathcal{E}_\mu J^\mu + \text{external heat}$). Assuming invariance under xy translations, we ask, how does the charge tend to arrange? To answer this question, we assume that $F = F_{tz}(t, z)dt \wedge dz$, which trivially solves the homogeneous Maxwell’s equations ($dF = 0$), and we write $\nabla_\mu F^{\nu\mu} = J^\nu$ explicitly, which gives

$$\partial_z \mathcal{E}^z = \rho_{\text{lte}} + \sigma_1 (gz)^{-1} \partial_t \mu, \\ \partial_t \mathcal{E}^z = \sigma_1 [\partial_z (gz\mu) - gz\mathcal{E}^z]. \quad (14)$$

To close the system, we need an equation relating ρ_{lte} and μ . We posit $\mu = -\mu_0 + b^2 \rho_{\text{lte}}$ (with μ_0 and b positive constants, estimated in Supplemental Material [58]), which is valid when ρ_{lte} is small and the positive charges are equally spaced. Moreover, instead of solving (14) exactly, we recall that $\sigma_1 \mathcal{K}^\mu \nabla_\mu \mu = \mathcal{O}(\tau^2)$, so the term $\partial_t \mu$ can be neglected, and we obtain a single parabolic equation:

$$\frac{\partial_t \mathcal{E}^z}{g\sigma_1} = -\mu_0 + b^2 \partial_z (z \partial_z \mathcal{E}^z) - z \mathcal{E}^z. \quad (15)$$

We solve this equation numerically, assuming local charge neutrality at $t = 0$, and requiring that no charge leaves the chunk of metal, so $\mathcal{E}^z(\text{outside}) = 0$. The resulting evolution is shown in Fig. 3. Over time, the electrons *fall behind*, due to their inertia, and they accumulate on the rear. This stops when a sufficiently strong electric field is formed, which counteracts the inertia. This is an instance of the Stewart-Tolman effect [1] in relativity.

Application 2: Time dilation and Joule heating—From Fig. 1, we know that a stationary current I in an accelerated circuit scales like $1/z$, by time dilation. But since the Joule power output grows like $I^2 \sim 1/z^2$, we expect wires to get warmer toward the rear of the rocket, by an amount that

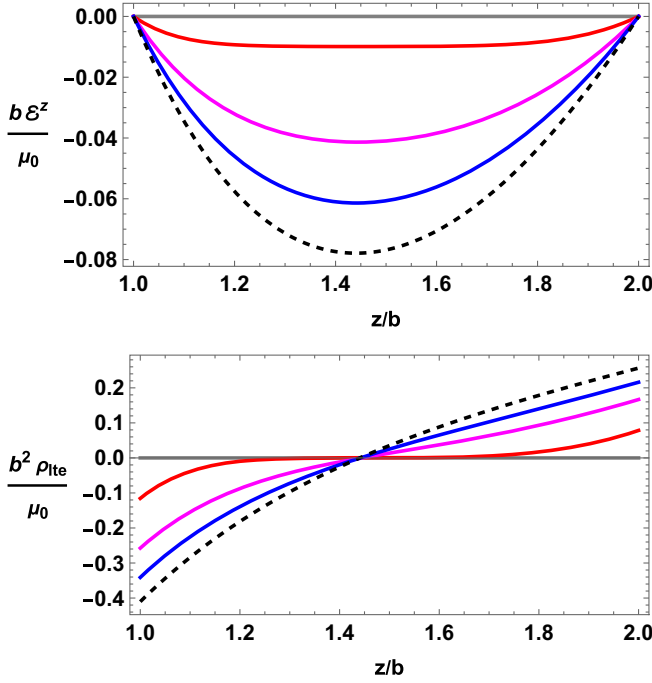


FIG. 3. Electric field (upper panel) and charge density (lower panel) of a solution of (15) with $\mathcal{E}^z(t=0) = \mathcal{E}^z(\text{boundary}) = 0$, for a piece of metal extending from $z = b$ to $z = 2b$. The curves represent snapshots at $g\sigma_1 b t = 0$ (gray), 0.01 (red), 0.05 (magenta), 0.1 (blue), and ∞ (dashed).

exceeds the redshift effect (which only predicts $T \sim 1/z$). Let us quantify this effect.

