

Beyond the Quantum Cramér-Rao Bound

J. R. Hervas¹, A. Z. Goldberg², A. S. Sanz¹, Z. Hradil³, J. Řeháček³, and L. L. Sánchez-Soto^{1,4,5}

¹*Departamento de Óptica, Facultad de Física, Universidad Complutense, 28040 Madrid, Spain*

²*National Research Council of Canada, 100 Sussex Drive, Ottawa, Ontario K1N 5A2, Canada*

³*Department of Optics, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic*

⁴*Institute for Quantum Studies, Chapman University, Orange, California 92866, USA*

⁵*Max-Planck-Institut für die Physik des Lichts, 91058 Erlangen, Germany*



(Received 8 July 2024; accepted 10 December 2024; published 10 January 2025)

The quantum Cramér-Rao bound (QCRB) stands as a cornerstone of quantum metrology. Yet, akin to its classical counterpart, it provides only local information and overlooks higher-order details. We leverage the theory of higher-order asymptotics to circumvent these issues, providing corrections to the performance of estimators beyond the QCRB. While the QCRB often yields a whole family of optimal states and several optimal measurements, our approach allows us to identify optimal states and measurements within the family that are otherwise equivalent according to the QCRB alone; these are requisite for optimal metrology before reaching the asymptotic limit. These results are especially pertinent when dealing with unitary processes.

DOI: [10.1103/PhysRevLett.134.010804](https://doi.org/10.1103/PhysRevLett.134.010804)

Introduction—Quantum metrology is a rapidly growing field that leverages the principles of quantum physics to achieve extremely precise measurements [1–6]. Today, quantum metrology offers significant advantages in terms of precision, sensitivity, and noise reduction. These benefits are driving remarkable advancements in both fundamental and applied research.

The cornerstone of frequentist quantum metrology, the quantum Cramér-Rao bound (QCRB), sets a lower bound on the minimum mean square error (MSE) of an estimator for a parameter encoded in a state [7,8]. The inverse of the QCRB is a measure for the state’s sensitivity under small variations of a parameter, known as the quantum Fisher information (QFI) [9].

Despite the prevalence of the QFI in quantum information, its implications have to be taken with a grain of salt. First, the associated QCRB is only asymptotically attainable, leading to apparent violations of the bound when few measurement samples are available [10,11]. More importantly, the QFI is tailored for local estimation scenarios rather than global ones, where both the location and size of the parameter neighborhood remain unknown. It has been noticed that the MSE of global estimation problems can significantly differ from local strategies [12–16]. Additionally, the QCRB suggests that the best strategy is

to use all available resources in a single measurement [17]. However, as the number of resources increases, so does the required prior knowledge about the parameter, a problem that can be circumvented by using a Bayesian approach [18–20].

In this Letter, we will address additional issues that, while highly relevant, have remained largely unexplored. For single-parameter estimation, multiple measurements can saturate the QCRB [21,22], raising the question of which one measurement to prefer over the rest. The QCRB also has limitations when determining the optimal state for sensing a unitary transformation. As a representative example, consider the simplest case of rotations in SU(2): a whole family of optimal states is known to exist when all rotation axes are taken into consideration [23–26].

This Letter handles these limitations by leveraging the theory of higher-order asymptotics. This theory extends beyond the central limit theorem, providing corrections based on the number of trials, which are crucial for a *bona fide* statistical inference [27]. By properly incorporating these corrections, we can distinguish between states and measurements that appear equivalent according to the QCRB. Additionally, the corrections are essential for evaluating performance in real-world scenarios with finite resources. Although tools from the theory of higher-order asymptotics are well established in the statistical literature and have been applied in the context of quantum hypothesis testing [28], their application in quantum metrology offers a ripe opportunity for progress.

Higher-order asymptotics—We consider the canonical scenario of quantum metrology in which we want to estimate the value of a single real parameter θ from the

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Open access publication funded by the Max Planck Society.

outcomes of ν independent measurements. Let $p(x|\theta)$ be the probability that in any given round the measurement will result in the outcome x , given that the value of the parameter is θ (we take x as a discrete random variable, but the continuous case can be dealt with in a similar way). In the quantum domain, the measurement setting is described by the positive operator-valued measure (POVM) $\Pi_x \geq 0$, with $\sum_x \Pi_x = 1$, and the probabilities are given by Born's rule $p(x|\theta) = \text{Tr}[\rho_\theta \Pi_x]$.

