

Fundamental Limit on the Power of Entanglement Assistance in Quantum Communication

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The optimal rate of reliable communication over a quantum channel can be enhanced by preshared entanglement. Whereas the enhancement may be unbounded in infinite-dimensional settings even when the input power is constrained, a long-standing conjecture asserts that the ratio between the entanglement-assisted and unassisted classical capacities is bounded in finite-dimensional settings [Bennett *et al.*, *IEEE Trans. Inf. Theory* **48**, 2637 (2002)]. In this Letter, we prove this conjecture by showing that their ratio is upper bounded by $o(d^2)$, where d is the input dimension of the channel. An application to quantum communication with noisy encoders and decoders is given.

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Introduction—The fact that preshared quantum entanglement between a sender and a receiver can lead to a boost in the amount of information that can be communicated over a quantum channel has long been a celebrated and widely studied phenomenon, which highlights the unique and novel possibilities that quantum phenomena inherently present for the development of new technology. The textbook example of this is the superdense coding protocol [1], where in a noiseless setting, preshared entanglement allows a sender to communicate two bits of classical information by only transmitting a single qubit to the receiver, a fact that has been demonstrated experimentally [2].

It turns out that, when communicating over a noisy channel, the advantage of using entanglement can become much more potent than for a noiseless channel, with even further practical implications [3]. For example, for a d -dimensional depolarizing channel, the ratio between its *entanglement-assisted classical capacity* C_E and its (ordinary) *classical capacity* C approaches $d + 1$ [4] as the depolarizing probability approaches 1, i.e., as the channel approaches the fully depolarizing channel (see Fig. 1). This

means that in the mentioned limit, the asymptotic rate at which classical bits can be reliably communicated over the channel with the help of quantum encoders and decoders will be $d + 1$ times as large if the sender and receiver have access to an unlimited amount of preshared entangled qubits. Thus, entanglement can grant an arbitrarily large boost to the classical communication rate of a sufficiently noisy channel as the input and output space dimension grow to infinity—a phenomenon with applications in, e.g., deep-space communication [5–7] and covert communication [8–10].

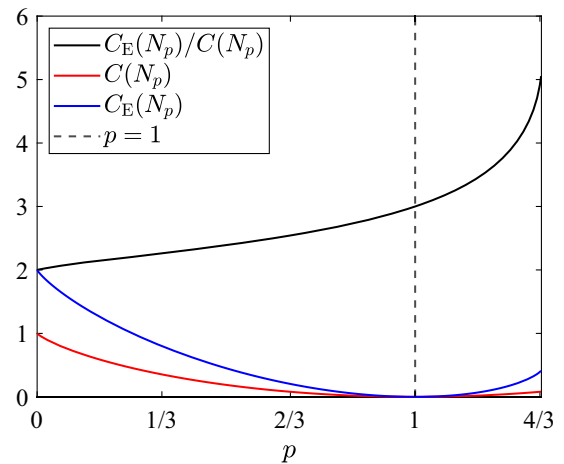


FIG. 1. The entanglement-assisted classical capacity $C_E(N_p)$, the classical capacity $C(N_p)$, and their ratio plotted as a function of p for the qubit depolarizing channel $N_p(\rho) = (1 - p)\rho + p\text{tr}(\rho)\frac{1}{2}\mathbb{I}_2$, which is a quantum channel for $p \in [0, \frac{4}{3}]$. $C_E(N_p)$ and $C(N_p)$ are calculated analytically in [11] (see also [12], Theorem 1). Note that $C_E(N_1) = C(N_1) = 0$, where their ratio converges to 3.

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For other families of quantum channels, it has similarly been observed that the ratio C_E/C is finite and bounded for finite input and output dimensions (see Ref. [4] for the erasure channel, [13] for the amplitude damping channel) and can become arbitrarily large as the input and output space dimensions are unbounded (see Ref. [14] for the lossy bosonic channel with power constraints). It is therefore natural to ask whether it is possible for this ratio to diverge for families of channels with fixed input and output dimension, or, conversely, whether the ratio is bounded by a function of input and output dimension for all quantum channels. This question was indeed asked in one of the earliest papers on entanglement assistance [13]. In this Letter, we show that the latter alternative occurs by proving that the ratio C_E/C is upper bounded by $O[d_A^2/\ln(d_A)]$, where d_A is the input dimension of the channel. This is the content of the theorem presented in the next section. We then offer a proof of the main results and discuss an application of our bound to fault-tolerant quantum communication.

