

Multitwist Trajectories and Decoupling Zeros in Conformal Field Theory

Alexandre Homrich^{1,2}, David Simmons-Duffin³, and Pedro Vieira^{2,4}

¹*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*

²*Perimeter Institute for Theoretical Physics, 31 Caroline Street, North Waterloo, Ontario N2L 2Y5, Canada*

³*Walter Burke Institute for Theoretical Physics, Caltech, Pasadena, California 91125, USA*

⁴*Instituto de Física Teórica, UNESP, ICTP South American Institute for Fundamental Research, Rua Dr Bento Teobaldo Ferraz 271, 01140-070, São Paulo, Brazil*



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Conformal Regge theory predicts the existence of analytically continued conformal field theory data for complex spin. How could this work when there are so many more operators with large spin compared to small spin? Using planar $N = 4$ SYM as a test ground, we find a simple physical picture. Operators do organize themselves into analytic families but the continuation of the higher families have zeros in their structure operator product expansion constants for lower integer spins. They thus decouple. Newton's interpolation series technique is perfectly suited to this physical problem and will allow us to explore the complex spin half-plane.

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Introduction—Certain conformal field theory (CFT) data can be analytically continued to complex values of spin S [1]. Specifically, for each four-point function $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$, the Lorentzian inversion formula produces functions $c_{1234}^{\pm}(\Delta, S)$ with the following properties. For integer S , they have simple poles at the locations of local spin- S operators $\mathcal{O}_i$ appearing in the operator product expansion (OPE), $\mathcal{O}_1 \mathcal{O}_2 \rightarrow \mathcal{O}_3 \mathcal{O}_4$:

$$c_{1234}^{(-1)^S}(\Delta, S) \sim \sum_i -\frac{f_{12i} f_{34i}}{\Delta - \Delta_i} \quad (S \in \mathbb{Z}), \quad (1)$$

where f_{12i}, f_{34i} are OPE coefficients. Furthermore, the $c_{1234}^{\pm}(\Delta, S)$ are analytic in S and bounded for $\text{Re}(S) > 1$, when $\Delta = (d/2) + i\nu$ is on the principal series. The latter condition is crucially tied to the boundedness of correlators in the Regge limit and conformal Regge theory [1,2].

Because the same local operators appear in every OPE (modulo global symmetries), different $c(\Delta, J)$ functions associated to different four-point functions have identical pole locations at integer S . We call this “local operator universality.”

What happens at noninteger S is less clear. In [3], it was argued that singularities in Δ at noninteger S should be interpreted as matrix elements of light-ray operators. However, little is known about the structure of such singularities. Is there “light-ray operator universality,”

where the same light-ray operators appear in different $c(\Delta, S)$ functions, or do different four-point functions see completely different light-ray operators?

Perhaps the simplest hypothesis is that there is light-ray operator universality—that is, local operators live on discretely-spaced Regge trajectories, whose dimensions $\Delta(S)$ and structure constants $C(S)$ are analytic in S [4].

However, this idea raises an obvious puzzle: There are usually many more local operators with large spin S than with small spin S . We illustrate this in Fig. 1, where we plot the leading-order dimensions and structure constants of twist-three operators in planar $\mathcal{N} = 4$ SYM with even spin (up to spin 60). These operators do seem to be nicely organized into smooth trajectories—or families. This is consistent with analyticity in the spin mentioned above. On the other hand, the contribution of the various families to

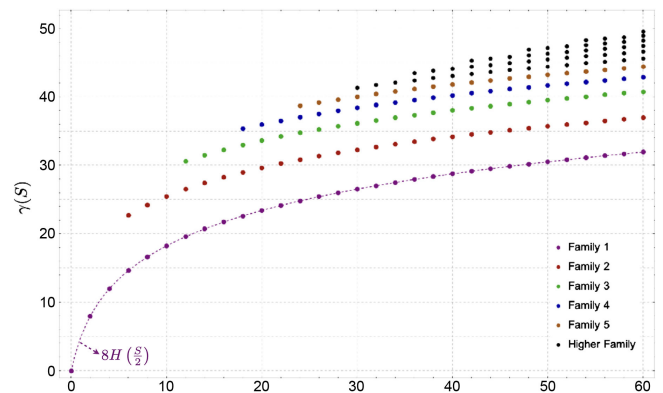


FIG. 1. Leading order anomalous dimensions $\gamma(S)$ where $\Delta(S) = 3 + S + g^2 \gamma(S)$ for the twist-three operators of even spin, with $g^2 = g_{\text{YM}}^2 N_c / 16\pi^2$.