To report a value of the parameter θ , a proper estimator $\hat{\theta}(x)$ must be used. This is a function of the obtained measurement results x . The quantity of central interest is the MSE of $\hat{\theta}(x)$, which for estimators that approach the classical Cramér-Rao bound (CCRB) can generally be expanded as [29]

$$\text{MSE}[\hat{\theta}(x)] = \frac{1}{\nu F_x(\theta)} + \frac{1}{\nu^2} \left[\Gamma(\theta) + b^2(\theta) + 2 \frac{b'(\theta)}{F_x(\theta)} \right] + \mathcal{O}\left(\frac{1}{\nu^3}\right), \quad (1)$$

where the classical Fisher information (CFI) is $F_x(\theta) = \sum_x p(x|\theta) [\partial_\theta p(x|\theta)/p(x|\theta)]^2$ and $b(\theta)$ denotes the bias to order $1/\nu$: $\langle \hat{\theta}(x) - \theta \rangle = b(\theta)/\nu + \mathcal{O}(1/\nu^2)$, with $\langle \hat{\theta}(x) \rangle = \sum_x p(x|\theta) \hat{\theta}(x)$. The function $\Gamma(\theta)$ depends on the particular estimator. Actually, a vast number of estimators have been formulated by proposing *ad hoc* functions $\hat{\theta}(x)$ [10]. One of the most widely adopted is the maximum likelihood estimator (MLE), defined as $\hat{\theta}_{\text{MLE}}(x) = \arg \max_\theta p(\theta|x)$. Here, the likelihood $p(\theta|x)$ must be interpreted as the probability of θ being the true value given the data x . For this estimator, the function $\Gamma(\theta)$ takes the value [29]

$$\Gamma(\theta) = -\frac{1}{F_x(\theta)} + \frac{\gamma_{02}(\theta) - 2\gamma_{21}(\theta) + \gamma_{40}(\theta)}{F_x^3(\theta)} + \frac{\frac{1}{2}\gamma_{11}^2(\theta) - [\gamma_{11}(\theta) - \gamma_{30}(\theta)]^2}{F_x^4(\theta)}, \quad (2)$$

and $b(\theta) = -\gamma_{11}/[2\nu F_x^2(\theta)]$, with

$$\gamma_{rs}(\theta) = \sum_x p(x|\theta) \left[\frac{\partial_\theta p(x|\theta)}{p(x|\theta)} \right]^r \left[\frac{\partial_{\theta\theta}^2 p(x|\theta)}{p(x|\theta)} \right]^s. \quad (3)$$

For unbiased estimators, the MSE reduces to the variance, and the $1/\nu$ term gives the time-honored classical Cramér-Rao bound (CCRB). Similar expressions for other estimators can be found in [29]; nonetheless, no estimator is universally guaranteed to minimize all the $1/\nu^2$ contributions in Eq. (1); which one is best depends on the particular problem at hand. This differs markedly from the usual asymptotic case, where the CCRB sets the fundamental limit, independent of the estimator.

To establish a fundamental limit, we must first restrict the allowed estimators. It is reasonable to compare estimators that are unbiased to order $1/\nu$; i.e., $b(\theta) = 0$. This is not a significant loss of generality, since for any $\hat{\theta}(x)$ one can always build a bias-corrected version as $\hat{\theta}^*(x) = \hat{\theta}(x) - b(\hat{\theta}(x))/\nu$. In this case, the MSE takes the following value

$$\text{MSE}[\hat{\theta}(x)] = \frac{1}{\nu F_x(\theta)} + \frac{\Gamma(\theta)}{\nu^2} + \mathcal{O}\left(\frac{1}{\nu^3}\right). \quad (4)$$

Interestingly, Rao [29–32] showed that the MLE minimizes $\Gamma(\theta)$, thus setting a fundamental limit similar to the CCRB and confirming that the MLE is distinctive among all estimators that reach the CCRB. With this in mind, we henceforth focus on the MLE.