A Bound on C_E/C —We consider Hilbert spaces A and B of finite dimension d_A and d_B , respectively, whose unit vectors represent the pure states of quantum systems with the same label, and, as usual, we set $AB = A \otimes B$, etc. We shall denote the set of (bounded) linear maps on A by $\mathcal{B}(A)$ and by I_A the identity operator on A . We let $\mathcal{S}(A)$ denote the compact convex set of *quantum states* (or density matrices), i.e., positive semidefinite elements of $\mathcal{B}(A)$ with trace 1. The set of pure states $\mathcal{S}_*(A)$ consists of the extremal points of $\mathcal{S}(A)$, being of the form $\rho_A = |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a unit vector in A . A *quantum channel* is a linear, completely positive, and trace preserving map $T: \mathcal{B}(A) \rightarrow \mathcal{B}(B)$. By id_A we denote the identity channel on $\mathcal{B}(A)$.

Given a quantum channel $T: \mathcal{B}(A) \rightarrow \mathcal{B}(B)$ the classical capacity $C(T)$ is defined as the maximal asymptotic rate (in bits per channel use) at which classical information can be transmitted reliably (with asymptotically vanishing probability of error) through the channel with the help of a quantum encoder and decoder [15,16]. In the absence of a single-letter formula for $C(T)$, it is useful for our purposes that

$$C(T) \geq C_H(T) \quad (1)$$

as a consequence of the Holevo-Schumacher-Westmoreland theorem [17,18], where $C_H(T)$ denotes the *Holevo quantity*, which is defined below in (3).

The entanglement-assisted classical capacity, $C_E(T)$, is defined as the maximum asymptotic rate of such reliable information transmission with the help of an unlimited supply of preshared entanglement between sender and receiver. Contrary to $C(T)$, this quantity can be expressed in a concise way in terms of relative entropies as shown in [13]; see (4) below (see Ref. [15], Chapter 8) for a

thorough discussion of various types of capacities of quantum channels).

Our main result is the following bound on the ratio $C_E(T)/C(T)$.

Theorem—Let $T: \mathcal{B}(A) \rightarrow \mathcal{B}(B)$ be an arbitrary quantum channel. Then,

$$C_E(T) \leq \frac{(4d_A - 3)(2d_A - \frac{5}{2})^2}{(2d_A - \frac{3}{2}) \ln(2d_A - \frac{3}{2}) - 2d_A + \frac{5}{2}} C(T).$$

For large d_A , the prefactor on the right-hand side scales as $[8d_A^2/\ln(d_A)][1 + o(1)]$. From the explicit capacity expressions from [11], we can obtain an achievable prefactor that scales as $2d_A[1 + o(1)]$ for large d_A and any uniform upper bound on the ratio must be lower-bounded by that expression.

Proof of the main result—

Definitions: The relative entropy of two states ρ, τ on the same space is defined as

$$D(\rho\|\tau) = \begin{cases} \text{tr}[\rho \ln \rho - \rho \ln \tau] & \text{if } \ker \tau \subseteq \ker \rho \\ \infty & \text{otherwise} \end{cases}. \quad (2)$$

We use the natural logarithm (instead of the conventional binary logarithm) in order to simplify the presentation of subsequent relative entropy bounds, but note that the main result is independent of this choice.

Given a channel $T: \mathcal{B}(A) \rightarrow \mathcal{B}(B)$, its Holevo quantity, $C_H(T)$, can be expressed as [19–21]

$$C_H(T) = \frac{1}{\ln 2} \inf_{\sigma_B} \sup_{\rho_A} D(T(\rho_A)\|\sigma_B), \quad (3)$$

where ρ_A and σ_B denote arbitrary quantum states on $\mathcal{S}(A)$ and $\mathcal{S}(B)$, respectively. Without loss of generality, σ_B can be restricted to the channel image [19].

The entanglement-assisted capacity of T is defined in [13] and shown to equal

$$C_E(T) = \frac{1}{\ln 2} \sup_{\rho_{A'A}} D((\text{id}_{A'} \otimes T)\rho_{A'A}\|\rho_{A'} \otimes T(\rho_A)), \quad (4)$$

where A' is an arbitrary auxiliary system with $d_{A'} = d_A$, $\rho_{A'A}$ is optimized over all pure states in $\mathcal{S}_*(A'A)$, and we use the standard notation $\rho_A = \text{tr}_{A'}[\rho_{A'A}]$, where $\text{tr}_{A'}$ denotes the partial trace with respect to A' . The relative entropy expression corresponds to the mutual information of the channel T . We further note that

$$\begin{aligned} D((\text{id}_{A'} \otimes T)\rho_{A'A}\|\rho_{A'} \otimes T(\rho_A)) \\ \leq D((\text{id}_{A'} \otimes T)\rho_{A'A}\|\rho_{A'} \otimes \sigma_B) \end{aligned} \quad (5)$$

for any state σ_B . In order to obtain a useful upper bound on C_E , we will make a judicious choice of $\sigma_B(\rho_{A'A})$, dependent

on $\rho_{A'A}$, in (5) and apply some basic estimates of $D(\cdot|\cdot)$ that we next proceed to establish.

