Interactions of Multiple Spin-2 Fields beyond Pairwise Couplings

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Thus far, all known ghost-free interactions of multiple spin-2 fields have involved at most pairwise couplings of the fields, which are direct generalizations of bimetric interactions. We present a class of spin-2 theories with genuine multifield interactions and explicitly demonstrate the absence of ghost instabilities. The construction involves integrating out a nondynamical field in a theory of spin-2 fields with only pairwise ghost-free interactions. The new multivierbein interactions generated are not always expressible in terms of the associated metrics.

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Introduction.—Interacting theories for multiple fields with spin 0, 1/2, and 1 are well understood and realized in nature via the standard model of particle physics, where the multiplets and their mixings are crucial for the viability of the theory. In contrast, general relativity is the simplest possible interacting theory of a single spin-2 field. It is a fundamental question as to whether, in analogy with lower spins, consistent theories of multiple spin-2 fields could exist. Such theories could have profound implications for the understanding of the gravitational force beyond general relativity, but they have not been easy to construct.

In a covariant setup, spin-2 fields have more components than physically needed, and generic theories do not have enough symmetries and constraints to remove the unphysical components. Some of these, if not eliminated, give rise to ghost instabilities, with an example being the Boulware-Deser ghost [1] of a massive spin-2 field. A few years ago the ghost-free theory of two interacting spin-2 fields was found [2], which also fulfills some other important consistency criteria [3,4]. This generalized previous work on a single massive spin-2 field in a fixed background [5–8]. The model is formulated in terms of two symmetric rank-2 tensors (or “metrics”) $g_{\mu\nu}$ and $f_{\mu\nu}$ interacting through a specific potential $V_{\text{bi}}(g, f)$ —hence the name “ghost-free bimetric theory.” For a recent review see Ref. [9].

Theories for more than two spin-2 fields are also strongly restricted by the absence of ghosts. From the analysis of Ref. [2], it is easy to see that certain ghost-free theories can be constructed as straightforward extensions of bimetric

theory by simply adding copies of the bimetric potentials $V_{\text{bi}}(g, f)$ for pairs of metrics, but without forming loops [10]. An example for four metrics $g_{\mu\nu}^I$ would be $V_{\text{bi}}(g^1, g^2) + V_{\text{bi}}(g^1, g^3) + V_{\text{bi}}(g^3, g^4)$. So far, these pairwise couplings have been the only known ghost-free interactions of multiple spin-2 fields.

An important class of multiple spin-2 theories was constructed in Ref. [11] using antisymmetrized products of the vierbein fields which appeared to be ghost-free. However, a more detailed analysis in Ref. [12] revealed that generically such multivierbein interactions contained ghosts. It was argued that the ghost-free subset consisted only of models where the vierbeins could be traded off for the metrics by virtue of a *vierbein symmetrization condition*, exactly as in bimetric theory. The only known models of this type are the pairwise interactions mentioned above and described in more detail in Ref. [13]. Reference [14] reached a similar conclusion for multiple spin-2 theories in three dimensions.

The question is whether there exist interactions between multiple spin-2 fields beyond the pairwise ones, and thus beyond the class conjectured in Ref. [12]. Here we show that this is indeed the case. These are genuine multiple spin-2 interactions in the sense that, similar to the gluon vertices in quantum chromodynamics, they lead to direct interactions among more than two dynamical vierbein fields and cannot be reduced to pairwise interactions through field redefinitions.

Summary of results: In this work we derive a class of ghost-free interactions for multiple spin-2 fields by integrating out a nondynamical field in a theory with ghost-free bimetric interactions. The result is an interaction term for $\mathcal{N}$ vierbeins $(e_I)_\mu^A$ of the form

$$S_{\text{multi}} = -M^4 \int d^4x \det \left(\sum_{I=1}^{\mathcal{N}} \beta^I e_I \right), \quad (1)$$

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involving a mass scale M and arbitrary dimensionless coefficients β^I , $I = 1, \dots, \mathcal{N}$. The kinetic terms of the vierbein fields have the standard Einstein-Hilbert form. For restricted vierbein configurations the multivierbein vertex can be expressed in terms of metrics, but this is not always possible. The interactions can involve up to four different vierbeins in each term and are therefore more general than the pairwise couplings known so far.