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physical observables is ostensibly nonanalytic, given the disappearance of physical operators at small spin. How can these two facts be reconciled?

There is a simple possible solution to this puzzle: Perhaps the analytically continued structure constants $C(S)$ of a higher family which starts at some large physical S_* will have zeros at all integer spins below S_* , where a local operator is “missing.” In other words, an infinite number of Regge trajectories of light-ray operators exists for complex S , all but finitely many of which have zeros at integer spins [7].

In this Letter we confirm this picture. Using an extrapolation technique found in Newton’s 1687 *Principia* we will continue the data at integer spins into the complex plane. What we will see can only be described as dark magic: these Newton series extrapolations will beautifully converge to functions with precisely these zeros at the locations of missing operators.

The twist-three data—We consider the one-loop anomalous dimensions $\gamma(S)$ for all primary operators of the form

$$\text{Tr}(D_+^S Z^3) + \text{permutations} \quad (2)$$

with even [9] spin S . These operators have three units of R charge and classical twist equal to 3.

The family of operators (2) will be a central player in this note. These operators are the *second* simplest spinning operators in this conformal gauge theory. The simplest would be the twist-two operators—simply replace Z^3 by Z^2 in (2)—which are however too simple for our purpose since there is a single lonely primary twist-two operator for each (even) spin S . The twist-two family is still very interesting, as reviewed in Sec. VI of the Supplemental Material [10].

At weak coupling, the primary operators (2) can be thought of as eigenstates of an $\text{SL}(2, R)$ quantum spin chain of length 3 with S excitations, the conformal dimension of the operators corresponding to the energy levels of the quantum spin chain [19,20]. This energy, in turn, can be determined by solving a simple electrostatic problem of S charges in a line with real positions u_j given by

$$\sum_{j \neq k} f(u_j - u_k) + F(u_j) - 2\pi n_j = 0, \quad (3)$$

where the *external force* $F(u) = 6 \arctan(2u)$, the *interaction forces* $f = 2 \arctan(u)$, and the mode numbers [21] n_j are distinct integers taking all values from $-S/2$ to $S/2$, skipping one value which we call n_* . The twist-three families depend on this missing mode number n_* also denoted as *the hole*. The lowest (dimension) family has $n_* = 0$ [22]. All other families are degenerate pairs with $n_* = \pm 1, \pm 2, \dots$

Solving the electrostatic equations (3) is trivial: *Mathematica*’s built-in FindRoot does the job. Once we find a $\{u_1, \dots, u_S\}$ we extract the energy—that is the one-loop

anomalous dimension of the corresponding operator—by adding up the energies of each particle

$$\gamma(S) \equiv \sum_{j=1}^S \frac{2}{u_j^2 + 1/4}. \quad (4)$$

This is how Fig. 1 was generated. Similarly, the structure constant $C(S)$ is also given by an expression in terms of the u_j . The explicit plug-in expressions are simple but not as simple as the spectrum expression: $C(S)$ is given by a ratio of determinants of matrices of size S built out of these u_j or equivalently in terms of integrals of so-called Baxter polynomials $Q(x) = \prod_{j=1}^S (x - u_j)$, as reviewed in the Supplemental Material [10]; see Sec. I there.

Of course, this integrability description of the conformal data is tailored for physical operators: It is hard to put a noninteger number of particles on a line or make sense of determinants of matrices with nonintegers sizes.

This concludes the description of how we obtained all the physical integer spin twist-three data. In the Supplemental Material [10] we discuss a few analytic things we know about it. We now need to go to the complex spin plane. Enter Newton.

Newton’s magic—Given the integer spin CFT data, we want to go to the complex plane. Normally this is not a well-posed problem; we can always add a bunch of trigonometric functions that vanish at the integers to any possible analytic continuation.