Relative entropy estimates: When $\phi \in \mathcal{B}(A)$ is positive definite, we denote by K_ϕ the quadratic form, defined on the self-adjoint part of $\mathcal{B}(A)$, corresponding to $\nabla \ln \phi$, i.e.,

$$K_\phi(\eta) = (\eta|\nabla_\eta \ln \phi) = \int_0^\infty dx \operatorname{tr}[\eta(\phi + x\mathbf{I}_A)^{-1}]^2, \quad (6)$$

where $(\cdot|\cdot)$ denotes the Hilbert-Schmidt inner product. See, e.g., the Appendix for elaborations on the equality above.

In the proof of the main theorem, we shall make use of the following estimates for $D(\rho|\tau)$ in terms of $K_\tau(\rho - \tau)$. (i) For arbitrary $\rho, \tau \in \mathcal{S}(A)$, where τ is positive definite, it holds that

$$D(\rho|\tau) \leq K_\tau(\rho - \tau). \quad (7)$$

(ii) For arbitrary $\rho, \tau \in \mathcal{S}(A)$, where τ is positive definite such that $k\tau \geq \rho$ for a constant $k \geq 1$, it holds that

$$D(\rho|\tau) \geq g(k)K_\tau(\rho - \tau) \quad (8)$$

where

$$g(k) = \frac{k \ln k - k + 1}{(k - 1)^2}.$$

The inequality in (i) is a well known consequence of the concavity of the logarithm; see, e.g., [22]. For full-rank ρ , the proof of both estimates can be found in [[23], Lemma 2.2]. For the benefit of the reader, we provide proofs of both (i) and (ii) in the Appendix.

For later reference, we note that, if the eigenvalues of the state τ are $\mu_1 \leq \dots \leq \mu_{d_A}$ with a corresponding orthonormal basis of eigenvectors $|h_1\rangle, \dots, |h_{d_A}\rangle$, then we have by (6) that

$$\begin{aligned} K_\tau(\eta) &= \int_0^\infty dx \sum_{k,l=1}^{d_A} |\langle h_k|\eta|h_l\rangle|^2 (x + \mu_k)^{-1} (x + \mu_l)^{-1} \\ &= \sum_{k,l=1}^{d_A} |\langle h_k|\eta|h_l\rangle|^2 \frac{\ln \mu_k - \ln \mu_l}{\mu_k - \mu_l} \end{aligned} \quad (9)$$

for any self-adjoint $\eta \in \mathcal{B}(A)$, where the expression $[(\ln \mu_k - \ln \mu_l)/(\mu_k - \mu_l)]$ is to be set equal to μ_k^{-1} in case $\mu_k = \mu_l$.

If τ is not positive definite, i.e., if $\mu_k = 0$ for some k with notation as above, we may replace τ by $\tau + \epsilon \mathbf{I}_A$, $\epsilon > 0$, in the previous arguments and observe that

$$D(\rho|\tau) = \lim_{\epsilon \rightarrow 0} [\rho \ln \rho - \rho \ln(\tau + \epsilon \mathbf{I}_A)]$$

to conclude that the inequalities (7) and (8) still hold, with K_τ given by (9) provided the conventions

$$\begin{aligned} |\langle h_k|\eta|h_l\rangle|^2 \frac{\ln \mu_k - \ln \mu_l}{\mu_k - \mu_l} &= \begin{cases} \infty & \text{if } \langle h_k|\eta|h_l\rangle \neq 0 \text{ and } \mu_k = 0 \text{ or } \mu_l = 0 \\ 0 & \text{if } \langle h_k|\eta|h_l\rangle = 0 \text{ and } \mu_k = 0 \text{ or } \mu_l = 0 \end{cases} \end{aligned}$$

are used. Thus, we can restrict the sum in (9) to include only pairs of nonzero eigenvalues of τ provided $\ker(\tau) \subseteq \ker(\rho)$ and still use the inequalities (7) and (8) in this case. This will be done without further notice in the remainder of this Letter.

