

Correlations Enable Lossless Ergotropy Transport

Rick P. A. Simon^{1,2,*}, Janet Anders^{1,2}, and Karen V. Hovhannisyan^{1,†}¹*University of Potsdam, Institute of Physics and Astronomy, Karl-Liebknecht-Str. 24-25, 14476 Potsdam, Germany*²*Department of Physics and Astronomy, University of Exeter, Stocker Road, Exeter EX4 4QL, United Kingdom*

(Received 25 June 2024; accepted 21 October 2024; published 7 January 2025)

“A battery powers a device” can be read as “work stored in the battery is being transported to the device.” In quantum batteries, the total amount of stored work can be measured by ergotropy, which is the maximal work extractable by unitary operations. Transporting ergotropy is fundamentally different from transporting energy, and here we find that ergotropy can be *gained* even when the transmission channel is strictly energy conserving. We show that, generically, ergotropy transport is lossy whenever the two systems start uncorrelated. In contrast, for a large class of correlated initial states, transport can be gainful. Furthermore, a single correlated state can be used multiple times, allowing to transport without losses an order of magnitude more work than the battery capacity. Correlations are thus a useful resource for ergotropy transport, and we quantify how this resource is consumed during gainful transport.

DOI: 10.1103/PhysRevLett.134.010408

Introduction—Batteries store work; it is their sole function. By storing work we mean storing energy that is available to be extracted in the form of work. Many fundamental aspects of quantum batteries’ storage capacity [1–5], charging speed [6–11], and stability [12–22] have been studied [23]. However, the no less fundamental matter of transporting a given amount of “charge” (i.e., work) from the battery to a consumption unit without external energetic intervention has so far received little attention [4,22,24–26]. A quantum battery autonomously powering a (quantum) device is a paradigmatic example of this scenario.

The charge level of a quantum battery, i.e., the total amount of work stored in it, is standardly quantified by ergotropy [1–10,12–14,17–32]. It is the maximal amount of work extractable from a system *unitarily* [33]. However, a battery performing work on a consumption unit means that the two systems interact with each other. And when they are of comparable size, the process will generally not be unitary for either system. This makes the problem of ergotropy transport highly nontrivial: When will this nonunitarity cause losses during transport? Can nonunitarity be leveraged to facilitate transport?

In this Letter, we put forth a rigorous framework for autonomous ergotropy transport between two quantum

systems. We establish that the efficiency of transport ultimately depends on the initial correlations between the battery and the consumption unit. If they start uncorrelated, ergotropy will generically be partially lost during transport. However, if they start correlated, the transport may even be gainful, i.e., the consumption unit may receive more ergotropy than the battery sends.

Ergotropy and its transport—If H is the Hamiltonian of a system that lives in a d -dimensional Hilbert space, and ρ is its state, then the ergotropy is [33]

$$\mathcal{E}(\rho) := \text{Tr}(\rho H) - \min_U \text{Tr}(U \rho U^\dagger H). \quad (1)$$

Here, the minimization is over *all* unitary operations.

In this Letter, we focus on the bipartite scenario of transporting ergotropy from system B (battery) to system C (consumption unit). To make the bookkeeping of energy and entropy as clean as possible, we choose the transport “channel” to be strictly energy-conserving and unitary. Namely, the operation that acts on BC to perform the transport is unitary, call it U_{BC} , and strict energy conservation means [34–38]

$$[U_{BC}, H_{BC}] = 0, \quad (2)$$

where 0 is the zero operator and

$$H_{BC} = H_B \otimes \mathbb{1}_C + \mathbb{1}_B \otimes H_C \quad (3)$$

is the total Hamiltonian of BC before and after the transport (during the transport, the Hamiltonian is varied in time so as to generate U_{BC}). Here, H_B and H_C are, respectively, the

*Contact author: rick.paul.axel.simon@gmail.com

†Contact author: karen.hovhannisyan@uni-potsdam.de

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](#). Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.

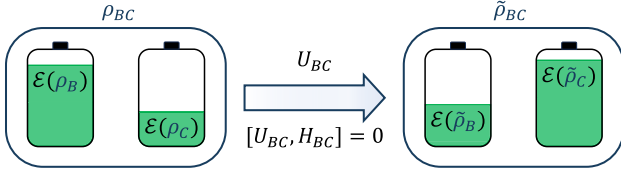


FIG. 1. Schematic of transporting ergotropy from battery (B) to consumption unit (C). Initially, the systems do not interact, and their ergotropies are $\mathcal{E}(\rho_B)$ and $\mathcal{E}(\rho_C)$. Then an energy-conserving unitary U_{BC} transforms the initial joint state ρ_{BC} into $\tilde{\rho}_{BC}$, changing local ergotropies to $\mathcal{E}(\tilde{\rho}_B)$ and $\mathcal{E}(\tilde{\rho}_C)$. If C gains more ergotropy than B loses, then the transport is gainful. Otherwise, it is lossy. One of our main results is that the amount of correlations between B and C is a key resource for achieving a gainful transport.

Hamiltonians of B and C , and $\mathbb{1}$ is the identity operator. This process is pictured in Fig. 1.

The local evolution on B and C induced by U_{BC} can be unitary only under very restrictive conditions [37,39–41]. In general, the individual evolutions of B and C will be (highly) nonunitary. Because of this, ergotropy transport is fundamentally different from energy transport [42–47] as well as the process of increasing one system’s average energy at the expense of another system’s ergotropy [48,49]. For example, when a thermal system is heated up, its energy increases, but its ergotropy remains zero. Conversely, it is possible to purify a system’s state without changing its energy, which means its ergotropy increases while its energy remains fixed.

The main figure of merit we consider is the quality of the transport. It is naturally characterized by how much the channel adds to the ergotropy it transports. We call the added amount *ergotropy gain*. By definition, it is the difference between the amount of ergotropy lost by B and that received by C ,

$$\mathcal{E}_G := \mathcal{E}(\tilde{\rho}_B) + \mathcal{E}(\tilde{\rho}_C) - \mathcal{E}(\rho_B) - \mathcal{E}(\rho_C), \quad (4)$$

where $\rho_B = \text{Tr}_C[\rho_{BC}]$ and $\rho_C = \text{Tr}_B[\rho_{BC}]$ are the marginals of the initial state of BC , ρ_{BC} . Similarly, $\tilde{\rho}_B$ and $\tilde{\rho}_C$ are the marginals of $\tilde{\rho}_{BC} = U_{BC}\rho_{BC}U_{BC}^\dagger$, which is the final, post-transport state of BC . When $\mathcal{E}_G < 0$, namely, when ergotropy is dissipated during transport, we say that the channel is lossy. When $\mathcal{E}_G \geq 0$, it is lossless, and gainful when $\mathcal{E}_G > 0$.