Generating new interactions.—The starting point is a theory for $(\mathcal{N} + 1)$ vierbeins, u_μ^A and $(e_I)_\mu^A$, $I = 1, \dots, \mathcal{N}$, with ghost-free bimetric interactions. Let us denote the corresponding metrics by $f_{\mu\nu}^0 = u_\mu^A \eta_{AB} u_\nu^B$ and $f_{\mu\nu}^I = (e_I)_\mu^A \eta_{AB} (e_I)_\nu^B$. The action has the following structure:

$$S[u, e_I] = \sum_{J=0}^{\mathcal{N}} S_{\text{EH}}[f^J] + S_{\text{int}}[u, e_I]. \quad (2)$$

It includes the Einstein-Hilbert kinetic terms

$$S_{\text{EH}}[f^J] = m_J^2 \int d^4x \sqrt{f^J} R(f^J), \quad (3)$$

where m_J are the $(\mathcal{N} + 1)$ Planck masses. S_{int} contains the simplest ghost-free bimetric interactions between u_μ^A and each one of the $(e_I)_\mu^A$,

$$S_{\text{int}}[u, e_I] = -2m^4 \int d^4x \det u \left(\beta_0 + \sum_{I=1}^{\mathcal{N}} \beta^I \text{Tr}(u^{-1} e_I) \right). \quad (4)$$

In addition to the traces $\text{Tr}(u^{-1} e_I) = u_\mu^A (e_I)_\mu^A$, we could also include the remaining ghost-free bimetric interactions, but we have chosen not to do so. For brevity, we set $\beta^I = 1$ for $I = 1, \dots, \mathcal{N}$, by scaling $e_I \rightarrow e_I/\beta^I$ and redefining the Planck masses m_I in Eq. (3) accordingly.

Lorentz constraints: Each vierbein contains six Lorentz parameters that drop out of the corresponding metric and hence appear in the action (2) only through the potential terms in S_{int} . Since these are nondynamical, their equations of motion are constraints. Specifically, these are the antisymmetric parts of the equations of motion for $(e_I)_\mu^A$ [11,15], precisely six equations per vierbein,

$$\frac{\delta S_{\text{int}}}{\delta (e_I)_\mu^A} \eta^{AB} (e_I)_B^\nu - \frac{\delta S_{\text{int}}}{\delta (e_I)_\nu^A} \eta^{AB} (e_I)_B^\mu = 0. \quad (5)$$

The corresponding equation for u_μ^A is a linear combination of Eq. (5) due to the overall Lorentz invariance of the action. For the potential in Eq. (4), the Lorentz constraints uniquely imply the following symmetrization conditions,

$$u_\mu^A \eta_{AB} (e_I)_\nu^B = u_\nu^A \eta_{AB} (e_I)_\mu^B, \quad I = 1, \dots, \mathcal{N}. \quad (6)$$

These allow us to express the potential in terms of metrics by replacing $u^{-1} e_I = \sqrt{f_0^{-1} f_I}$ in addition to $(\det u) = \sqrt{|\det f_0|}$. It is then straightforward to verify that the arguments for the absence of a ghost in bimetric theory [2] extend to this case.

Multiple spin-2 action: Let us set the coupling $m_0 = 0$ to freeze the kinematics of the vierbein u_μ^A . We can then eliminate u_μ^A algebraically using its equation of motion to obtain an action for the remaining $\mathcal{N}$ vierbeins (note that the “massive gravity” limit of $m_0 \rightarrow \infty$ gives a similar starting action, but with a fixed background u_μ^A which cannot be varied and eliminated; see Ref. [16]). Indeed, varying the action (2) for $m_0 = 0$ with respect to the inverse vierbein u_μ^A gives an equation with the unique solution

$$u_\mu^A = -\frac{3}{\beta_0} \sum_{I=1}^{\mathcal{N}} (e_I)_\mu^A. \quad (7)$$

Using this to eliminate u_μ^A in Eq. (4) gives the new interactions for the remaining $\mathcal{N}$ vierbeins,

$$S_{\text{int}}[e_I] = -M^4 \int d^4x \det(e_1 + e_2 + \dots + e_{\mathcal{N}}), \quad (8)$$

with $M^4 \equiv -54m^4\beta_0^{-3}$. These are not reducible to linear combinations of the pairwise bimetric potentials. That the starting action (2) was ghost-free does not by itself guarantee the absence of ghosts in the final theory since the procedure changes the number of dynamical and nondynamical fields. However, we will explicitly confirm the absence of ghosts below.