Proof of the bound on C_E/C : As both sides of the stated inequality vanish if $d_A = 1$, we can assume in the following that $d_A \geq 2$.

Recalling (5) and (3), our goal is to establish an upper bound on $D((\operatorname{id}_{A'} \otimes T)\rho_{A'A} \| \rho_{A'} \otimes \sigma_B(\rho_{A'A}))$ for all $\rho_{A'A} \in \mathcal{S}_*(A'A)$ and suitable $\sigma_B(\rho_{A'A})$ in terms of $D(T(\rho'_A) \| \sigma'_B)$, $\sigma'_B \in \mathcal{S}(B)$, for some $\rho'_A \in \mathcal{S}(A)$. $\rho_{A'A}$ will by assumption be pure, i.e., $\rho_{A'A} = |v\rangle\langle v|$ for some $|v\rangle \in \mathcal{H}_{A'A}$, $\langle v|v\rangle = 1$. Given a state σ_B , let the replacement channel $R_{\sigma_B}: \mathcal{B}(A) \rightarrow \mathcal{B}(B)$ be defined by

$$R_{\sigma_B}(\phi) = \operatorname{tr}(\phi)\sigma_B, \quad \phi \in \mathcal{B}(A).$$

Then $(\operatorname{id}_{A'} \otimes R_{\sigma_B})(\rho_{A'A}) = \rho_A \otimes \sigma_B$ for any $\rho_{A'A} \in \mathcal{S}(A'A)$. Introducing the notation $\Delta T = T - R_{\sigma_B}$, we have $\Delta T(\rho_A) = T(\rho_A) - \sigma_B$ and

$$(\operatorname{id}_{A'} \otimes \Delta T)(\rho_{A'A}) = (\operatorname{id}_{A'} \otimes T)(\rho_{A'A}) - \rho_{A'} \otimes \sigma_B$$

for any $\rho_{A'A} \in \mathcal{S}(A'A)$. By (i), we hence have

$$\begin{aligned} D((\operatorname{id}_{A'} \otimes T)(\rho_{A'A}) \| \rho_{A'} \otimes \sigma_B) \\ \leq K_{\rho_{A'} \otimes \sigma_B}((\operatorname{id}_{A'} \otimes \Delta T)(\rho_{A'A})). \end{aligned} \quad (10)$$

We next proceed to estimate the last expression in terms of quantities of the form $K_{\sigma_B}[\Delta T(\rho'_A)]$ for suitable pure states $\rho'_A \in \mathcal{S}_*(A)$. As is well known, we can write $|v\rangle$ in the form $|v\rangle = \sum_{k=1}^{d_A} \alpha_k |e_k\rangle \otimes |f_k\rangle$, where $|e_1\rangle, \dots, |e_{d_A}\rangle$ and $|f_1\rangle, \dots, |f_{d_A}\rangle$ are orthonormal bases for A' and A , respectively, and $\alpha_1 \geq \dots \geq \alpha_{d_A}$ are non-negative real numbers whose squares sum to 1. We then have

$$\rho_{A'A} = \sum_{k,l=1}^{d_A} \alpha_k \alpha_l |e_k\rangle\langle e_l| \otimes |f_k\rangle\langle f_l|, \quad (11)$$

$$(\operatorname{id}_{A'} \otimes \Delta T)(\rho_{A'A}) = \sum_{k,l=1}^{d_A} \alpha_k \alpha_l |e_k\rangle\langle e_l| \otimes \Delta T(|f_k\rangle\langle f_l|). \quad (12)$$

Moreover, $\rho_{A'} = \sum_{k=1}^{d_A} \alpha_k^2 |e_k\rangle\langle e_k|$, which implies

$$\rho_{A'} \otimes \sigma_B = \sum_{k=1}^{d_A} \sum_{r=1}^{d_B} \alpha_k^2 \mu_r |e_k\rangle\langle e_k| \otimes |g_r\rangle\langle g_r|,$$

where the eigenvalues of σ_B have been denoted by $\mu_1 \geq \dots \geq \mu_{d_B} \geq 0$ with a corresponding orthonormal basis of eigenvectors $|g_1\rangle, \dots, |g_{d_B}\rangle$. Using this diagonal form of $\rho_{A'} \otimes \sigma_B$ in (9) with the system A replaced by $A'B$ and $\tau = \rho_{A'} \otimes \sigma_B$ and $\eta = (\text{id}_{A'} \otimes \Delta T)(\rho_{A'A})$ given by (12), we get