Consider an isolated, stationary, nonequilibrium system, where two pieces of equipment located, respectively, at z_1 and z_2 , and having the same redshifted temperature ($z_1 T_1 = z_2 T_2$), are connected by a straight cable crossed by a uniform redshifted current density $zJ^z = \text{const} \equiv \mathcal{J}$. Let us estimate the temperature along the cable. Assuming $\sigma_3 = 0$ and $\kappa_3 = \text{const}$, Eqs. (8) and (9) become

$$z s^z = -\frac{\kappa_3}{T} \partial_z(zT), \quad \frac{T}{z} \partial_z(z s^z) = \frac{\mathcal{J}^2}{\sigma_1 z^2} + \frac{\kappa_3}{T z^2} [\partial_z(zT)]^2. \quad (16)$$

Combining them together, we arrive at the equation

$$\partial_z[z \partial_z(zT)] = -\frac{\mathcal{J}^2}{\sigma_1 \kappa_3}, \quad (17)$$

to be solved with boundary conditions $T(z_1) = T_1$ and $T(z_2) = T_2$. In Fig. 4, we provide two solutions: one where the acceleration is negligible ($z_1 \rightarrow \infty$), and one where it matters ($z_1 \rightarrow 0$). In the former, the cable's hottest spot lies at its midpoint, which is the farthest point from the thermostatic boundaries, where the cable gives off its heat. In the latter, the peak in redshifted temperature has moved notably to the left, as predicted.

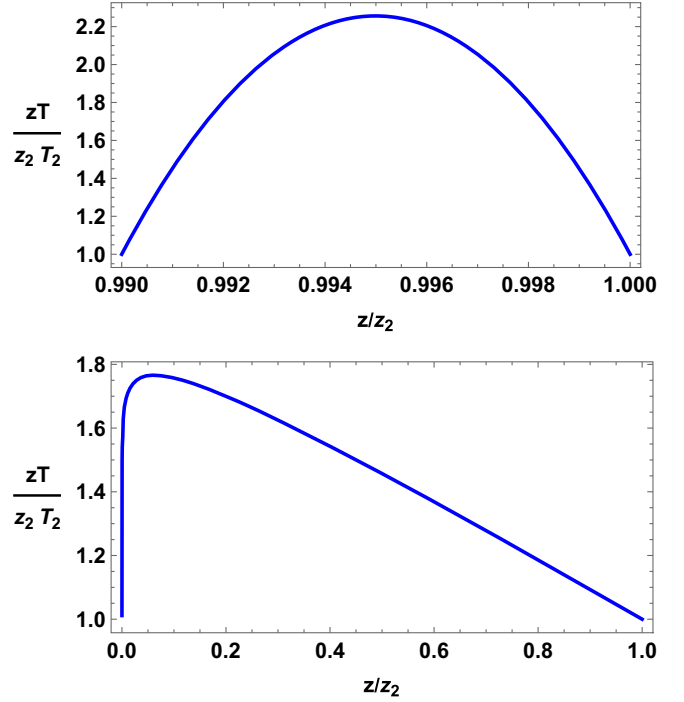


FIG. 4. Solutions of (17) with boundary conditions $z_1 T(z_1) = z_2 T(z_2)$. In the upper panel, we set $z_1/z_2 = 0.99$ and $\mathcal{J}^2/[\sigma_1 \kappa_3 T(z_2)] = 10^5$. In the lower panel, we set $z_1/z_2 = 10^{-7}$ and $\mathcal{J}^2/[\sigma_1 \kappa_3 T(z_2)] = 1$.

Application 3: Redshifted magnetic diffusion—Let us see how the magnetic diffusion equation (11) changes on our rocket. We consider a transversal electromagnetic wave whose four potential is $A = A_x(t, z)dx$. Then, the Faraday tensor is $F = \partial_t A_x dt \wedge dx + \partial_z A_x dz \wedge dx$, and the only relevant Maxwell equation is $J^x = \nabla_\mu F^{x\mu}$. Since nothing depends on x , we have $J^x = \sigma_1 \mathcal{E}^x$, and thus

$$\partial_t A_x = \frac{1}{\sigma_1} \partial_z [gz \partial_z A_x] - \frac{1}{\sigma_1 gz} \partial_z^2 A_x, \quad (18)$$