The $\mathcal{N}$ symmetrization constraints (6), with u_μ^A given by Eq. (7), provide a set of constraint equations for the new multiple spin-2 theory. Equivalently, one can derive these from the multiple spin-2 action (8) using Eq. (5). In matrix notation these new Lorentz constraints are

$$\sum_{I=1}^{\mathcal{N}} e_I^T \eta e_I = \sum_{I=1}^{\mathcal{N}} e_I^T \eta e_J, \quad J = 1, \dots, \mathcal{N}. \quad (9)$$

Note that if we sum Eq. (9) over J , the resulting equation is identically satisfied. Hence, there are $\mathcal{N} - 1$ independent matrix relations among the antisymmetric parts of the combinations $(e_J)_\mu^A \eta_{AB} (e_I)_\nu^B$. These can be used to eliminate $6(\mathcal{N} - 1)$ nondynamical components of the vierbeins. Another six components are removed by the overall Lorentz invariance of the action. This leaves us with $16\mathcal{N} - 6\mathcal{N} = 10\mathcal{N}$ independent components, which is the same as the number of components in $\mathcal{N}$ symmetric rank-2 tensor fields.

Direct ghost proof.—In the following we will use $3 + 1$ metric variables to demonstrate the existence of the constraints that remove the Boulware-Deser ghosts from the

physical spectrum. This choice of variables is convenient for isolating the nondynamical fields in the Einstein-Hilbert actions, but their use requires some justification. In general, $\mathcal{N}$ Lorentzian metrics may not admit compatible notions of space and time and compatible $3+1$ decompositions. However, the vierbeins e_I are not fully independent since the parent theory gives rise to constraints (6). These, as shown in Ref. [3], ensure that simultaneous $3+1$ decompositions exist for the pair (u, e_I) for each I . Geometrically, the null cone of each e_I shares some common timelike and spacelike directions with the null cone of u [3]. Therefore, with further mild restrictions on the ranges of metric variables, there always exist large classes of configurations where all e_I share a common spatial hypersurface with u and, hence, admit simultaneous $3+1$ decompositions. In the new multivierbein theory, the null cones are further correlated by Eq. (7) for u . The ghost proof below assumes such configurations admitting simultaneous $3+1$ decompositions. It may be possible to further generalize it to a covariant analysis along the lines of Refs. [17,18].

We decompose a vierbein e_μ^A into a gauge-fixed vierbein E_μ^A , rotated by a Lorentz transformation Λ_B^A ,

$$e_\mu^A = \Lambda_B^A E_\mu^B, \quad (10)$$

and use the following $3+1$ parametrization [11,12,19]:

$$E_\mu^A = \begin{pmatrix} N & 0 \\ E_j^a N^j & E_i^a \end{pmatrix}, \quad \Lambda_B^A = \begin{pmatrix} \gamma & v_c \\ v^a & \delta_c^a + \frac{1}{1+\gamma} v^a v_c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\Omega)_b^c \end{pmatrix}. \quad (11)$$

Here N is the lapse, N^i is the shift vector, and E_i^a is a gauge-fixed spatial vierbein with six independent components. The Lorentz transformation has been further decomposed into spatial rotations parametrized by an $SO(3)$ matrix Ω (containing three parameters), and Lorentz boosts parametrized by the three-dimensional (rescaled) boost vector v^a , with $\gamma \equiv \sqrt{1 + v_a v^a}$.

In the action, we write $(\mathcal{N} - 1)$ of the vierbeins $(e_I)_\mu^A$ using the $3+1$ parametrization given above. The overall Lorentz invariance of the action allows us to take the last vierbein to be of the same form but with $v^a = 0$ and $\Omega = \mathbb{1}$. Since the Lorentz parameters do not show up in the kinetic terms, they appear without derivatives in the action. It was shown in Ref. [20] that the Lorentz constraints (5) are equivalent to the equations of motion of the Lorentz parameters.