$$\begin{aligned} K_{\rho_{A'} \otimes \sigma_B}[(\text{id}_{A'} \otimes \Delta T)(\rho_{A'A})] &= \sum_{k,l,r,s} \frac{\ln \alpha_k^2 \mu_r - \ln \alpha_l^2 \mu_s}{\alpha_k^2 \mu_r - \alpha_l^2 \mu_s} \alpha_k^2 \alpha_l^2 |\langle g_r | \Delta T(|f_k\rangle\langle f_l|) | g_s \rangle|^2 \\ &\leq \sum_{k,l,r,s} \frac{\ln \mu_r - \ln \mu_s}{\mu_r - \mu_s} \max\{\alpha_k^2, \alpha_l^2\} |\langle g_r | \Delta T(|f_k\rangle\langle f_l|) | g_s \rangle|^2, \end{aligned} \quad (13)$$

where the inequality follows by using that $\ln$ is a concave function and hence the slope $[(\ln x - \ln y)/(x - y)]$ of the secant connecting the points $(x, \ln x)$ and $(y, \ln y)$ is a decreasing function of both x and y .

The states $\chi_{k,l}^a := \frac{1}{2}(|f_k\rangle + i^a |f_l\rangle)(\langle f_k| + (-i)^a \langle f_l|)$, $k \neq l$ and $a = 0, 1, 2, 3$, are pure and are seen to satisfy the identities $|f_k\rangle\langle f_l| + |f_l\rangle\langle f_k| = \chi_{k,l}^0 - \chi_{k,l}^2$ and $|f_k\rangle\langle f_l| - |f_l\rangle\langle f_k| = i(\chi_{k,l}^1 - \chi_{k,l}^3)$. By linearity of ΔT , these imply

$$\begin{aligned} &|\langle g_r | \Delta T(|f_k\rangle\langle f_l|) | g_s \rangle|^2 + |\langle g_r | \Delta T(|f_l\rangle\langle f_k|) | g_s \rangle|^2 \\ &\leq \sum_{a=0}^3 |\langle g_r | \Delta T(\chi_{k,l}^a) | g_s \rangle|^2. \end{aligned}$$

Using this estimate in (13) and recalling (9) we find that

$$\begin{aligned} K_{\rho_{A'} \otimes \sigma_B}[(\text{id}_{A'} \otimes \Delta T)(\rho_{A'A})] &\leq \sum_k \alpha_k^2 K_{\sigma_B}[\Delta T(|f_k\rangle\langle f_k|)] \\ &\quad + \frac{1}{2} \sum_{a=0}^3 \sum_{k \neq l} \max\{\alpha_k^2, \alpha_l^2\} \\ &\quad \times K_{\sigma_B}[\Delta T(\chi_{k,l}^a)]. \end{aligned} \quad (14)$$

We shall choose σ_B below such that there exists a $k_A \geq 1$, depending only on d_A , fulfilling

$$k_A \sigma_B \geq T(\chi_{k,l}^a) \quad \text{and} \quad k_A \sigma_B \geq T(|f_k\rangle\langle f_k|) \quad (15)$$

for all a and $k \neq l$. Hence, by applying (ii) to (14) and using $\max\{\alpha_k^2, \alpha_l^2\} \leq \alpha_k^2 + \alpha_l^2$, it follows that

$$\begin{aligned} &K_{\rho_{A'} \otimes \sigma_B}[(\text{id}_{A'} \otimes \Delta T)(\rho_{A'A})] \\ &\leq g(k_A)^{-1} \left(\sum_k \alpha_k^2 D[T(|f_k\rangle\langle f_k|) \| \sigma_B] \right. \\ &\quad \left. + \frac{1}{2} \sum_{a=0}^3 \sum_{k \neq l} (\alpha_k^2 + \alpha_l^2) D[T(\chi_{k,l}^a) \| \sigma_B] \right), \end{aligned} \quad (16)$$

holds for all $\rho_{A'A}$ given by (11) and any state σ_B fulfilling (15).