Problem setting—To gain further insights into these corrections, we consider an SU(2) rotation $R = \exp(i\theta J_n)$, parametrized by its rotation axis $\mathbf{n}$ and angle θ . Here, $J_n = \mathbf{J} \cdot \mathbf{n}$, with $\mathbf{J}$ being the generators of the rotation, which satisfy the commutation relations of the Lie algebra $\mathfrak{su}(2)$. The irreducible representations of this algebra are labeled by J and are spanned by the states $|J, m\rangle$, which are the simultaneous eigenstates of $\mathbf{J}^2$ and J_3 . These states form a basis for a $(2J+1)$ -dimensional Hilbert space, denoted by $\mathcal{H}_J$. In what follows, we assume that we have a pure state $|\psi\rangle \in \mathcal{H}_J$ that undergoes the rotation R , resulting in $|\psi_\theta\rangle = R|\psi\rangle$.

QCRB-optimal states—If the axis of rotation is fixed, we can always take $\mathbf{n}$ as being directed along the z axis. The optimal probe states according to the QCRB for estimating the angle are those maximizing the variance of J_3 : they are the NOON states $R(|J, J\rangle + |J, -J\rangle)/\sqrt{2}$ [33].

When the rotation axis is an arbitrary $\mathbf{n}$, one can find states that minimize the QCRB averaged over all possible rotation axes

$$\frac{1}{4\pi} \int_{\mathcal{S}_2} \frac{1}{4\mathbf{m}_2(\mathbf{n})} d\Omega, \quad (5)$$

where $d\Omega$ is the surface area element on the unit sphere $\mathcal{S}_2$ and we use the notation $\mathbf{m}_k(\mathbf{n}) = \langle (J_n - \langle J_n \rangle)^k \rangle$ for the k th central moment (that, in general, depends on $\mathbf{n}$ and on the initial state $|\psi\rangle$). Here, we have taken into account that the QFI, which is obtained by optimizing the CFI over all possible measurements, reduces to $Q_\psi(\mathbf{n}) = 4\mathbf{m}_2(\mathbf{n})$ for pure states.

The optimal states minimizing the averaged QCRB are those that satisfy $\langle J_n \rangle = 0$ and $\langle J_n^2 \rangle = \text{constant}$: $\mathbf{m}_2(\mathbf{n})$ is thus isotropic and the states do not carry information in their first two moments. In general, for a given J , various states can fulfill these conditions, making the optimal states not unique [26]. Furthermore, there are states with isotropic higher-order moments $\mathbf{m}_k(\mathbf{n})$, raising the question of whether the QCRB is overlooking relevant higher-order information [34,35].

Another motivation for states with isotropic higher-order moments comes from a connection between QFI and fidelity. Under unitary evolution, a state's fidelity with itself is given, to lowest order, by $|\langle\psi|e^{i\theta J_n}|\psi\rangle|^2 = 1 - \theta^2 Q_\psi(\mathbf{n})$, such that states with the largest QFI are those that change the most rapidly with small unitaries; as was previously established. If we extend the fidelity calculation to higher orders and average over all axes $\mathbf{n}$, we now find that the fidelity that deviates the most rapidly from unity to next order in θ is the one for states with $\mathbf{m}_k$ independent from $\mathbf{n}$ for the next order k (see Sec. I in Supplemental Material [36]). This is true for any unitary transformation (e.g., crowning Fock states for displacement sensing) and suggests that higher-order isotropic states emerge in a natural and unexplored way.

QCRB-optimal measurements—We study these higher-order corrections using a measurement whose CFI attains the QFI, so they can be understood as higher-order corrections to the QCRB. The requirements for a measurement to attain the QFI in the pure state case were given in a particularly simple form in the Appendix E of Ref. [22].