It is easy to show that the potential (8) is linear in all lapses N_I and shifts N_I^i before integrating out the Lorentz fields v_I^a and Ω_I . The potential is $\det(u)$, where the matrix u is given by Eq. (7). The N_I and N_I^i appear only in the first column u_0^A . Since $\det(u)$ is antisymmetric in the columns of u , each term in it contains only one factor of u_0^A —hence the

linearity. The Einstein-Hilbert terms are also linear in these variables; hence, the entire action has the form

$$S = \sum_{I=1}^{\mathcal{N}} \int d^4x [(\pi_I)_a^i (\dot{E}_I)_i^a - N_I \mathcal{C}_I - N_{Ii} \mathcal{C}_I^i]. \quad (12)$$

Here, $(\pi_I)_a^i$ are the canonical momenta conjugate to the $(E_I)_i^a$, and all other fields are nondynamical. The functions $\mathcal{C}_I$ and $\mathcal{C}_I^i$ contain the E_I , the π_I , and the Lorentz fields, v_I^a and Ω_I , but not the lapses and shifts. We note that thus far in this section the considerations, including the form of the action in Eq. (12), are not restricted to the model (8) but also apply to the general class of multivierbein theories introduced in Ref. [11]. However, the difference is crucial for the argument that follows.

The dynamical fields $(E_I)_i^a$ contain the ghost modes. These must be eliminated by the constraints that arise from the equations of motion for the N_I and N_{Ii} , that is, $\mathcal{C}_I = 0$ and $\mathcal{C}_I^i = 0$, as well as from Eq. (5) for the Lorentz fields, and the gauge-fixing conditions for general covariance. In particular it is necessary that, after solving for the Lorentz fields, $\mathcal{C}_I = 0$ become equations for the E_I and π_I alone, remaining independent of the lapses. Then they can eliminate the ghost fields in favor of the remaining dynamical variables. The difficulty is that for the general multivierbein interactions of Ref. [11] which also have the form of Eq. (12), the Lorentz constraints (5) render the spatial rotations Ω_I , and hence the $\mathcal{C}_I$, dependent on the N_I [12]. Then $\mathcal{C}_I = 0$ can be solved for the N_I rather than eliminating the ghost fields which will remain in the spectrum. We now show that the multivierbein interaction in Eq. (8) circumvents this problem.

First, $(\mathcal{N} - 1)$ combinations of the $\mathcal{N}$ vector equations $\mathcal{C}_I^i = 0$ can be used to determine the $(\mathcal{N} - 1)$ boost vectors v_I^a as $v_I^a(E, \pi, \Omega)$. The unused combination of the $\mathcal{C}_I^i$, say $\mathcal{C}^i$, are the three constraints of the spatial diffeomorphisms. To determine the rotation matrices Ω_I , consider the Lorentz constraints that, for the model (8), give the $(\mathcal{N} - 1)$ independent matrix equations in Eq. (9). Their spatial parts are the $3(\mathcal{N} - 1)$ equations

$$\sum_{I=1}^{\mathcal{N}} (e_I)_i^A \eta_{AB} (e_J)_j^B = 0, \quad J = 1, \dots, \mathcal{N} - 1. \quad (13)$$

Crucially, the $(e_I)_i^A$ do not contain the lapses or the shifts since, from Eq. (10), it follows that

$$e_i^A = \begin{pmatrix} v_b \Omega_c^b E_i^c \\ (\delta_b^a + \frac{1}{1+\gamma} v^a v_b) \Omega_c^b E_i^c \end{pmatrix}. \quad (14)$$

Therefore, the constraints (13), along with the solutions for v_I^a , determine the $3(\mathcal{N} - 1)$ parameters of the rotation

matrices as $\Omega_I(E, \pi)$. This insures that C_I depend only on the E_I and π_I , as desired.

Finally, the $0i$ components of the Lorentz constraints provide $3(\mathcal{N} - 1)$ equations linear in the lapses and shifts. These determine $(\mathcal{N} - 1)$ of the shift vectors N_I^i as linear functions of the N_I and of the $\mathcal{N}$ th shift vector, say, $N_{\mathcal{N}}^i$. The linearity ensures that on using these solutions, the action is linear in N_I and $N_{\mathcal{N}}^i$. At this stage, the only other variables in the action are the E_I and the π_I .

Of the remaining constraints, one combination of the C_I , say, C , together with the unused C^i , form a set of four first class constraints associated with general covariance, as in Ref. [21]. The remaining $\mathcal{N} - 1$ combinations of the C_I are second class constraints. Their preservation in time, $\dot{C}_I = 0$, gives another set of $\mathcal{N} - 1$ constraints. Although we have not proven this here, their existence can be argued in the parent theory (2), where the calculations are very similar to the bimetric case explicitly analyzed in Ref. [21].