Here, however, physics come to the rescue to render the continuation unique. Conformal Regge theory ensures that the $c(\Delta, S)$ functions discussed in the Introduction should be bounded and free of singularities in the complex S plane to the right of $S = 1$ [1,2]. In a planar theory, this requirement is slightly weakened: singularities are excluded to the right of $S = 2$, as a consequence of the bound on chaos [23]. We hypothesize that this requirement extends to the individual structure constants $C(S)$ of each Regge trajectory [which are residues of $c(\Delta, S)$ at the locations of the poles]. A rough argument is that different $C(S)$ can be accessed as different residues of $c(\Delta, S)$ in Δ . As long as the Δ are not degenerate, these residues should be individually finite. Finally, in our case, supersymmetry relates the twist-three operators of interest to operators with higher spin by two units; see [24,25] for the precise Ward identities in the twist-two case. Thus, we expect that $C(S)$ should be bounded and free of singularities to the right of $S_* = 0$.

(At the same time, in the left half plane we do expect a host of cuts and poles.)

Together, these conditions suffice for Carlson’s theorem, which ensures the uniqueness of the analytic continuation of this data [26].

What is perhaps less well known is that the unique extension alluded to in Carlson’s theorem can be explicitly

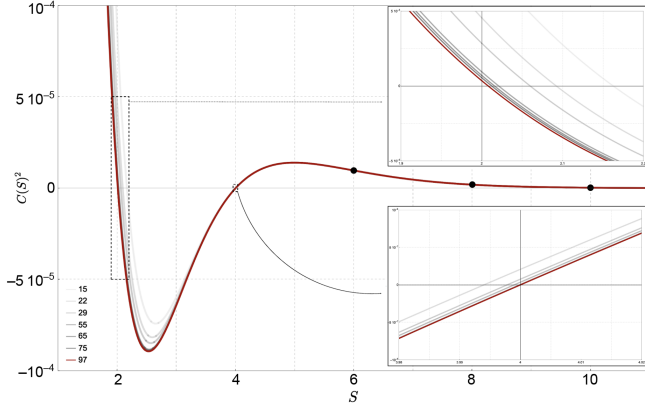


FIG. 2. Newton's continuation of the structure constants for the second family converging as we include more and physical operators for $N = 15, \dots, 97$. The outcome of Newton's interpolation is almost spooky: The physical structure constants are *growing* as we come from the right approaching $S = 6$ and the continuation reverses this growth and dives perfectly through zero surgically at (the even integer) 4 coming back up through zero again precisely at (the even integer) 2. If this is not dark magic, what is dark magic?.

constructed by a beautiful interpolation series written down by Newton in 1687's *Principia Mathematica* [27]. Newton's series

$$f_N(z) \equiv \sum_{j=0}^N \binom{z}{j} \sum_{i=0}^j \binom{j}{i} (-1)^{j-i} f(i). \quad (5)$$

converges to the proper extension $f(z)$ as $N \rightarrow \infty$. Note that the z dependence only enters through the (extension of the) binomial coefficients. For us, f could be taken to be either the energy, $\gamma(S)$, or the structure constants, $C(S)$, and the argument would be $z = S/2 - S_n/2$ where S_n is the first value for which a given family exists. That is $S_1 = 0$ for the lowest family, $S_2 = 6$ for the second family, $S_3 = 12$ for the third family, etc. Then physical values correspond to $z = 0, 1, 2, \dots$ and we include N of these in the interpolation. Newton's series is very powerful: it is supposed to converge to the right of the first allowed singularity, so for us it should converge in the full right half complex spin plane [28].