As Eq. (4) suggests, the ergotropy gain can also be interpreted as the difference between final and initial values of the “locally extractable ergotropy.” The latter is the maximal work extractable by unitary operations of the form $U_B \otimes U_C$ and is obviously equal to $\mathcal{E}(\rho_B) + \mathcal{E}(\rho_C)$. Here, “local” refers to quantities pertaining to individual systems (B or C), to contrast with BC -wide (“global”) quantities. This observation allows us to connect the ergotropy gain to the so-called “ergotropic gap” [50], defined as the

difference between the global and locally extractable ergotropies,

$$\delta_{BC}^\mathcal{E} := \mathcal{E}(\rho_{BC}) - \mathcal{E}(\rho_B) - \mathcal{E}(\rho_C) \geq 0. \quad (5)$$

The ergotropic gap, $\delta_{BC}^\mathcal{E}$, gives the amount of work hidden from local operations.

Now, as directly follows from Eq. (1), the action of an energy-conserving unitary on a state does not change its ergotropy. Hence, since U_{BC} is energy conserving, $\mathcal{E}(\tilde{\rho}_{BC}) = \mathcal{E}(\rho_{BC})$. Therefore, the ergotropy gain in Eq. (4) can be cast as a change in the ergotropic gap,

$$\mathcal{E}_G = \delta_{BC}^\mathcal{E} - \tilde{\delta}_{BC}^\mathcal{E}. \quad (6)$$

Thus, a positive ergotropy gain is achieved when the energy-conserving unitary U_{BC} “redistributes” the ergotropy in the total system so that more of the global ergotropy becomes locally extractable. The resource expended for achieving the gain is the ergotropic gap. Note that U_{BC} acts globally, and therefore $\tilde{\rho}_X$ is generally not reachable from ρ_X by local unitary transformations.

An immediate consequence of Eq. (6) and the fact that the ergotropic gap is a nonnegative quantity is the following lemma.

Lemma 1—Ergotropy transport cannot be gainful, i.e., one always has $\mathcal{E}_G \leq 0$, whenever $\delta_{BC}^\mathcal{E} = 0$.

Thus, for gainful transport to be possible, the initial ergotropic gap must be positive. Importantly, the presence of a nonzero ergotropic gap is linked to the correlations in the system [2,50–53]. It can be used to witness entanglement [2,51–53] or quantum discord [50]. In general, however, the relation between ergotropic gap and correlations is not exact: there exist uncorrelated states for which $\delta_{BC}^\mathcal{E} > 0$ and correlated states for which $\delta_{BC}^\mathcal{E} = 0$. An example of the former situation is the phenomenon of activation, where two uncorrelated copies of a passive state ρ turn out nonpassive (“active”) when considered jointly [54,55].

Activation can happen only when $d_{BC} > 4$. When $d_{BC} = 4$, namely, each system is a qubit, all passive states are Gibbs states (i.e., are $\propto e^{-\beta H}$, $\beta > 0$), so no activation is possible [54,55]. In fact, the following lemma shows that *any* two-qubit product state leads to nongainful ergotropy transport.

Lemma 2—For two qubits, one has $\mathcal{E}_G \leq 0$ whenever the initial state is uncorrelated, i.e., $\rho_{BC} = \rho_B \otimes \rho_C$.

Importantly, Lemma 2 does not imply a “deterministic” relation between $\mathcal{E}_G$ and correlations—their presence is not sufficient for $\mathcal{E}_G > 0$. For a pertinent example, as well as a proof of Lemma 2, see the Appendix.

As noted above, one cannot extend Lemma 2 to higher Hilbert-space dimensions (see the Appendix for an explicit example). However, as our extensive numerics show, even in higher dimensions correlations play a central role in

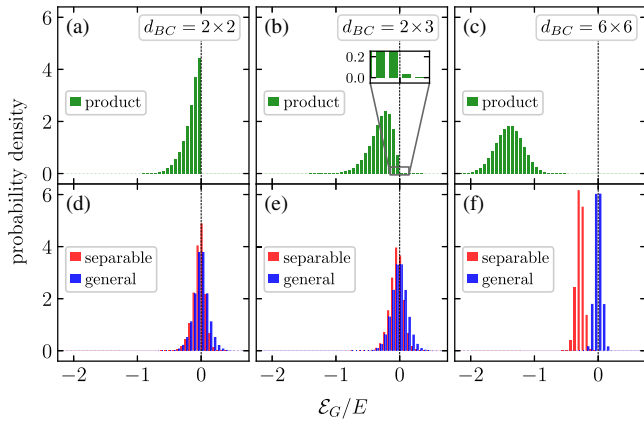


FIG. 2. Distribution of ergotropy gain $\mathcal{E}_G/E$ for randomly sampled initial states, Hamiltonians H_{BC} , and energy-conserving unitaries U_{BC} . In panels (a)–(c), ρ_{BC} is a product of random ρ_B and ρ_C . The states are drawn from the Hilbert–Schmidt measure, H_{BC} are sampled from the Gaussian Unitary Ensemble, and U_{BC} are sampled from the Haar measure. We see that the ensemble average of $\mathcal{E}_G/E$ decreases as the Hilbert space dimension d_{BC} grows. Each of the panels (d), (e), and (f) shows two histograms of $\mathcal{E}_G/E$: one for separable and one for general states of BC (H_{BC} and U_{BC} sampled as above). Each histogram depicts 10^6 points. We see that, in all dimensions, the distance between product and general states is significantly larger than the distance between separable and general states.

ergotropy transport. This is vividly seen in Fig. 2. Panels (a)–(c) are histograms of $\mathcal{E}_G/E$ calculated for a large sample of randomly sampled product states on randomly generated H_{BC} , acted upon by random U_{BC} . The quantity E is a characteristic energy scale of the system. For qubits, E is simply the energy gap. In higher dimensions, however, multiple energy scales exist, and no single scale can be picked out. This makes defining dimensionless energetic quantities more involved, and $\mathcal{E}_G/E$ is a shorthand notation for the dimensionless ergotropy gain (see Supplemental Material [56], Sec. I.B, for details). On panel (a), for B and C being two qubits, we see that $\mathcal{E}_G/E$ never becomes positive, in accordance with Lemma 2. Panels (b) and (c) show that, for higher-dimensional B and C , although $\mathcal{E}_G/E$ is sometimes positive, its bulk becomes more and more negative as d_{BC} grows. We numerically analyze this in detail in Sec. I.D of [56]. We find that the ensemble average of $\mathcal{E}_G/E$ is indeed negative and its magnitude grows polynomially with d_{BC} .