where we may neglect the term with the second time derivative. Applying ∂_z on both sides, and recalling that $\mathcal{B}^y = -\epsilon^{yzt} F_{zx} u_t = \partial_z A_x$ [48], we arrive at

$$\partial_t \mathcal{B}^y = \frac{1}{\sigma_1} \partial_z^2 (gz \mathcal{B}^y), \quad (19)$$

which is the magnetic diffusion equation we were looking for. Its stationary solutions take the form $\mathcal{B}^y = c_1 + c_2/z$, so both $\mathcal{B}^y = \text{const}$ (a uniform magnetic field) and $gz \mathcal{B}^y = \text{const}$ (a uniform redshifted magnetic field) are admissible steady states. The question is then which of these represents the *true* equilibrium, i.e., the one that does not rely on continuous dissipation to be sustained. The answer becomes evident once we observe that the solution of (18) consistent with $\mathcal{B}^y = c_1 + c_2/z$ is $A_x = c_1(z + gt/\sigma_1) + c_2 \ln z$, implying that the entropy production is

$\nabla_\mu s^\mu \propto (J_x)^2 \propto (\mathcal{E}_x)^2 \propto (\partial_t A_x)^2 \propto (c_1)^2$. This shows that keeping $\mathcal{B}^y$ uniform necessarily entails a continuous dissipation of heat, while the true diffusive equilibrium is instead the configuration in which the *redshifted* magnetic field $gz\mathcal{B}^y$ is uniform.

Application 4: Charged relativistic stars—Let us step out of our rocket, and land on a problem of astrophysical interest. In most descriptions of charged compact objects, one assumes either $\rho_{\text{lte}} = \text{const} \times \varepsilon_{\text{lte}}$ (for bulk charge), or $\rho_{\text{lte}} \sim e^{-(r-R_\star)^2/b^2}$ (for surface charge) [18–20,22,23,25]. These choices are made for convenience, since modeling the actual charge stratification would require a fully relativistic Thomas-Fermi equation for the electron cloud. Our theory produces such an equation naturally.

We work in a static, spherically symmetric background spacetime ($ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2d^2\Omega$), and we assume that the Faraday tensor is also spherically symmetric (but not necessarily static), so that $F = F_{tr}(t,r)dt \wedge dr$, which automatically fulfills $dF = 0$. Then, the nontrivial Maxwell equations are $\nabla_\mu F^{\mu\nu} = J^\nu$ and $\nabla_\mu F^{\mu\nu} = J^\nu$, which explicitly read

$$\begin{aligned} \frac{e^{-\Lambda}}{r^2} \partial_r (r^2 e^\Lambda \mathcal{E}^r) &= \rho_{\text{lte}} + \sigma_1 e^{-\Phi} \partial_t \mu + \sigma_3 e^{-\Phi} \partial_t T, \\ e^{-\Phi} \partial_t \mathcal{E}^r &= \sigma_1 [e^{-\Phi} \partial_r (e^\Phi \mu) - \mathcal{E}^r] + \sigma_3 e^{-\Phi} \partial_r (e^\Phi T). \end{aligned} \quad (20)$$

In a time-independent scenario, one can combine the two lines of (20) into a single equation:

$$\frac{e^{-\Lambda}}{r^2} \partial_r \left\{ r^2 e^{-\Lambda-\Phi} \left[\partial_r (e^\Phi \mu) + \frac{\sigma_3}{\sigma_1} \partial_r (e^\Phi T) \right] \right\} = \rho_{\text{lte}}, \quad (21)$$

which is of Thomas-Fermi type [59,60]. Gradients of $e^\Phi T$ are often present since stars cool down or accrete. The Seebeck coefficient $-\sigma_3/\sigma_1$ (or “thermopower”) has been computed from microphysics in several references [56,61–64], but the most common expression in neutron star models is [65] (n_e = electron number density)

$$\frac{\sigma_3}{\sigma_1} = - \left(\frac{\pi^4}{3n_e e^3} \right)^{1/3} T. \quad (22)$$

In the cold limit, we can neglect the Seebeck correction, and (21) becomes a differential equation for μ alone, to be solved with boundary conditions $[\partial_r \mu]_{r=0} = 0$ and $4\pi R_\star^2 [e^{-\Lambda-\Phi} \partial_r (e^\Phi \mu)]_{r=R_\star} = Q = \text{total charge}$.