Applying this method to the anomalous dimensions of the various trajectories results in rapid convergence. We present the outcome of this analysis in Fig. S1 of the Supplemental Material [10]. This confirms the picture that, indeed, each trajectory $\gamma(S)$ is free of singularities in the right half-plane (RHP), as otherwise the method would not have converged. Of course, given convergence, one could simply scan over the complex plane and observe that no degeneracies occur away from the real line. In Sec. IV of the Supplemental Material [10] we discuss an alternative integrability-based method that allows for the computation of the energies directly at complex spin. It confirms the

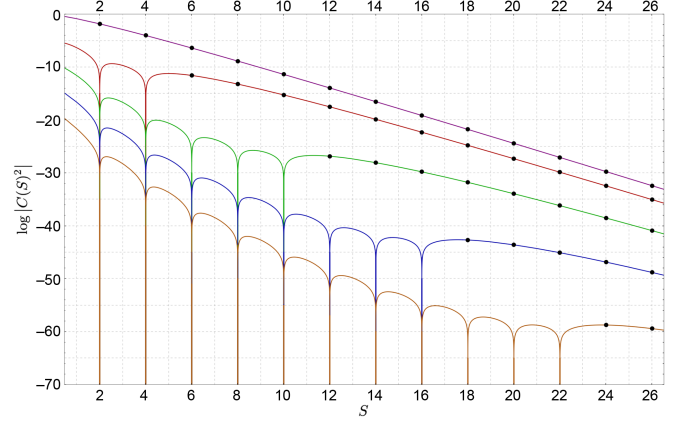


FIG. 3. To extend the analytic continuation for all $S > 0$ we first run the interpolation for a family starting at S_n and check that a zero emerges dynamically as $S_n - 2$ as we increase N . Once this happens we can put it in by hand and continue instead the Newton's series interpolation of $C^2(S) \times (S - S_n + 2)$ and see if a zero at $S_n - 4$ emerges; if so we include it and interpolate $C^2(S) \times (S - S_n + 2) \times (S - S_n + 4)$ and so on, always checking for stability of the procedure.

absence of operator mixing in the RHP and the numerical accuracy of the Newton series.

Having confirmed the lack of degeneracies, let us now jump to the most exciting application of this method: The study of the structure constants for the second family, which starts at spin $S = 6$. We first apply (5) using the first $N = 15$ physical spins; then we include more and more physical operators in the interpolation up to $N = 97$; see Fig. 2. As we increase N we see that the analytic continuation rapidly converges [29] and precisely predicts zeros at the lower spins $S = 4$ and $S = 2$ as mentioned in the Introduction.

Repeating (a refinement of) this analysis for all families we end up with Fig. 3 [30]. The picture put forward in the Introduction with the decoupling of the various higher families at lower physical spins is neatly realized.

Discussion—The complex spin plane is a formidable universe and planar $\mathcal{N} = 4$ SYM, being a solvable higher dimensional conformal field theory is a perfect laboratory for its exploration [31].

In this Letter, we focused on the right half-plane $\text{Re}(S) > 0$. On this side, conformal Regge theory predicts controllable growth at large spin and no singularities at finite spin. As such, there is not much room for surprises in the RHP. The notable exception are the *missing zeros* we explored here. We found that the analytic continuation of structure constants for higher trajectories develops zeros at integer positive spins, explaining their decoupling from correlation functions and resolving the puzzle raised in the Introduction.

We conjecture this decoupling picture to be a general mechanism for any higher dimensional conformal field theory. Of course, testing it is not easy. Here, the data

obtained in planar $\mathcal{N} = 4$ SYM through integrability was precious. To convincingly observe these zeros, we used about 100 physical data points computed with hundreds of digits of precision and plugged those into Newton's formula. Of course, this is not a luxury we can afford in the 3D Ising model where we might have at most a few physical operators with a few digits of precision.

In $\mathcal{N} = 4$ SYM, we could also explore other sectors and learn about analyticity (or lack thereof) in several other parameters such as R charges, mixed spins (S_1, S_2) for more complicated spinning operators, twist, etc. A very nice study of a strong coupling double trace operator decoupling as one analytically continues their R -charge representations can be found in [32].