This thus demonstrates that lack of correlations in the initial state is unfavorable for ergotropy transport. This motivates two questions: Which type of correlations matter the most? Is there a quantitative link between their strength and ergotropy gain? Figures 2(d)–2(f) shed light on the first question. Each of these panels contains histograms of two ensembles of initial states: random (general) states of BC and random separable states of BC . As above, here H_{BC} and U_{BC} are also sampled randomly. We see that, for low-dimensional B and C , in terms of ergotropy gain, separable states perform almost as well as all possible bipartite states,

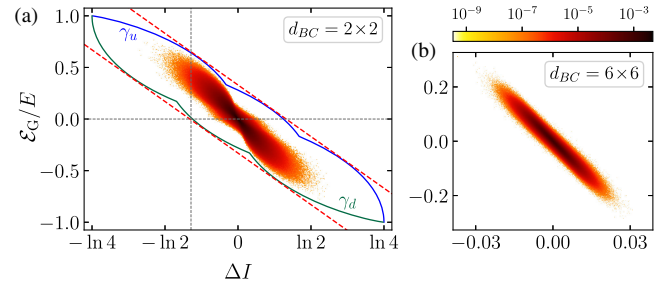


FIG. 3. 2D histogram of $\mathcal{E}_G/E$ vs ΔI for general ρ_{BC} , H_{BC} , and U_{BC} randomly generated as in Fig. 2. Panel (a) is for ergotropy transport between two qubits. Here, the points distribute in a “propeller shape.” The blue and green curves γ_u and γ_d provide tight upper and lower analytical bounds on the distribution. The cumbersome closed-form expressions for them are derived in Sec. IV of [56]. The red dashed lines, which are the tightest linear bounds, show that $\mathcal{E}_G/E$ is guaranteed to be positive whenever $\Delta I < -2 \ln(5/4)$; the dotted gray lines cross at this turning point. Panel (b) illustrates what happens in higher dimensions, showcasing the $d_B = d_C = 6$ case. Compared to panel (a), the distribution is more concentrated around 0. This reflects the fact that both $\mathcal{E}_G/E$ and ΔI asymptotically concentrate around their ensemble averages.

including the entangled ones. In higher dimensions, the distance between the bulks of separable and general states is much smaller than the distance between general and product states. Thus, although entangled states are more advantageous for ergotropy transport, especially in high dimensions, the presence of any type of correlations has a much bigger impact on $\mathcal{E}_G/E$ than their type. Note that we plot separable vs general, and not separable vs entangled, because efficiently characterizing the set of entangled states is impossible for $d_{BC} > 6$ [83]. However, it is possible to sample the separable set in arbitrary dimensions. In Sec. I of [56], based on a pure-state decomposition of separable states from Ref. [84], we construct an efficient algorithm of generating random separable states that reasonably uniformly cover the whole separable set.

Now, having determined that the general presence of correlations is the primary factor affecting the ergotropy gain, we are ready to answer the second question above. To do so, we use the standard quantifier of “general” (or “total”) correlations—the quantum mutual information [85]: $I(\rho_{BC}) = S(\rho_B) + S(\rho_C) - S(\rho_{BC})$. Since $\mathcal{E}_G$ is given by the change in the ergotropy gap [see Eq. (6)], we will compare $\mathcal{E}_G$ with the *change* in the mutual information during transport: $\Delta I = I(\tilde{\rho}_{BC}) - I(\rho_{BC})$. In view of the unitary invariance of the von Neumann entropy, we have

$$\Delta I = S(\tilde{\rho}_B) + S(\tilde{\rho}_C) - S(\rho_B) - S(\rho_C). \quad (7)$$

To reveal how the two quantities are related, we, as above, randomly sample a large ensemble of ρ_{BC} , H_{BC} , and U_{BC} , and plot $\mathcal{E}_G/E$ against ΔI . This is done in Fig. 3, where 2×2 [panel (a)] and 6×6 [panel (b)] dimensional cases are

depicted. For the two qubits, it reveals a propellerlike shape for the distribution. Its width is nonzero; therefore there is no deterministic relation between ΔI and $\mathcal{E}_G/E$.

In higher dimensions, the ranges of ΔI and $\mathcal{E}_G/E$ shrink, as is evidenced by Fig. 3(b). In Sec. II of [56], we show that this phenomenon is general by proving Levy-type measure-concentration bounds [86,87] for both $\mathcal{E}_G/E$ and ΔI considered as functions of the initial state ρ_{BC} . There, and throughout the Letter, we sample the states from the Hilbert–Schmidt measure [88,89]. Our proof is based on the fact that both the ergotropy and von Neumann entropy, as functions of the state, are Lipschitz continuous with respect to the Bures metric [90,91]. Our concentration bounds establish that the widths of the distributions of ΔI and $\mathcal{E}_G/E$ decrease polynomially with d_B and d_C . Furthermore, as we numerically demonstrate in Sec. III of [56], the relation between ΔI and $\mathcal{E}_G/E$ becomes increasingly “deterministic” even when adjusted for the shrinkage due to concentration.

The concentration of ΔI and $\mathcal{E}_G$ means that, on the one hand, tailoring initial states and transfer unitaries such that ergotropy gain is significant becomes increasingly more complex as the dimensions grow higher. On the other hand, generic transport is robust in that both losses (and gains) are severely restricted.

Multiple transports and channel reusability—As we saw above, correlations generically endow the initial state of BC with a positive ergotropic gap $\delta_{BC}^{\mathcal{E}}$, allowing one to avoid losses during transport. As Eq. (6) shows, each time $\mathcal{E}_G > 0$, this resource gets used up by that exact amount. While for a single transport having the largest $\mathcal{E}_G$ is best, the situation changes when one wishes to perform multiple transports.