For unitary evolution with generator J_n , the optimal measurement is characterized by a set of two projectors $\Pi_\pm = |\Phi_\pm\rangle\langle\Phi_\pm|$

$$|\Phi_\pm(\theta)\rangle = \frac{1}{2} \left[(1 + e^{i\phi_\pm}) |\psi_\theta\rangle + \frac{(1 - e^{i\phi_\pm})}{\sqrt{\mathbf{m}_2}} (J_n - \langle J_n \rangle) |\psi_\theta\rangle \right], \quad (6)$$

that depends on a single free parameter $\phi_+ = \phi_- + \pi$ (see Sec. II in [36]). This represents a family of measurements that are equivalently optimal according to the QCRB, but that may be further optimized when considering higher-order asymptotics. Note that due to periodicity, it is enough to consider the range $[-\pi/2, 0]$.

There is a subtlety often overlooked in quantum metrology: when considering $p(x|\theta)$, the value of θ on which the measurement depends must be treated as a constant, say θ_0 . For example, $p(x_+|\theta) = |\langle\psi_\theta|\Phi_+(\theta_0)\rangle|^2$, and the dependence on θ is only through the state, not the measurement, even when we assume $\theta_0 = \theta$. It follows that the resulting probabilities, $|\langle\psi_\theta|\Phi_\pm(\theta_0)\rangle|^2$, only sum up to one when $\theta = \theta_0$. This implies that we have to extend our measurement, because the MLE presupposes a valid probability distribution that must arise from a true POVM. However, modifying the POVM can affect the performance of real estimators, such as the MLE, even when $\theta = \theta_0$. This is because the MLE relies on $p(x|\theta)$ as a function of θ , and any modification to the POVM alters this dependence.

We will consider two ways of addressing this issue, which are essentially exhaustive. The first method is to replace one of the initial projectors, say $|\Phi_-(\theta_0)\rangle\langle\Phi_-(\theta_0)|$, with $\Pi_2 = \mathbb{1} - |\Phi_+(\theta_0)\rangle\langle\Phi_+(\theta_0)|$. When $\theta = \theta_0$, this is equivalent to the $\Pi_\pm$ measurement, but when $\theta \neq \theta_0$, Π_2 absorbs the missing probability.

The second approach involves extending the two initial measurements by introducing a set of POVM elements $\{\Pi_3, \Pi_4, \dots\}$, with associated probabilities $\{p(x_3|\theta), p(x_4|\theta), \dots\}$ that are zero when $\theta = \theta_0$. Interestingly, the performance at $\theta = \theta_0$ is unaffected by this extension. To see why, note that the MLE is the value of θ that maximizes $\sum_x \nu_x \ln[p(x|\theta)]$, where ν_x is the number of measurement outcomes for x . At $\theta = \theta_0$, only the outcomes ν_+ and ν_- are observed, meaning the MLE is independent of the extended elements. In principle, it is possible to imagine other types of extensions, but they would be a combination of these two. While these two extensions both approach the QCRB, we will see that their second-order corrections are different.

Results—In what follows, we assume that $\theta_0 = 0$ without loss of generality. First, we study the extension $\Pi_2 = \mathbb{1} - |\Phi_+(0)\rangle\langle\Phi_+(0)|$. In this case, $\Gamma(0) = \cot^2 \phi_+ / 8\mathbf{m}_2$, which is minimal and equal to zero when $\phi_+ = -\pi/2$, revealing the optimal measurement. The optimal ϕ_+ is the one where $p(x_+|0) = p(x_-|0) = 1/2$, that is, the “fair coin” case. To understand why this is the case, note that CCRB is attained in the asymptotic limit because the MLE tends to a Gaussian distribution. Convergence to a Gaussian is faster for a fair coin than for a biased coin. This rate of convergence is reflected in the $1/\nu^2$ correction terms.

As the term $1/\nu^2$ cancels out for all second-order unpolarized states, we are naturally interested in the $1/\nu^3$ corrections to look for optimal probe states among the QCRB-optimal family. These next-order corrections have not been explored in detail in the statistical literature, so we calculate them explicitly. For this, we employ the method described in [40]. This involves expanding the equation for the MLE in powers of $\hat{\theta}_{\text{MLE}}$ and then using the Lagrange-Bürmann inversion formula [41] to obtain the inverse series for $\hat{\theta}_{\text{MLE}}$ as a function of the experimental output. Details are provided in [36] (Sec. III). Noting that $\phi_+ = -\pi/2$, which is fixed by $\Gamma(0)$, we get