The expression on the right-hand side of this inequality is a linear combination of relative entropies with positive coefficients that can be further estimated by applying the fact that, for any given convex combination $\rho = \sum_{i=1}^n p_i \rho_i$ of states $\rho_1, \dots, \rho_n \in \mathcal{S}(B)$, the quantity $\sum_{i=1}^n p_i D(\rho_i \| \sigma)$ as a function of $\sigma \in \mathcal{S}(B)$ assumes a unique minimum at $\sigma = \rho$. This follows immediately from the identity [24]

$$\sum_{i=1}^n p_i D(\rho_i \| \sigma) = \sum_{i=1}^n p_i D(\rho_i \| \rho) + D(\rho \| \sigma)$$

that is straightforward to verify, and often called *Donald's identity*. Choosing $\sigma_B(\rho_{A'A})$ so as to minimize the right-hand side of (16), we thus have

$$\begin{aligned} M_A \sigma_B(\rho_{A'A}) &= \sum_k \alpha_k^2 T(|f_k\rangle\langle f_k|) + \frac{1}{2} \sum_{a=0}^3 \sum_{k \neq l} (\alpha_k^2 + \alpha_l^2) T(\chi_{k,l}^a) \\ &= \sum_k \alpha_k^2 T(|f_k\rangle\langle f_k|) + 2 \sum_{k \neq l} (\alpha_k^2 + \alpha_l^2) T(|f_k\rangle\langle f_k|), \end{aligned} \quad (17)$$

where we have used that $\chi_{k,l}^0 + \chi_{k,l}^2 = \chi_{k,l}^1 + \chi_{k,l}^3 = |f_k\rangle\langle f_k| + |f_l\rangle\langle f_l|$, and M_A is a normalization factor given by

$$M_A = \sum_k \alpha_k^2 + 2 \sum_{k \neq l} (\alpha_k^2 + \alpha_l^2) = 4d_A - 3,$$

where the last equality follows by recalling that $\alpha_1^2 + \dots + \alpha_{d_A}^2 = 1$. Moreover, from (17) we get that

$$\begin{aligned} M_A \sigma_B(\rho_{A'A}) &= 2 \sum_k T(|f_k\rangle\langle f_k|) + (2d_A - 3) \sum_k \alpha_k^2 T(|f_k\rangle\langle f_k|) \\ &\geq 2T(I_A), \end{aligned}$$

from which it follows that (15) holds for $\sigma_B(\rho_{A'A})$ with $k_A = \frac{1}{2} M_A = 2d_A - \frac{3}{2}$, as $T(I_A) \geq T(\rho_A)$ for any state ρ_A . In conjunction with the definition of $\sigma_B(\rho_{A'A})$ this shows by (16) that

$$\begin{aligned}
 & K_{\rho_{A'} \otimes \sigma_B(\rho_{A'A})}[(\text{id}_{A'} \otimes \Delta T)(\rho_{A'A})] \\
 & \leq \frac{M_A}{g(\frac{1}{2}M_A)} \left(\sum_k p_k D(T(|f_k\rangle\langle f_k|) \| \sigma_B(\rho_{A'A})) \right. \\
 & \quad \left. + \sum_{a=0}^3 \sum_{k \neq l} p_{k,l}^a D(T(\chi_{k,l}^a) \| \sigma_B(\rho_{A'A})) \right) \\
 & \leq \frac{M_A}{g(\frac{1}{2}M_A)} \left(\sum_k p_k D(T(|f_k\rangle\langle f_k|) \| \tau_B) \right. \\
 & \quad \left. + \sum_{a=0}^3 \sum_{k \neq l} p_{k,l}^a D(T(\chi_{k,l}^a) \| \tau_B) \right)
 \end{aligned}$$

for all states $\tau_B \in \mathcal{S}(B)$, where we introduced the probability distribution $\{p_k\}_k \cup \{p_{k,l}^a\}_{k \neq l}^a$ by normalizing the α -dependent coefficients with M_A . Estimating $D(T(\chi_{k,l}^a) \| \tau_B)$ by $\sup_{\omega_A \in \mathcal{S}(A)} D[T(\omega_A) \| \tau_B]$, we obtain

$$\begin{aligned}
 & K_{\rho_{A'} \otimes \sigma_B(\rho_{A'A})}[(\text{id}_{A'} \otimes \Delta T)(\rho_{A'A})] \\
 & \leq \frac{M_A}{g(\frac{1}{2}M_A)} \sup_{\omega_A \in \mathcal{S}(A)} D(T(\omega_A) \| \tau_B). \quad (18)
 \end{aligned}$$

Finally, recalling the definition (3) of $C_H(T)$ and using (10), we conclude by taking the infimum over $\tau_B \in \mathcal{S}(B)$ on the right-hand side of (18) that

$$D((\text{id}_{A'} \otimes T)(\rho_{A'A}) \| \rho_{A'} \otimes \sigma_B(\rho_{A'A})) \leq \frac{M_A \ln 2}{g(\frac{1}{2}M_A)} C_H(T)$$

holds for all $\rho_{A'A} \in \mathcal{S}_*(A'A)$. By evaluating $g(\frac{1}{2}M_A)$ this proves the theorem as a consequence of (4), (5), and (1).