The latter is arguably the most relevant practical scenario, where our setup is a wire through which third parties send ergotropy to fourth parties, aiming for the highest number of lossless transports. Thus, having $\mathcal{E}_G$ *positive but small* would be desirable, so that the initial supply of $\delta_{BC}^{\mathcal{E}}$ is consumed slowly. A key quantity here is the total amount of ergotropy, $\mathcal{E}_{\text{tot}}^+$, that one can losslessly transport through the channel in multiple iterations. Labeling transport iterations by the index ι ,

$$\mathcal{E}_{\text{tot}}^+ := \sum_{\iota: \mathcal{E}_G^{(\iota)} \geq 0} [\mathcal{E}(\tilde{\rho}_C^{(\iota)}) - \mathcal{E}(\rho_C^{(\iota)})]. \quad (8)$$

A paradigmatic protocol realizing this scenario is as follows. The system starts locally empty: both $\rho_B^{(0)}$ and $\rho_C^{(0)}$ are passive [$\rho_{BC}^{(0)}$ may be correlated]. Then, a “charging” unitary U_{ch} acts locally on B , injecting some ergotropy into it. Next, an energy-conserving U_{BC} is performed on the total system, transporting ergotropy from B to C . Finally, a “draining” unitary U_{dr} acts locally on C , extracting (part of) its ergotropy. This comprises a full “transport cycle” of the

channel, and is realized by the unitary operation

$$U_{\text{cyc}} = (\mathbb{1}_B \otimes U_{\text{dr}})U_{BC}(U_{\text{ch}} \otimes \mathbb{1}_C). \quad (9)$$

As noted above, the ergotropic gap $\delta^{(0)} := \delta_{BC}^{\mathcal{E}}(\rho_{BC}^{(0)})$ of the initial “locally empty” state is *the* resource that will fuel the gainfulness of the transport and be consumed over time. Since the ergotropic gap depends only on the local and global spectra of the state [see Eq. (A2) in the Appendix], the consumption of $\delta^{(0)}$ is only determined by U_{BC} .

For a transparent exemplification of this situation, we again turn to qubits and take $\rho_{BC}^{(0)} = |0\rangle_B \langle 0| \otimes |0\rangle_C \langle 0|$, $U_{\text{ch}} = U_{\text{dr}} = \sigma_X$, and $U_{BC} = \text{SWAP}$, where σ_X is the Pauli X matrix. In this idealized scenario, the state returns to $\rho_{BC}^{(0)}$ after each transport cycle, and although $\mathcal{E}_G = 0$, an E amount of ergotropy is injected into B and the same E amount of ergotropy is extracted from C each time. Thus, this noiseless, idealized channel can be used infinitely many times to losslessly transport an infinite amount of ergotropy.

We caution, however, that this behavior is not stable: a small deviation of U_{BC} from a swap will make the transport lossy. Indeed, let us introduce a small ε error in the transport unitary,

$$U_{BC}^{(\varepsilon)} = \text{SWAP} + 2\nu_{BC}^{(\varepsilon)} \sin(\varepsilon/2). \quad (10)$$

Here, $\nu_{BC}^{(\varepsilon)} = \bar{\sigma}_Z \cos(\varepsilon/2) - \bar{\sigma}_X \sin(\varepsilon/2)$, where $\bar{\sigma}_X$ and $\bar{\sigma}_Z$ are the Pauli X and Z matrices in the subspace spanned by $|0\rangle_B \otimes |1\rangle_C$ and $|1\rangle_B \otimes |0\rangle_C$. Keeping $U_{\text{ch}} = U_{\text{dr}} = \sigma_X$, it is easy to calculate that, upon iteration ι of the transport cycle, the ergotropy gain $\mathcal{E}_G^{(\iota)} = -2E \sin(2\iota\varepsilon - \varepsilon) \sin \varepsilon$, which is negative for $1 \leq \iota \leq \pi/(2\varepsilon) + 1/2$.

To combat this lossiness, let us take an initial state with a positive ergotropic gap. For example, consider $\rho_{BC}^{(0)} = |\phi^{(0)}\rangle \langle \phi^{(0)}|$ with $|\phi^{(0)}\rangle = \cos\kappa|0\rangle_B \otimes |0\rangle_C + \sin\kappa|1\rangle_B \otimes |1\rangle_C$, where $0 < \kappa \leq \pi/4$ controls the initial ergotropic gap: $\delta^{(0)} = 2E \sin^2\kappa > 0$. In Sec. V of [56], we show that, in this case,

$$\mathcal{E}_G^{(\iota)} = 2E \sin(2\kappa + \varepsilon - 2\iota\varepsilon) \sin \varepsilon. \quad (11)$$

This quantity is positive for the first $\lfloor \frac{\kappa}{\varepsilon} + \frac{1}{2} \rfloor$ iterations ($\lfloor x \rfloor$ is the floor function). For example, for $\varepsilon = 0.03$ and $\kappa = \pi/8$, the transport is gainful during the first 13 iterations. As a result, $\mathcal{E}_{\text{tot}}^+ \approx 11.84E$ is collected from C . Thus, a single slightly correlated state can be used to losslessly transport an order of magnitude more work ($\mathcal{E}_{\text{tot}}^+$) than it can store (E).

We emphasize that, although the transport is executed multiple times, the correlated state $\rho_{BC}^{(0)}$ is prepared only once. This is important because correlations rarely come for free—creating them costs energy [92–95]. For the example

above, the work cost of creating $|\phi^{(0)}\rangle$ from the uncorrelated state $|0\rangle_B \otimes |0\rangle_C$ by unitary operations is $2E \sin^2 \kappa = \delta^{(0)} \approx 0.29E$. And it is remarkable that this small investment unlocks ≈ 40.82 times as much losslessly transportable ergotropy (recall that $\mathcal{E}_{\text{tot}}^+ \approx 11.84E$).

Lastly, we note that our multiple transfer setup is fundamentally different from that considered in Ref. [26]. Using our nomenclature, only uncorrelated, incoherent (diagonal) initial states are considered in Ref. [26], and a fresh battery is used upon each iteration.

Summary and outlook—We introduced a paradigmatic setup of a quantum battery autonomously transporting ergotropy to a quantum device (consumption unit). We established that initial battery–device correlations can significantly enhance the transport. Despite the channel being energy-conserving, the device may get more ergotropy than the battery loses. Such gainful transport is counterintuitive, as the natural expectation is that, when transported, a useful resource should suffer dissipation.

We proved that, between two qubits, gainful transport is possible *only* if they are initially correlated. Although for higher-dimensional systems the correlation–ergotropy–gain relation becomes fuzzier, we showed that there, too, correlations maintain a central role. We also established that ΔI and $\mathcal{E}_G$ concentrate with increasing dimensions, meaning that ergotropy transport between large systems is generically robust—while large gains are not to be expected, large losses are also unlikely.

Finally, we showed that a one-time creation of correlations between B and C can have a long-lasting loss prevention effect when, subsequently, multiple iterations of an imperfect channel transport ergotropy through BC . Even a modest amount of correlations between B and C can enable lossless transport of an order of magnitude more work than the energetic resources needed to create the initial correlated state.

The framework developed here may serve as a stepping stone for further research into ergotropy transport under realistic conditions. A significant step would be accounting for environmental noise during transport, e.g., by considering nonunitary energy-conserving channels [38].

Acknowledgments—We thank Carsten Henkel and Marcin Łobejko for useful discussions. K. V. H. and J. A. are grateful for support from the University of Potsdam. J. A. gratefully acknowledges funding from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Grants No. 384846402 and No. 513075417 and from the Engineering and Physical Sciences Research Council (EPSRC) (Grant No. EP/R045577/1) and thanks the Royal Society for support.