In Fig. 5, we test the assumption $\rho_{\text{lte}} = \text{const} \times \varepsilon_{\text{lte}}$ against our theory. Specifically, we consider a low temperature, constant density star ($\varepsilon_{\text{lte}} = \text{const}$) with compactness $C = 0.33$, corresponding to the metric [66], Sec. 11.6,

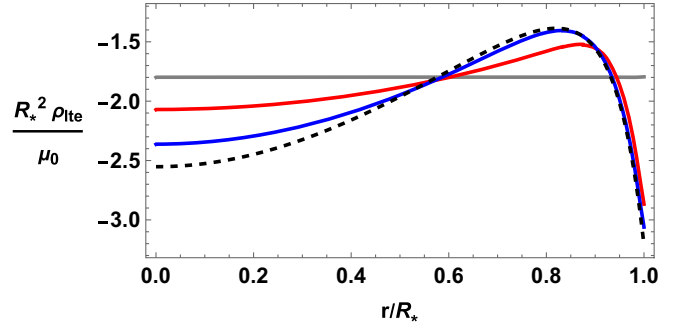


FIG. 5. Density profile of a solution of (20) with $\rho(t=0) = \text{const}$ and $\mathcal{E}_r(r=R_\star) = \text{const}$ (no charge escapes the star), both adjusted so that $Q/(R_\star \mu_0) = -10$. The curves represent snapshots at $\sigma_1 t = 0$ (gray), 1 (red), 3 (blue), and ∞ (dashed). The dashed line also solves (21).

$$\begin{aligned} e^{-\Lambda(r)} &= \sqrt{1 - 2C \frac{r^2}{R_\star^2}}, \\ e^{\Phi(r)} &= \frac{3}{2} e^{-\Lambda(R_\star)} - \frac{1}{2} e^{-\Lambda(r)}, \end{aligned} \quad (23)$$

and we “prepare” the electron cloud so that $\rho_{\text{lte}} = \text{const}$. Then, we let the system evolve according to (20), with $\sigma_1 = \text{const}$ and $\mu = -\mu_0 + b^2 \rho_{\text{lte}}$ ($\mu_0 = \text{const}$, $b = 0.1 R_\star$).

As expected, the electrons spontaneously migrate to a solution of the Thomas-Fermi equation (21). The final configuration is a balance between two competing effects: the familiar tendency of charge to accumulate near the surface, within a layer of thickness b , due to electrostatic repulsion, and the opposing tendency of electrons to concentrate at the center under the influence of gravity.

We emphasize that, since μ_0 and b are functions of the ion number density (see Supplemental Material [58]), taking them to be constant is equivalent to assuming a uniform positive-ion density throughout the star, which is consistent with the constant-density idealization. In a more realistic model, the radial profiles $\mu_0(r)$ and $b(r)$ should instead be determined by the ion density obtained from solving the Tolman-Oppenheimer-Volkoff equations for the equation of state of interest.

Conclusions—We have presented a robust framework for modeling electrothermal effects in strong gravitational fields, providing a unified description of diverse phenomena, ranging from the magnetothermal evolution of neutron stars to the electric stratification of quark stars. Our framework can be straightforwardly extended to incorporate bound charges, neutrino luminosities, and anisotropies arising from strong magnetic fields or the solid lattice. The modified equations with a strong magnetic field are provided in Supplemental Material [58].

Beyond its astrophysical relevance, the theory is of fundamental interest even in the context of inertial systems in flat spacetime. In fact, classical electrodynamics is the

archetype of a causal, special-relativistic field theory with hyperbolic equations of motion. Coupling it to an acausal parabolic matter sector (as is done in textbooks) breaks this link with relativity. Our framework restores full consistency, while still reproducing the known phenomenology of conductors in the appropriate limits.

Owing to the assumption of Born rigidity, this theory applies most naturally to solids. It can, nevertheless, be viewed as a dynamical sector of resistive magnetohydrodynamics, describing thermoelectric processes in which the fluid motion is approximated by a Killing flow.

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