It is also an important open problem to understand how to directly compute the structure constants at any complex spin S from integrability without resorting to Newton's magic—as done for the spectrum in Sec. IV of the Supplemental Material [10] using Baxter's Q functions. In [33] new formulas were put forward for structure constants in this theory in terms of separation of variables like integrals [34–36] of Q functions so one might wonder if we cannot simply plug the Q functions found in the spectrum study there. Sadly, we do not know how to do it. Take for example the simplest possible leading order structure constant in the $SL(2)$ sector corresponding to twist-two operators. There is no operator degeneracy and a single family of operators in this case for which Dolan and Osborn [37,38] extracted $C(S)_{\text{twist-two LO}}^2 = \{S!^2 / [(2S)!]\}$. In integrability terms this can be obtained from [33]

$$C(S)_{\text{twist-two LO}}^2 = \frac{S!^2}{(2S+1)!} \frac{1}{\int_{-\infty}^{\infty} du \frac{\pi}{2} \frac{Q(u)^2}{\cosh(\pi u)^2}}; \quad (6)$$

see the Supplemental Material [10], Sec. VI. For any even integer spin the Q function is a polynomial so that integral is perfectly convergent; it evaluates to $1/(2S+1)$ [33]; see the Supplemental Material, Sec. III, for a derivation [10]. When S is not an integer, the Baxter function Q grows as $e^{2\pi u}$ at infinity [39], however, and the integral no longer converges. It would be interesting to generalize [33] so that it could apply not just for physical local operators of integer spin but also for analytically continued spins.

A related question is whether we can explicitly write down the light-ray operators corresponding to the Regge trajectories we found in this work, in the spirit of [40–50]. Such light-ray operators should take the form

$$\int d\alpha_1 d\alpha_2 d\alpha_3 \psi(\alpha_1, \alpha_2, \alpha_3) \text{Tr}[Z(\alpha_1)Z(\alpha_2)Z(\alpha_3)]. \quad (7)$$

Here, $Z(\alpha)$ is an insertion of Z on a Wilson line lying along future null infinity, at retarded time α . The three Z insertions are integrated against a wave function $\psi(\alpha_i)$,

which must be translationally invariant in retarded time to describe a primary light-ray operator. The homogeneity of ψ is related to the spin S [51].

In trying to directly construct such operators, we run into puzzles. Naively, at fixed S , the wave function ψ can depend in an arbitrary way on the translationally invariant homogeneity-zero combination $\chi = \alpha_{12}/\alpha_{13}$. In other words, there seem to be a *continuous infinity* of primary light-ray operators we can write down at each spin S , parametrized by an arbitrary function of χ .

What reduces this continuous infinity of choices down to a discrete set of Regge trajectories? In our case, the mode number n_* gave a natural way to define trajectories. This should translate into some natural conditions on the wave function ψ , and we unveil this story in [52]; see also [53] for a similar story in Wilson-Fisher theory. Explicit expressions for the light-ray operators manifest the missing zeros.

Everything we observed in this Letter at leading order in perturbation theory should hold at finite coupling [54]. Repeating the leading-order analysis at next-to-leading order should be straightforward, for instance. More ambitiously, can the missing zeros be useful in searching for a finite coupling version of [33]?

Finally, we have the left-half plane of S , where there is a host of exciting physics to explore. In this left-hand side of the complex spin plane there will be dragons. In perturbation theory, we expect there to be a host of different singularities from poles at negative spins which would open up into cuts at finite coupling, creating an intricate infinitely sheeted Riemann surface. This part of the plane contains so-called “horizontal trajectories” of Balitsky-Fadin-Kuraev-Lipatov type [45,46,61] whose tree-level spin S is fixed, but whose dimension Δ can vary. When the coupling is turned on, horizontal trajectories can recombine with the more traditional 45° trajectories discussed in this work [62,63]. In perturbation theory, such recombinations manifest as poles in anomalous dimensions and structure constants, which satisfy compatibility conditions between the two combining branches, see e.g. [2,62]. A class of horizontal trajectories that should recombine with the twist-three operators considered here was explored in [64]. It would be very interesting to describe the recombinations explicitly at the level of operators, using the techniques of [50]. We also expect branch cuts in perturbation theory where different twist-three trajectories recombine [52].

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at that time (albeit using different tools/ideas). We thank several participants of the 2022 Simons bootstrap collaboration meeting for reminding us of the important Ref. [8]. Research at the Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI. This work was additionally supported by a grant from the Simons Foundation (P. V.: No. 488661) and FAPESP Grants No. 2016/01343-7 and No. 2017/03303-1. D. S. D. is supported by Simons Foundation Grant No. 488657 (Simons Collaboration on the Nonperturbative Bootstrap) and a DOE Early Career Award under Grant No. DE-SC0019085.

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