Data availability—The code used to produce the data shown in Figs. 2 and 3 and all the figures in the Supplemental Material [56] is available upon reasonable request to K. V. H.

- [1] R. Alicki and M. Fannes, Entanglement boost for extractable work from ensembles of quantum batteries, *Phys. Rev. E* **87**, 042123 (2013).
- [2] M. Perarnau-Llobet, K. V. Hovhannisyan, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, Extractable work from correlations, *Phys. Rev. X* **5**, 041011 (2015).
- [3] G. M. Andolina, M. Keck, A. Mari, M. Campisi, V. Giovannetti, and M. Polini, Extractable work, the role of correlations, and asymptotic freedom in quantum batteries, *Phys. Rev. Lett.* **122**, 047702 (2019).
- [4] S. Tirone, R. Salvia, S. Chessa, and V. Giovannetti, Quantum work capacitances: Ultimate limits for energy extraction on noisy quantum batteries, *SciPost Phys.* **17**, 041 (2024).
- [5] X. Yang, Y.-H. Yang, M. Alimuddin, R. Salvia, S.-M. Fei, L.-M. Zhao, S. Nimmrichter, and M.-X. Luo, Battery capacity of energy-storing quantum systems, *Phys. Rev. Lett.* **131**, 030402 (2023).
- [6] K. V. Hovhannisyan, M. Perarnau-Llobet, M. Huber, and A. Acín, Entanglement generation is not necessary for optimal work extraction, *Phys. Rev. Lett.* **111**, 240401 (2013).
- [7] F. C. Binder, S. Vinjanampathy, K. Modi, and J. Goold, Quantacell: Powerful charging of quantum batteries, *New J. Phys.* **17**, 075015 (2015).
- [8] F. Campaioli, F. A. Pollock, F. C. Binder, L. Céleri, J. Goold, S. Vinjanampathy, and K. Modi, Enhancing the charging power of quantum batteries, *Phys. Rev. Lett.* **118**, 150601 (2017).
- [9] D. Ferraro, M. Campisi, G. M. Andolina, V. Pellegrini, and M. Polini, High-power collective charging of a solid-state quantum battery, *Phys. Rev. Lett.* **120**, 117702 (2018).
- [10] J.-Y. Gyhm, D. Šafránek, and D. Rosa, Quantum charging advantage cannot be extensive without global operations, *Phys. Rev. Lett.* **128**, 140501 (2022).
- [11] J.-Y. Gyhm and U. R. Fischer, Beneficial and detrimental entanglement for quantum battery charging, *AVS Quantum Sci.* **6**, 012001 (2024).
- [12] A. C. Santos, B. Çakmak, S. Campbell, and N. T. Zinner, Stable adiabatic quantum batteries, *Phys. Rev. E* **100**, 032107 (2019).
- [13] D. Farina, G. M. Andolina, A. Mari, M. Polini, and V. Giovannetti, Charger-mediated energy transfer for quantum batteries: An open-system approach, *Phys. Rev. B* **99**, 035421 (2019).
- [14] F. Barra, Dissipative charging of a quantum battery, *Phys. Rev. Lett.* **122**, 210601 (2019).
- [15] J. Liu, D. Segal, and G. Hanna, Loss-free excitonic quantum battery, *J. Phys. Chem. C* **123**, 18303 (2019).
- [16] F. Pirmoradian and K. Mølmer, Aging of a quantum battery, *Phys. Rev. A* **100**, 043833 (2019).
- [17] K. V. Hovhannisyan, F. Barra, and A. Imparato, Charging assisted by thermalization, *Phys. Rev. Res.* **2**, 033413 (2020).
- [18] S. Gherardini, F. Campaioli, F. Caruso, and F. C. Binder, Stabilizing open quantum batteries by sequential measurements, *Phys. Rev. Res.* **2**, 013095 (2020).
- [19] J. Q. Quach and W. J. Munro, Using dark states to charge and stabilize open quantum batteries, *Phys. Rev. Appl.* **14**, 024092 (2020).
- [20] F. H. Kamin, F. T. Tabesh, S. Salimi, F. Kheirandish, and A. C. Santos, Non-Markovian effects on charging