$$\begin{aligned} \text{MSE} \left[\hat{\theta}_{\text{MLE}}^*(x) \right]_{\theta=0} &= \frac{1}{4\mathbf{m}_2\nu} \\ &+ \frac{138\mathbf{m}_2^4 + 79\mathbf{m}_2^2\mathbf{m}_4 + \mathbf{m}_2(30\mathbf{m}_3^2 - 3\mathbf{m}_6) + 26\mathbf{m}_4^2}{768\mathbf{m}_2^5\nu^3} \\ &+ \mathcal{O}\left(\frac{1}{\nu^4}\right). \end{aligned} \quad (7)$$

Note that this MSE depends on $\mathbf{n}$. This expression can be readily evaluated to assess how quickly one approaches the QCRB and a method's performance before reaching the asymptotic limit without the need for numerical methods. We also provide the formulas for the MLE (without bias correction) in [36] (Sec. III).

We proceed with the family of QCRB-optimal states, with $\langle J_n \rangle = 0$ and $\langle J_n^2 \rangle = \text{constant}$. The average over the

sphere of the $1/\nu^3$ term in Eq. (7) is minimized by states with $\langle J_n^3 \rangle = 0$ and $\langle J_n^4 \rangle = \text{constant}$ (see Sec. IV in [36]), confirming our intuition of the superiority of states whose higher-order moments $\mathbf{m}_k$ are independent from $\mathbf{n}$ among the family of states that are isotropic to second order. We have thus found general strategies for selecting optimal measurements and states for preasymptotic metrology from among those that are optimal in the asymptotic limits.

We next consider the second extension that adds extra POVM elements with vanishing probabilities at $\theta = 0$; as aforementioned, the performance does not depend on their choice. Using Eq. (1) yields an incorrect expression for the second-order corrections. This is because, in general, they are not applicable at points where probabilities approach zero [note that $\Gamma(0)$ diverges at $\phi_+ = 0$ where $p(x_-|0) = 0$ in the previous POVM extension]. We discuss this in more detail in [36], Sec. V. As Eq. (1) fails, we perform an asymptotic expansion as before to find, up to order $1/\nu^2$

$$\text{MSE}[\hat{\theta}_{\text{MLE}}(x)]_{\theta=0} = \frac{1}{4\mathbf{m}_2\nu} + \frac{\mathbf{m}_2^3(14\text{csc}^2\phi_+ - 5) - \mathbf{m}_2\mathbf{m}_4 + 3\mathbf{m}_3^2}{32\mathbf{m}_2^4\nu^2} + \mathcal{O}\left(\frac{1}{\nu^3}\right). \quad (8)$$

We observe that the optimal measurement and states correspond to $\phi_+ = -\pi/2$ and $\langle J_n^3 \rangle = 0$, respectively, as before, although there is no longer an advantage in having $\langle J_n^4 \rangle = \text{constant}$. While the statistical literature considers probabilities approaching zero to be highly pathological, this is a common scenario in quantum metrology. Note also that the definition of the fundamental quantity $\Gamma(\theta)$ loses its meaning, as it relies on Eq. (1) being valid.

We tested this expression numerically. For that, we calculated the average over a grid of $\mathbf{n}$ values of the second order corrections

$$\nu^2 \left[\langle \hat{\theta}_{\text{MLE}}^2(x) \rangle - \frac{1}{4\mathbf{m}_2\nu} \right]. \quad (9)$$

The results are shown in Fig. 1, where we can see how the numerical MSE average converges to the theoretical value given by Eq. (8).

As we have seen, these techniques are useful because they reveal information not captured by the QCRB: they provide the rate of convergence to the QCRB and allow for a better assessment of the performance before reaching the asymptotic limit, as well as identify optimal measurements and states that appear equivalent according to the QCRB. This additional information comes with certain costs: there is some dependence on the estimators, the need to specify how the POVM is extended, and the breakdown of the general theory when probabilities approach zero. Regarding the first issue, it is inherent to the problem—after all, the QCRB (and also the CCRB) is a simplification of a complex estimation process. If we want more detailed information, we must