The degree of freedom count is now easy. The fields $(E_I)_i^a$ and the momenta $(\pi_I)_a^i$ contain $2 \times 6\mathcal{N} = 12\mathcal{N}$ phase space variables, including the $\mathcal{N}$ ghost fields and their $\mathcal{N}$ conjugate momenta. The $\mathcal{N} - 1$ second class constraints and their associated time preservation conditions eliminate $2(\mathcal{N} - 1)$ ghost fields and ghost momenta. The four first class constraints and the associated symmetry eliminate eight phase space variables, including the last pair of ghost variables, just as in general relativity. The physical phase space thus consists of $12\mathcal{N} - (2\mathcal{N} + 6) = 10\mathcal{N} - 6$ variables, corresponding to one massless field (with four phase space modes) and $(\mathcal{N} - 1)$ massive fields of spin-2 (each with ten phase space modes).

Existence of metric formulations.—Ghost-free multivierbein theories with only pairwise interactions can be expressed in terms of the corresponding metrics, although some extra information contained in the vierbeins is retained (corresponding to the choices of square-root matrices that appear in the theory). Here we explore the existence of a similar metric formulation for the non-pairwise interaction in Eq. (8). We show that the additional vierbein information cannot be encoded as simple requirements on familiar matrix functions unless additional constraints are imposed.

By extracting a factor of $\det e_1$ and using $g_{\mu\nu} \equiv (e_1)_\mu^A \eta_{AB} (e_1)_\nu^B$, the interaction becomes

$$S_{\text{int}} = -M^4 \int d^4x \sqrt{g} \det \left(\mathbb{1} + g^{-1} \sum_{J=2}^{\mathcal{N}} e_1^T \eta e_J \right). \quad (15)$$

Defining the antisymmetric matrices A_{IJ} as

$$A_{IJ} \equiv \frac{1}{2} (e_I^T \eta e_J - e_J^T \eta e_I), \quad (16)$$

the action depends on the linear combination $\sum_{J=1}^{\mathcal{N}} A_{IJ}$. Also, in terms of these, the Lorentz constraints (9) read

$$\sum_{J=1}^{\mathcal{N}} A_{IJ} = 0, \quad I = 1, \dots, \mathcal{N}. \quad (17)$$

Hence, the antisymmetric parts drop out of Eq. (15). For any pair of vierbeins, if we set $A_{IJ} = 0$, then one could replace $e_I^{-1} e_J = (g_I^{-1} g_J)^{-1/2}$, where $g_I = e_I^T \eta e_I$. Then a covariant formulation in terms of the metrics would be possible if $A_{IJ} = 0$ for all J . But the constraints (17) that arise in the theory are weaker. Hence, although the A_{IJ} drop out of the action, the theory has no *equivalent* metric formulation. However, metric formulations exist under mild restrictions.

Metric formulation for three fields: We now consider metric representations for the case $\mathcal{N} = 3$, following an analysis carried out in Ref. [20]. Let us introduce three Lorentz matrices $(\bar{L}_I)_B^A$ as Stückelberg fields for the three vierbeins $(e_I)_\mu^A$, that is, we write every vierbein as a Lorentz rotation of a gauge-fixed vierbein $(\bar{e}_I)_\mu^A$,

$$(e_I)_\mu^A = (\bar{L}_I)_B^A (\bar{e}_I)_\mu^B. \quad (18)$$

Defining $(L_I)_\mu^\nu = (\bar{e}_I)_C^\nu (\bar{L}_I)_B^C (\bar{e}_I)_\mu^B$, we can also write

$$(e_I)_\mu^A = (\bar{e}_I)_\nu^A (L_I)_\mu^\nu. \quad (19)$$

Let us choose the gauge-fixed vierbeins such that

$$(\bar{e}_I)_\mu^A \eta_{AB} (\bar{e}_J)_\nu^B = (\bar{e}_I)_\nu^A \eta_{AB} (\bar{e}_J)_\mu^B \quad (20)$$

for $I, J = 1, 2, 3$. General vierbeins may not be gauge fixed in this way [22]. Such gauge fixing is possible only for restricted configurations for which the null cones associated with the I and J vierbein intersect [3]. Note that this is a stronger requirement than the existence of simultaneous $3 + 1$ decompositions assumed for the ghost analysis. Now the Lorentz constraints (9) provide the following two independent sets of equations for the L_I which determine two of the three Lorentz matrices:

$$\begin{aligned} L_1^T \bar{e}_1^T \eta \bar{e}_2 L_2 - (1 \leftrightarrow 2) &= L_3^T \bar{e}_3^T \eta \bar{e}_1 L_1 - (1 \leftrightarrow 3), \\ L_2^T \bar{e}_2^T \eta \bar{e}_3 L_3 - (2 \leftrightarrow 3) &= L_3^T \bar{e}_3^T \eta \bar{e}_1 L_1 - (1 \leftrightarrow 3). \end{aligned} \quad (21)$$