- and self-discharging process of quantum batteries, *New J. Phys.* **22**, 083007 (2020).
- [21] F. Barra, K. V. Hovhannissyan, and A. Imparato, Quantum batteries at the verge of a phase transition, *New J. Phys.* **24**, 015003 (2022).
- [22] W.-L. Song, H.-B. Liu, B. Zhou, W.-L. Yang, and J.-H. An, Remote charging and degradation suppression for the quantum battery, *Phys. Rev. Lett.* **132**, 090401 (2024).
- [23] F. Campaioli, S. Gherardini, J. Q. Quach, M. Polini, and G. M. Andolina, Colloquium: Quantum batteries, *Rev. Mod. Phys.* **96**, 031001 (2024).
- [24] J. Monsel, M. Fellous-Asiani, B. Huard, and A. Auffèves, The energetic cost of work extraction, *Phys. Rev. Lett.* **124**, 130601 (2020).
- [25] S. Tirone, R. Salvia, and V. Giovannetti, Quantum energy lines and the optimal output ergotropy problem, *Phys. Rev. Lett.* **127**, 210601 (2021).
- [26] M. Łobejko, P. Mazurek, and M. Horodecki, The asymptotic emergence of the second law for a repeated charging process, [arXiv:2209.05339](https://arxiv.org/abs/2209.05339).
- [27] D. Gelbwaser-Klimovsky, R. Alicki, and G. Kurizki, Work and energy gain of heat-pumped quantized amplifiers, *Europhys. Lett.* **103**, 60005 (2013).
- [28] W. Niedenzu, D. Gelbwaser-Klimovsky, A. G. Kofman, and G. Kurizki, On the operation of machines powered by quantum non-thermal baths, *New J. Phys.* **18**, 083012 (2016).
- [29] S. Seah, S. Nimmrichter, and V. Scarani, Work production of quantum rotor engines, *New J. Phys.* **20**, 043045 (2018).
- [30] W. Niedenzu, M. Huber, and E. Boukobza, Concepts of work in autonomous quantum heat engines, *Quantum* **3**, 195 (2019).
- [31] B. Çakmak, Ergotropy from coherences in an open quantum system, *Phys. Rev. E* **102**, 042111 (2020).
- [32] S. Seah, M. Perarnau-Llobet, G. Haack, N. Brunner, and S. Nimmrichter, Quantum speed-up in collisional battery charging, *Phys. Rev. Lett.* **127**, 100601 (2021).
- [33] A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, Maximal work extraction from finite quantum systems, *Europhys. Lett.* **67**, 565 (2004).
- [34] M. H. Partovi, Quantum thermodynamics, *Phys. Lett. A* **137**, 440 (1989).
- [35] A. B. Brailovskii, V. L. Vaks, and V. V. Mityugov, Quantum models of relaxation, *Phys. Usp.* **39**, 745 (1996).
- [36] D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth, Thermodynamic cost of reliability and low temperatures: Tightening Landauer's principle and the second law, *Int. J. Theor. Phys.* **39**, 2717 (2000).
- [37] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource theory of quantum states out of thermal equilibrium, *Phys. Rev. Lett.* **111**, 250404 (2013).
- [38] G. Chiribella and Y. Yang, Optimal quantum operations at zero energy cost, *Phys. Rev. A* **96**, 022327 (2017).
- [39] J. Åberg, Catalytic coherence, *Phys. Rev. Lett.* **113**, 150402 (2014).
- [40] J. A. Vaccaro, S. Croke, and S. M. Barnett, Is coherence catalytic?, *J. Phys. A* **51**, 414008.
- [41] M. P. Woods, R. Silva, and J. Oppenheim, Autonomous quantum machines and finite-sized clocks, *Ann. Henri Poincaré* **20**, 125 (2019).
- [42] M. H. Partovi, Entanglement versus stosszahlansatz: Disappearance of the thermodynamic arrow in a high-correlation environment, *Phys. Rev. E* **77**, 021110 (2008).
- [43] D. Jennings and T. Rudolph, Entanglement and the thermodynamic arrow of time, *Phys. Rev. E* **81**, 061130 (2010).
- [44] S. Jevtic, D. Jennings, and T. Rudolph, Maximally and minimally correlated states attainable within a closed evolving system, *Phys. Rev. Lett.* **108**, 110403 (2012).
- [45] I. Henao and R. M. Serra, Role of quantum coherence in the thermodynamics of energy transfer, *Phys. Rev. E* **97**, 062105 (2018).
- [46] G. M. Andolina, D. Farina, A. Mari, V. Pellegrini, V. Giovannetti, and M. Polini, Charger-mediated energy transfer in exactly solvable models for quantum batteries, *Phys. Rev. B* **98**, 205423 (2018).
- [47] P. Lipka-Bartosik, G. F. D'otalevi, and P. Bakhshinezhad, Fundamental limits on anomalous energy flows in correlated quantum systems, *Phys. Rev. Lett.* **132**, 140402 (2024).
- [48] M. Łobejko, P. Mazurek, and M. Horodecki, Thermodynamics of minimal coupling quantum heat engines, *Quantum* **4**, 375 (2020).
- [49] T. Biswas, M. Łobejko, P. Mazurek, K. Jałowiecki, and M. Horodecki, Extraction of ergotropy: Free energy bound and application to open cycle engines, *Quantum* **6**, 841 (2022).
- [50] A. Mukherjee, A. Roy, S. S. Bhattacharya, and M. Banik, Presence of quantum correlations results in a nonvanishing ergotropic gap, *Phys. Rev. E* **93**, 052140 (2016).
- [51] M. Alimuddin, T. Guha, and P. Parashar, Bound on ergotropic gap for bipartite separable states, *Phys. Rev. A* **99**, 052320 (2019).
- [52] K. Sen and U. Sen, Local passivity and entanglement in shared quantum batteries, *Phys. Rev. A* **104**, L030402 (2021).
- [53] S. Puliyl, M. Banik, and M. Alimuddin, Thermodynamic signatures of genuinely multipartite entanglement, *Phys. Rev. Lett.* **129**, 070601 (2022).
- [54] W. Pusz and S. L. Woronowicz, Passive states and KMS states for general quantum systems, *Commun. Math. Phys.* **58**, 273 (1978).
- [55] A. Lenard, Thermodynamical proof of the Gibbs formula for elementary quantum systems, *J. Stat. Phys.* **19**, 575 (1978).
- [56] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.134.010408>, which includes Refs. [57–82], for further details on the relation of ergotropy transport with quantum marginal problem, specifics of our numerical simulations, proof of concentration bounds for $\mathcal{E}_G/E$ and ΔI , numerics for higher-dimensional systems, bounds on $\mathcal{E}_G$ for two qubits, and calculations for the multiple transport scenario.
- [57] V. Trávníček, K. Bartkiewicz, A. Černoč, and K. Lemr, Experimental measurement of the Hilbert–Schmidt distance between two-qubit states as a means for reducing the complexity of machine learning, *Phys. Rev. Lett.* **123**, 260501 (2019).
- [58] N. Liu and P. Wittek, Vulnerability of quantum classification to adversarial perturbations, *Phys. Rev. A* **101**, 062331 (2020).
- [59] M. Cerezo, A. Arrasmith, R. Babbush, S. C. Benjamin, S. Endo, K. Fujii, J. R. McClean, K. Mitarai, X. Yuan,