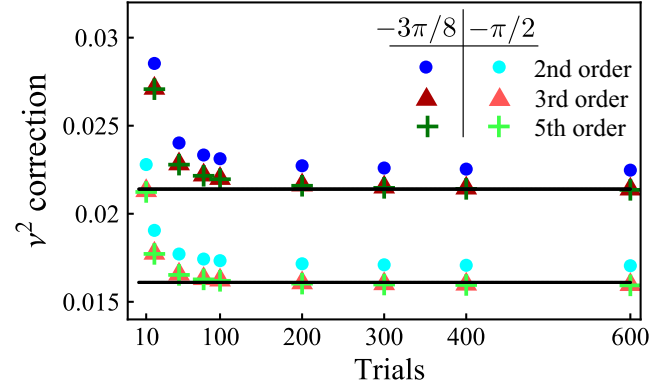


FIG. 1. $1/\nu^2$ corrections at $\theta = 0$, for a different number of trials, of the MSE averaged over the sphere for the second POVM extension. We compare states isotropic to the second, third, and fifth order, and measurements with $\phi_+ = -\pi/2$ and $\phi_+ = -3\pi/8$. The black lines denote the average over all $\mathbf{n}$ of Eq. (8) for third order isotropic states, and the two measurements ($\phi_+ = -\pi/2$ bottom, $\phi_+ = -3\pi/8$ top). We used 300 grid points and the states belong to $J = 6$.

accept these caveats. As for the other two issues, a few comments can be made: when dealing with full-rank density matrices, there is no need to extend the measurement, and probabilities generally do not approach zero, thus eliminating the problem. Moreover, it is possible to derive expressions that remain valid when probabilities approach zero, as we have shown, making it feasible to extend Eq. (1) to account for this.

We have presented two possible ways to extend the POVM, raising the question of which method is superior or what the differences are. If the goal is to achieve optimal measurement across a wide range of θ values, it is advisable to have a POVM consisting of many elements. These elements should likely include projections onto the subspaces of $J_n^2|\psi_\theta\rangle$, $J_n^3|\psi_\theta\rangle$, $J_n^4|\psi_\theta\rangle$, and so on. This approach ensures that even when θ varies significantly, information can still be extracted from the new state, which will have components in these subspaces. This concept was demonstrated in the problem of incoherent imaging [42,43], where expanding the POVM with projectors onto subspaces of powers of the generator allowed for achieving the QCRB over a broader region. Therefore, the second extension is our recommended approach.

On the other hand, when we are very close to the true value, the first extension shows that we can eliminate the $1/\nu^2$ terms, resulting in extremely fast convergence to the QCRB. However, this extension fails to attain the QCRB if we move far away from the true value because

$$\frac{1}{F_x(\theta)} = \frac{1}{4\mathbf{m}_2} + \frac{1}{4} \left(\frac{\mathbf{m}_4}{\mathbf{m}_2^2} - 1 \right) \theta^2 + \mathcal{O}(\theta^3), \quad (10)$$

and the θ^2 term is always positive and the average is independent of the state.

In the first-order approach, the QCRB serves as a metric for comparing two states in a straightforward manner. However, the QCRB provides only a oversimplified view. Extending beyond the first order offers a more comprehensive perspective, albeit at the expense of simplicity.

Concluding remarks—The QCRB, while having vital importance in quantum metrology, also exhibits certain deficiencies due to its local character. In this Letter, we have focused on three aspects: its inability to distinguish among a set of measurements that saturate it, its incapacity to differentiate optimal states for an arbitrary unitary transformation, and the fact that it provides information exclusively about the asymptotic limit.

We have considered higher-order corrections to the QCRB that allow us to assess performance before reaching the asymptotic limit. These corrections also enabled us to identify an optimal measurement and distinguish between states that appeared indistinguishable according to the QCRB.

Our results apply to any unitary transformation, with the general case handled by considering the generators of the group $SU(n)$ instead of the usual angular momentum operators.