Note that $C(T)$ can be replaced with $C_H(T)$ in the theorem.

Conclusion—In realistic quantum communication, the encoders and decoders of sender and receiver are subject gate-level noise. Considering a fixed level of gate noise $p > 0$ independent of the block length of the codes leads to new notions of capacity, in particular raising the question whether the traditional capacity is approached for small p [25].

In this context, the bound

$$C_E(T) \geq C_E(T, p) \geq C_E(T) - f(p) \frac{C_E(T)}{C(T)},$$

has been obtained for the fault-tolerant entanglement-assisted capacity $C_E(T, p)$ [26] (see also [27]). The function $f(p)$ goes to zero as p goes to zero and depends only on the dimension of the channel. This continuity bound is therefore *a priori* channel-dependent and it was left open in [26] whether a bound uniform for channels in a given dimension can be obtained. The uniform upper

bound on $C_E(T)/C(T)$ presented in this Letter now implies such a uniform bound.

A question for future investigations is whether one can improve our $O(d_A^2/\ln d_A)$ bound to $O(d_A)$, which would be tight in view of the existing lower bounds.

We also leave open whether there exists a bound on the ratio of capacities that only depends on the output dimension, or conversely, whether there exists a series of quantum channels with fixed output dimension (and necessarily, as a consequence of our Letter, with increasing input dimension) for which the ratio diverges.

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End Matter

Appendix: Proof of relative entropy inequalities— Before proving the relative entropy estimates (i) and (ii), we shall derive a useful integral representation for the relative entropy $D(\rho\|\tau)$, relating it to the quadratic form K_ϕ given in (6).

Making use of the well known integral representation of the matrix logarithm

$$\ln \phi = \int_0^\infty dx ((1+x)^{-1} \mathbf{I}_A - (\phi + x \mathbf{I}_A)^{-1}),$$

together with the resolvent identity

$$\begin{aligned} (\phi + x \mathbf{I}_A)^{-1} - (\psi + x \mathbf{I}_A)^{-1} \\ = (\phi + x \mathbf{I}_A)^{-1} (\psi - \phi) (\psi + x \mathbf{I}_A)^{-1}, \end{aligned}$$

it follows that

$$\ln \psi - \ln \phi = \int_0^\infty dx (\phi + x \mathbf{I}_A)^{-1} (\psi - \phi) (\psi + x \mathbf{I}_A)^{-1} \quad (\text{A1})$$

for any positive definite operators ϕ, ψ in $\mathcal{B}(A)$. Setting $\psi = \phi + t \cdot \eta$, $t \in \mathbb{R}$, in (A1) and differentiating with respect to t at $t = 0$, it follows that the derivative of $\ln$ at ϕ in the direction $\eta \in \mathcal{B}(A)$ is given by

$$\nabla_\eta \ln(\phi) = \int_0^\infty dx (\phi + x \mathbf{I}_A)^{-1} \eta (\phi + x \mathbf{I}_A)^{-1}.$$

Thus, given any smooth curve ρ_t , $0 \leq t \leq 1$, of positive definite states in $\mathcal{S}(A)$ such that $\rho_0 = \tau$ and $\rho_1 = \rho$, we have

$$\ln \rho - \ln \tau = \int_0^1 dt \int_0^\infty dx (\rho_t + x \mathbf{I}_A)^{-1} \dot{\rho}_t (\rho_t + x \mathbf{I}_A)^{-1},$$

where $\dot{\rho}_t$ denotes the derivative of ρ_t . In particular, setting $\rho_t = t\rho + (1-t)\tau$, we have $\dot{\rho}_t = \rho - \tau$, and using this expression in the definition of $D(\rho\|\tau)$ we obtain

$$D(\rho\|\tau) = \int_0^1 dt \int_0^\infty dx \operatorname{tr} \rho (\rho_t + x \mathbf{I}_A)^{-1} (\rho - \tau) (\rho_t + x \mathbf{I}_A)^{-1}.$$