- L. Cincio, and P. J. Coles, Variational quantum algorithms, *Nat. Rev. Phys.* **3**, 625 (2021).
- [60] V. V. Dodonov, O. V. Man'ko, V. I. Man'ko, and A. Wünsche, Hilbert–Schmidt distance and non-classicality of states in quantum optics, *J. Mod. Opt.* **47**, 633 (2000).
- [61] P. Pandya, O. Sakarya, and M. Wieśniak, Hilbert–Schmidt distance and entanglement witnessing, *Phys. Rev. A* **102**, 012409 (2020).
- [62] D. Pérez-García, M. M. Wolf, D. Petz, and M. B. Ruskai, Contractivity of positive and trace-preserving maps under l_p norms, *J. Math. Phys. (N.Y.)* **47**, 083506 (2006).
- [63] M. Ozawa, Entanglement measures and the Hilbert–Schmidt distance, *Phys. Lett. A* **268**, 158 (2000).
- [64] F. Mezzadri, How to generate random matrices from the classical compact groups, *Not. Am. Math. Soc.* **54**, 592 (2007).
- [65] D. Bures, An extension of Kakutani's theorem on infinite product measures to the tensor product of semifinite w^* -algebras, *Trans. Am. Math. Soc.* **135**, 199 (1969).
- [66] A. Uhlmann, The “transition probability” in the state space of a $*$ -algebra, *Rep. Math. Phys.* **9**, 273 (1976).
- [67] D. Petz, Monotone metrics on matrix spaces, *Linear Algebra Appl.* **244**, 81 (1996).
- [68] S. L. Braunstein and C. M. Caves, Statistical distance and the geometry of quantum states, *Phys. Rev. Lett.* **72**, 3439 (1994).
- [69] S.-i. Amari and H. Nagaoka, *Methods of Information Geometry* (Oxford University Press, New York, 2000).
- [70] D. Petz and C. Sudár, Geometries of quantum states, *J. Math. Phys. (N.Y.)* **37**, 2662 (1996).
- [71] H.-J. Sommers and K. Życzkowski, Bures volume of the set of mixed quantum states, *J. Phys. A* **36**, 10083 (2003).
- [72] P. B. Slater, Quantum Fisher–Bures information of two-level systems and a three-level extension, *J. Phys. A* **29**, L271 (1996).
- [73] P. B. Slater, Comparative noninformativities of quantum priors based on monotone metrics, *Phys. Lett. A* **247**, 1 (1998).
- [74] J. Dittmann, On the Riemannian metric on the space of density matrices, *Rep. Math. Phys.* **36**, 309 (1995).
- [75] M. J. W. Hall, Random quantum correlations and density operator distributions, *Phys. Lett. A* **242**, 123 (1998).
- [76] A. Sarkar and S. Kumar, Bures–Hall ensemble: Spectral densities and average entropies, *J. Phys. A* **52**, 295203 (2019).
- [77] P. B. Slater, Numerical and exact analyses of Bures and Hilbert–Schmidt separability and PPT probabilities, *Quantum Inf. Process.* **18**, 312 (2019).
- [78] L. Wei and N. Witte, Quantum interpolating ensemble: Bi-orthogonal polynomials and average entropies, *Random Matrices Theory Appl.* **12**, 2250055 (2023).
- [79] A. R. Willms, Uniform sampling on the standard simplex, *Missouri J. Math. Sci.* **33**, 119 (2021).
- [80] L. Devroye, *Non-Uniform Random Variate Generation* (Springer-Verlag, New York, 1986).
- [81] A. Edelman and N. R. Rao, Random matrix theory, *Acta Numerica* **14**, 233 (2005).
- [82] G. T. Toussaint, Solving geometric problems with the rotating calipers, in *Proceedings of the MELECON'83, Athens, Greece* (1983).
- [83] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* **81**, 865 (2009).
- [84] P. Horodecki, Separability criterion and inseparable mixed states with positive partial transposition, *Phys. Lett. A* **232**, 333 (1997).
- [85] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010).
- [86] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Mathematics Vol. 1200 (Springer, Berlin, 2001).
- [87] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).
- [88] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, New York, 2006).
- [89] K. Życzkowski and H.-J. Sommers, Induced measures in the space of mixed quantum states, *J. Phys. A* **34**, 7111 (2001).
- [90] P. Sekatski, J.-D. Bancal, X. Valcarce, E. Y.-Z. Tan, R. Renner, and N. Sangouard, Device-independent quantum key distribution from generalized CHSH inequalities, *Quantum* **5**, 444 (2021).
- [91] K. V. Hovhannisyanyan, R. P. A. Simon, and J. Anders, Concentration of ergotropy in many-body systems, *arXiv*: 2412.19801.
- [92] M. Huber, M. Perarnau-Llobet, K. V. Hovhannisyanyan, P. Skrzypczyk, C. Klöckl, N. Brunner, and A. Acín, Thermodynamic cost of creating correlations, *New J. Phys.* **17**, 065008 (2015).
- [93] M. N. Bera, A. Riera, M. Lewenstein, and A. Winter, Generalized laws of thermodynamics in the presence of correlations, *Nat. Commun.* **8**, 2180 (2017).
- [94] G. Vitagliano, C. Klöckl, M. Huber, and N. Friis, Trade-off between work and correlations in quantum thermodynamics, in *Thermodynamics in the Quantum Regime* (Springer International Publishing, New York, 2018), pp. 731–750.
- [95] F. Bakhshinezhad, F. Clivaz, G. Vitagliano, P. Erker, A. Reza khani, M. Huber, and N. Friis, Thermodynamically optimal creation of correlations, *J. Phys. A* **52**, 465303 (2019).
- [96] B. Dakić, V. Vedral, and Č. Brukner, Necessary and sufficient condition for nonzero quantum discord, *Phys. Rev. Lett.* **105**, 190502 (2010).
- [97] S. Bravyi, Requirements for compatibility between local and multipartite quantum states, *Quantum Inf. Comput.* **4**, 12 (2004).
- [98] A. A. Klyachko, Quantum marginal problem and N -representability, *J. Phys. Conf. Ser.* **36**, 72 (2006).
- [99] M. Christandl and G. Mitchison, The spectra of quantum states and the Kronecker coefficients of the symmetric group, *Commun. Math. Phys.* **261**, 789 (2005).

End Matter

Appendix: Proof of Lemma 2—Before proceeding to the core proof of Lemma 2, let us first show that the ergotropic gap $\delta_{BC}^{\mathcal{E}}$ depends only on the spectra of the global and local states whenever B and C do not interact [i.e., the total Hamiltonian is of the form given by Eq. (3)]. This important property of the ergotropic gap will become useful in what follows. To prove it, let us first invoke the well-known fact that the minimum in Eq. (1) is delivered by the unitary that rotates ρ to the so-called passive state $\rho^\downarrow$ [33,54,55], and thus

$$\mathcal{E}(\rho) = \text{Tr}[(\rho - \rho^\downarrow)H]. \quad (\text{A1})$$

The passive state $\rho^\downarrow = \sum_k \lambda_k |E_k\rangle\langle E_k|$, where $|E_k\rangle$ are the eigenvectors of H for which the corresponding eigenvalues are ordered increasingly ($E_0 \leq \dots \leq E_{d-1}$) and $\lambda_0 \geq \dots \geq \lambda_{d-1} \geq 0$ are the eigenvalues of ρ in decreasing order. Note that, whenever ρ or H have degenerate spectra, $\rho^\downarrow$ will not be unique.

Now, using Eq. (A1), it is straightforward to show that

$$\delta_{BC}^{\mathcal{E}} = \text{Tr}[(\rho_B^\downarrow \otimes \rho_C^\downarrow - \rho_{BC}^\downarrow)H_{BC}], \quad (\text{A2})$$

where $\rho_{BC}^\downarrow$ is passive with respect to H_{BC} and $\rho_X^\downarrow$ is passive with respect to H_X ($X = B, C$).

To prove Lemma 2, let us without loss of generality set the energies of the ground states of B and C , $|0\rangle_B$ and $|0\rangle_C$, to zero. With this convention, the Hamiltonians write as $H_B = E_B |1\rangle_B\langle 1|$ and $H_C = E_C |1\rangle_C\langle 1|$.