This work has only scratched the surface of the potential uses of higher-order asymptotic theory in quantum metrology. While our primary focus was on frequentist estimation, these techniques are equally applicable in the Bayesian framework. Extensions to the multiparameter case are also feasible and are expected to yield interesting results. Additionally, moving beyond the examination of pure states and unitary transformations, a promising avenue for future research involves extending these concepts to encompass general transformations and density matrices. Moreover, a more thorough investigation of scenarios where probabilities approach zero—a common instance in quantum metrology—looks intriguing. The multitude of potential applications suggests that these techniques can play a crucial role in advancing quantum metrology.

Acknowledgments—We acknowledge discussions with A. Smerzi, L. Pezzè, A. Datta, M. Barbieri, and M. Evans at an early stage of this work. A. Z. G. acknowledges that the NRC headquarters is located on the traditional unceded territory of the Algonquin Anishinaabe and Mohawk people, as well as support from the NRC Quantum Sensors Challenge program. This publication has received financial support from the QuantEra program (Project ApresSF) and from the Agencia Española de Investigación (Grant No. PID2021-127781NB-I00). L. L. S. S. was supported in part by Grant No. NSF PHY-1748958 to the Kavli Institute for Theoretical Physics (KITP).

- [1] M. G. A. Paris, Quantum estimation for quantum technology, *Int. J. Quantum. Inform.* **07**, 125 (2009).
- [2] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, *Nat. Photonics* **5**, 222 (2011).

- [3] M. Szczykulska, T. Baumgratz, and A. Datta, Multiparameter quantum metrology, *Adv. Phys. X* **1**, 621 (2016).
- [4] J. S. Sidhu and P. Kok, Geometric perspective on quantum parameter estimation, *AVS Quantum Sci.* **2**, 014701 (2020).
- [5] E. Polino, M. Valeri, N. Spagnolo, and F. Sciarrino, Photonic quantum metrology, *AVS Quantum Sci.* **2**, 024703 (2020).
- [6] F. Albarelli, M. Barbieri, M. G. Genoni, and I. Gianani, A perspective on multiparameter quantum metrology: From theoretical tools to applications in quantum imaging, *Phys. Lett. A* **384**, 126311 (2020).
- [7] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
- [8] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, Quaderni Monographs No. 1 (Ed. della Normale, Pisa, 2011).
- [9] D. Petz and C. Ghinea, Introduction to quantum Fisher information, in *Quantum Probability and Related Topics* (World Scientific, Singapore, 2011), Vol. 27, pp. 261–281.
- [10] S. M. Kay, *Fundamentals of Statistical Signal Processing* (Prentice-Hall, Englewood Cliffs, NJ, 1993).
- [11] H. L. Van Trees, *Detection, Estimation, and Modulation Theory* (Wiley, New York, 2001).
- [12] M. Hayashi, Comparison between the Cramer-Rao and the mini-max approaches in quantum channel estimation, *Commun. Math. Phys.* **304**, 689 (2011).
- [13] M. J. W. Hall and H. M. Wiseman, Heisenberg-style bounds for arbitrary estimates of shift parameters including prior information, *New J. Phys.* **14**, 033040 (2012).
- [14] W. Górecki, R. Demkowicz-Dobrzański, H. M. Wiseman, and D. W. Berry, π -corrected Heisenberg limit, *Phys. Rev. Lett.* **124**, 030501 (2020).
- [15] W. Górecki and R. Demkowicz-Dobrzański, Multiparameter quantum metrology in the Heisenberg limit regime: Many-repetition scenario versus full optimization, *Phys. Rev. A* **106**, 012424 (2022).
- [16] G. Chesi, A. Riccardi, R. Rubboli, L. Maccone, and C. Macchiavello, Protocol for global multiphase estimation, *Phys. Rev. A* **108**, 012613 (2023).
- [17] M. Zwierz, C. A. Pérez-Delgado, and P. Kok, General optimality of the Heisenberg limit for quantum metrology, *Phys. Rev. Lett.* **105**, 180402 (2010).
- [18] J. Rubio, P. Knott, and J. Dunningham, Non-asymptotic analysis of quantum metrology protocols beyond the Cramér–Rao bound, *J. Phys. Comm.* **2**, 015027 (2018).
- [19] J. Rubio and J. Dunningham, Quantum metrology in the presence of limited data, *New J. Phys.* **21**, 043037 (2019).
- [20] M. Gessner and A. Smerzi, Hierarchies of frequentist bounds for quantum metrology: From Cramer-Rao to Barankin, *Phys. Rev. Lett.* **130**, 260801 (2023).
- [21] S. L. Braunstein and C. M. Caves, Statistical distance and the geometry of quantum states, *Phys. Rev. Lett.* **72**, 3439 (1994).
- [22] S. Kurdzialek and R. Demkowicz-Dobrzański, Measurement noise susceptibility in quantum estimation, *Phys. Rev. Lett.* **130**, 160802 (2023).
- [23] P. Kolenderski and R. Demkowicz-Dobrzański, Optimal state for keeping reference frames aligned and the platonic solids, *Phys. Rev. A* **78**, 052333 (2008).