Writing the first factor of the integrand as $\rho = (1-t)(\rho - \tau) + \rho_t$ and using

$$\begin{aligned} & \int_0^1 dt \int_0^\infty dx \operatorname{tr} \rho_t (\rho_t + x \mathbf{I}_A)^{-1} (\rho - \tau) (\rho_t + x \mathbf{I}_A)^{-1} \\ &= \int_0^1 dt \int_0^\infty dx \operatorname{tr} \rho_t (\rho_t + x \mathbf{I}_A)^{-2} (\rho - \tau) \\ &= \int_0^1 dt \left[-\operatorname{tr} \rho_t (\rho_t + x \mathbf{I}_A)^{-1} (\rho - \tau) \right]_{x=0}^{x=\infty} \\ &= \int_0^1 dt \operatorname{tr} \rho_t \rho_t^{-1} (\rho - \tau) = 0, \end{aligned}$$

since $\operatorname{tr} \rho = \operatorname{tr} \tau = 1$, it follows that

$$D(\rho\|\tau) = \int_0^1 dt(1-t) \int_0^\infty dx \operatorname{tr}[(\rho - \tau)(\rho_t + xI_A)^{-1}]^2. \quad (\text{A2})$$

Making use of the quadratic form K_ϕ defined in (6), the integral representation (A2) can be written as

$$D(\rho\|\tau) = \int_0^1 dt(1-t) K_{\rho_t}(\rho - \tau). \quad (\text{A3})$$

We further note that K_ϕ is monotonically decreasing in ϕ in the sense that

$$K_\phi(\eta) \geq K_\psi(\eta) \quad \text{if } \phi \leq \psi, \quad (\text{A4})$$

for arbitrary positive definite operators ϕ, ψ and any self-adjoint operator η in $\mathcal{B}(A)$. The proof of this property makes use of the well known fact that τ^{-1} is a decreasing operator function of $\tau > 0$ and that $\eta\tau\eta \geq 0$, whenever $\tau \geq 0$ and η is self-adjoint. Using positivity and cyclicity of the trace, the latter property implies, for arbitrary non-negative operators τ and ζ , that

$$\operatorname{tr}\zeta\tau = \operatorname{tr}\tau^{\frac{1}{2}}\zeta\tau^{\frac{1}{2}} \geq 0,$$

and hence $\operatorname{tr}\zeta\tau$ is an increasing function of both $\tau \geq 0$ and $\zeta \geq 0$. Using this fact, we get, for $0 \leq \phi \leq \psi$ and $x > 0$,

$$\begin{aligned} & \operatorname{tr}\eta(\phi + xI_A)^{-1}\eta(\phi + xI_A)^{-1} \\ & \geq \operatorname{tr}\eta(\psi + xI_A)^{-1}\eta(\psi + xI_A)^{-1}, \end{aligned}$$

which together with (6) proves the claimed monotonicity.

We are now ready to prove the relative entropy estimates (i) and (ii), which are reproduced below. (i) For arbitrary $\rho, \tau \in \mathcal{S}(A)$, where τ is positive definite, it holds that

$$D(\rho\|\tau) \leq K_\tau(\rho - \tau).$$

(ii) For arbitrary $\rho, \tau \in \mathcal{S}(A)$, where τ is positive definite such that $k\tau \geq \rho$ for a constant $k \geq 1$, it holds that

$$D(\rho\|\tau) \geq g(k)K_\tau(\rho - \tau)$$

where $g(k) = [(k \ln k - k + 1)/(k - 1)^2]$.

Proof of (i) and (ii)—Suppose first that ρ is positive definite. Statement (i) then follows by using that $\rho_t \geq (1-t)\tau$ and hence $K_{\rho_t}(\eta) \leq K_{(1-t)\tau}(\eta)$ by (A4) which by use of (A3) gives

$$\begin{aligned} D(\rho\|\tau) & \leq \int_0^1 (1-t) dt \int_0^\infty dx \operatorname{tr}[(\rho - \tau)((1-t)\tau + xI_A)^{-1}]^2 \\ & = K_\tau(\rho - \tau), \end{aligned}$$

where the last equality follows by a change of variable $x = (1-t)y$, whereby the t dependence of the integrand disappears.

Statement (ii) follows similarly by using $\rho_t \leq (kt + 1 - t)\tau$. A change of variable $x = (kt + 1 - t)y$ then gives the claimed lower bound with

$$g(k) = \int_0^1 dt \frac{1-t}{kt + 1 - t} = \frac{k \ln k - k + 1}{(k - 1)^2}.$$

In case ρ has zero eigenvalues, the inequalities follow as both sides are continuous functions of ρ . ■