There are two qualitatively different cases with regard to the structure of the energy-conserving unitary: $E_B \neq E_C$ and $E_B = E_C := E$. We treat them separately.

Case 1— $E_B \neq E_C$. In this case, $H_{BC} = H_B \otimes \mathbb{1}_C + \mathbb{1}_B \otimes H_C$ is nondegenerate, and therefore only operators that are diagonal in the energy eigenbasis can satisfy Eq. (2) in the main text. Thus, in the energy eigenbasis, energy-conserving unitaries are necessarily of the form

$$U_{BC} = \text{diag}(e^{i\phi_0}, e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}). \quad (\text{A3})$$

It is now a simple exercise to check that the elements of the post-transfer state of B , $\tilde{\rho}_B = \text{Tr}_C[U_{BC}\rho_B \otimes \rho_C U_{BC}^\dagger]$, are

$$\begin{aligned} (\tilde{\rho}_B)_{00} &= (\rho_B)_{00}, \\ (\tilde{\rho}_B)_{01} &= (\rho_B)_{01}[(\rho_C)_{00}e^{i(\phi_0-\phi_2)} + (\rho_C)_{11}e^{i(\phi_1-\phi_3)}]. \end{aligned}$$

Due to the triangle inequality,

$$|(\rho_C)_{00}e^{i(\phi_0-\phi_2)} + (\rho_C)_{11}e^{i(\phi_1-\phi_3)}| \leq (\rho_C)_{00} + (\rho_C)_{11} = 1.$$

Therefore,

$$|(\tilde{\rho}_B)_{01}| \leq |(\rho_B)_{01}|. \quad (\text{A4})$$

Similarly, for $\tilde{\rho}_C = \text{Tr}_B[U_{BC}\rho_B \otimes \rho_C U_{BC}^\dagger]$, we have

$$\begin{aligned} (\tilde{\rho}_C)_{00} &= (\rho_C)_{00}, \\ (\tilde{\rho}_C)_{01} &= (\rho_C)_{01}[(\rho_B)_{00}e^{i(\phi_0-\phi_1)} + (\rho_B)_{11}e^{i(\phi_2-\phi_3)}], \end{aligned}$$

and

$$|(\tilde{\rho}_C)_{01}| \leq |(\rho_C)_{01}|. \quad (\text{A5})$$

Now, let us recall that the ergotropy of a qubit with the Hamiltonian $E|1\rangle\langle 1|$ is

$$\mathcal{E}(\rho) = E\left(\frac{1}{2} - \rho_{00} + \sqrt{\frac{1}{4} - \rho_{00}\rho_{11} + |\rho_{01}|^2}\right). \quad (\text{A6})$$

Thus, $\mathcal{E}$ is a monotonically increasing function of $|\rho_{01}|$. Therefore, in view of Eqs. (A4) and (A5),

$$\mathcal{E}(\tilde{\rho}_B) \leq \mathcal{E}(\rho_B) \quad \text{and} \quad \mathcal{E}(\tilde{\rho}_C) \leq \mathcal{E}(\rho_C), \quad (\text{A7})$$

which in turn means that, according to its definition in Eq. (4) in the main text, the ergotropy gain $\mathcal{E}_G \leq 0$.

Case 2— $E_B = E_C := E$. Here, the spectrum of H_{BC} is degenerate— $\text{Spec}(H_{BC}) = \{0, E, E, 2E\}$ —and hence there exist nontrivial energy-conserving unitaries. However, in this case, $\rho_B^\downarrow \otimes \rho_C^\downarrow$ is passive with respect to H_{BC} , and therefore $\delta_{BC}^{\mathcal{E}} = 0$ due to Eq. (A2). Indeed, denoting the eigenvalues of ρ_B and ρ_C by, respectively, $\lambda_0^B \geq \lambda_1^B$ and $\lambda_0^C \geq \lambda_1^C$, we have that

$$\rho_B^\downarrow \otimes \rho_C^\downarrow = \text{diag}(\lambda_0^B \lambda_0^C, \lambda_0^B \lambda_1^C, \lambda_1^B \lambda_0^C, \lambda_1^B \lambda_1^C)$$

in the energy eigenbasis. And since $\lambda_0^B \lambda_0^C \geq \lambda_0^B \lambda_1^C \geq \lambda_1^B \lambda_1^C$ and $\lambda_0^B \lambda_0^C \geq \lambda_1^B \lambda_0^C \geq \lambda_1^B \lambda_1^C$, this state is passive with respect to $H_{BC} = \text{diag}(0, E, E, 2E)$. Thus, due to Lemma 1, ergotropy transport by energy-conserving unitaries acting on product states cannot be gainful. This concludes the proof of Lemma 2.

Two remarks are due here. First, while Lemma 2 proves that correlations are necessary for $\mathcal{E}_G > 0$, their presence is not sufficient for achieving positive ergotropy gain. For example, when $E_B = E_C := E$, the state

$$\frac{1}{2}|0\rangle_B\langle 0| \otimes |0\rangle_C\langle 0| + \frac{1}{2}|\Psi^+\rangle\langle \Psi^+|,$$

where $|\Psi^+\rangle = (|0\rangle_B \otimes |1\rangle_C + |1\rangle_B \otimes |0\rangle_C)/\sqrt{2}$, is entangled but has zero ergotropic gap. Note that, since entangled states always have nonzero quantum discord

[96], this example may appear to contradict the claim of Ref. [50] that states with nonzero discord have positive ergotropic gap. However, there is no contradiction: in our case, the qubits have identical energy spectra, whereas the result of Ref. [50] applies only when the spectra are different.

Second, due to the activation phenomenon mentioned in the main text, we know that Lemma 2 cannot be directly generalized to higher dimensions. The minimal counterexample is found in 2×3 dimensions. Take B and C with Hamiltonians $H_B = E|1\rangle_B\langle 1|$ and $H_C = E|1\rangle_C\langle 1| + E|2\rangle_C\langle 2|$. Then, $\rho_B \otimes \rho_C$, with

$\rho_B = \frac{1}{2}1_B$ and $\rho_C = \frac{1}{2}|0\rangle_C\langle 0| + \frac{1}{2}|1\rangle_C\langle 1|$, is not passive despite both ρ_B and ρ_C being passive. In the extreme case when C is a large thermal bath, a natural family of such counterexamples was identified in Ref. [49].

Finally, in the Sec. VI of [56], we provide an alternative proof of Lemma 2 by making a connection to the so-called quantum marginal problem [97–99] and using its solution for two qubits [97]. In fact, we go even deeper and show that this connection can be useful in both ways by deriving a family of nontrivial inequalities for quantum marginals *in arbitrary dimensions* from the simple fact that $\delta_{BC}^{\mathcal{E}} \geq 0$.