- [24] G. Björk, A. B. Klimov, P. De La Hoz, M. Grassl, G. Leuchs, and L. L. Sánchez-Soto, Extremal quantum states and their Majorana constellations, *Phys. Rev. A* **92**, 031801 (R) (2015).
- [25] D. Baguette, F. Damanet, O. Giraud, and J. Martin, Anticoherence of spin states with point-group symmetries, *Phys. Rev. A* **92**, 052333 (2015).
- [26] A. Z. Goldberg, A. B. Klimov, G. Leuchs, and L. L. Sánchez-Soto, Rotation sensing at the ultimate limit, *J. Phys. Photonics* **3**, 022008 (2021).
- [27] J. K. Ghosh, *Higher Order Asymptotics*, NSF-CBMS Regional Conference Series in Probability and Statistics (Institute of Mathematical Statistics, Alexandria, VA, 1994).
- [28] N. Datta, Y. Pautrat, and C. Rouzé, Second-order asymptotics for quantum hypothesis testing in settings beyond I.I.D.—Quantum lattice systems and more, *J. Math. Phys. (N.Y.)* **57**, 062207 (2016).
- [29] C. R. Rao, Criteria of estimation in large samples, *Sankhya* **25**, 189 (1963).
- [30] C. R. Rao, Asymptotic efficiency and limiting information, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics* (University of California Press, 1961), Vol. 4, pp. 531–546.
- [31] C. R. Rao, Apparent anomalies and irregularities in maximum likelihood estimation (with discussion), *Sankhya* **24**, 73 (1962), <http://hdl.handle.net/10263/296>.
- [32] C. R. Rao, Efficient estimates and optimum inference procedures in large samples, *J. R. Stat. Soc. Ser. B* **24**, 46 (1962).
- [33] J. P. Dowling, Quantum optical metrology—The lowdown on high-N00N states, *Contemp. Phys.* **49**, 125 (2008).
- [34] J. Martin, S. Weigert, and O. Giraud, Optimal detection of rotations about unknown axes by coherent and anticoherent states, *Quantum* **4**, 285 (2020).
- [35] I. Gianani, M. G. Genoni, and M. Barbieri, Assessing data postprocessing for quantum estimation, *IEEE J. Sel. Top. Quantum Electron.* **26**, 1 (2020).
- [36] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.134.010804>, which includes Refs. [37–39].
- [37] W.-K. Tung, *Group Theory in Physics* (World Scientific, Philadelphia, 1985).
- [38] M. Tsang, Conservative classical and quantum resolution limits for incoherent imaging, *J. Mod. Opt.* **65**, 1385 (2018).
- [39] Y. Ye and X.-M. Lu, Quantum Cramér-Rao bound for quantum statistical models with parameter-dependent rank, *Phys. Rev. A* **106**, 022429 (2022).
- [40] J. B. S. Haldane and S. M. Smith, The sampling distribution of a maximum-likelihood estimate, *Biometrika* **43**, 96 (1956).
- [41] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs and Mathematical Tables* (Dover, New York, 1972).
- [42] M. Tsang, R. Nair, and X.-M. Lu, Quantum theory of superresolution for two incoherent optical point sources, *Phys. Rev. X* **6**, 031033 (2016).
- [43] J. Rehacek, M. Pař, B. Stoklasa, Z. Hradil, and L. L. Sánchez-Soto, Optimal measurements for resolution beyond the Rayleigh limit, *Opt. Lett.* **42**, 231 (2017).