Multiplying the first by L_1^{-1T} from the left and L_1^{-1} from the right and the second by L_3^{-1T} from the left and L_3^{-1} from the right, it is obvious that the covariant solution is $L_1 L_2^{-1} = L_1 L_3^{-1} = L_3 L_2^{-1} = \mathbb{1}$, which implies $L_1 = L_2 = L_3$. Then the multiple spin-2 potential becomes

$$V = M^4 \det (\bar{e}_1 + \bar{e}_2 + \bar{e}_3) \quad (22)$$

since the undetermined overall Lorentz matrix drops out due to Lorentz invariance. We can extract a factor of $\det e_1$ to obtain

$$V = M^4 \det(e_1) \det(\mathbb{1} + \bar{e}_1^{-1} \bar{e}_2 + \bar{e}_1^{-1} \bar{e}_3). \quad (23)$$

Let us introduce the metrics $g_{\mu\nu}$, $f_{\mu\nu}$, and $h_{\mu\nu}$, for the vierbeins $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$, respectively. Because of the symmetrization constraint (20), we have

$$\bar{e}_1^{-1} \bar{e}_2 = \sqrt{g^{-1}f}, \quad \bar{e}_1^{-1} \bar{e}_3 = \sqrt{g^{-1}h}, \quad (24)$$

Then the multiple spin-2 potential can be written in terms of metrics as

$$V = M^4 \sqrt{g} \det(\mathbb{1} + \sqrt{g^{-1}f} + \sqrt{g^{-1}h}). \quad (25)$$

We could have chosen to extract the determinant of a vierbein other than e_1 and correspondingly obtain

$$\begin{aligned} V &= M^4 \sqrt{f} \det\left(\mathbb{1} + \sqrt{f^{-1}g} + \sqrt{f^{-1}h}\right) \\ &= M^4 \sqrt{h} \det\left(\mathbb{1} + \sqrt{h^{-1}g} + \sqrt{h^{-1}f}\right). \end{aligned} \quad (26)$$

Since the parent theory with nondynamical u_μ^A possesses a metric formulation, we also could have tried to integrate out the metric $f_{\mu\nu}^0 = u_\mu^A \eta_{AB} u_\nu^B$ directly. However, to do this in a way consistent with the vierbein conditions, we would have had to assume the existence of two additional square-root matrices, just as in the above considerations.

The analysis partially generalizes to $\mathcal{N} > 3$. Now the gauge choices (20) cannot be made for all vierbeins since these are $\mathcal{N}(\mathcal{N} - 1)/2$ conditions, while there are at most $\mathcal{N}$ Lorentz transformations to implement the symmetrizations. The remaining conditions may be imposed by hand on the $\bar{e}_I$, but this is not necessary. To find a metric formulation similar to, say, Eq. (25), it is enough to impose Eq. (20) only on the $(\bar{e}_1)_\mu^A \eta_{AB} (\bar{e}_I)_\nu^B$, which gives $\mathcal{N} - 1$ gauge conditions. Then, with appropriate restrictions and on choosing a similar solution for the L_I , one obtains an expression which is a direct generalization of Eq. (25) to $\mathcal{N}$ metrics.

Finally, it is important to note that, while the vierbeins were restricted by hand to obtain a metric formulation, such restrictions are inbuilt in the final multimetric theory. Hence, the resulting multimetric theories can be considered in their own right, independent of the starting vierbein formulations.

Discussion.—To summarize, we have constructed non-trivial interactions of multiple spin-2 fields, beyond the known pairwise potentials, and have demonstrated the existence of constraints that eliminate the extra ghost modes. The interactions are given in terms of the $\mathcal{N}$ vierbeins $(e_I)_\mu^A$ in Eq. (1). On expressing the determinant in terms of the wedge products of the one-forms e_I^A , one gets

$$\sum_{I,J,K,L=1}^{\mathcal{N}} \beta^{IJKL} \epsilon_{ABCD} e_I^A \wedge e_J^B \wedge e_K^C \wedge e_L^D \quad (27)$$

with $\beta^{IJKL} = \beta^I \beta^J \beta^K \beta^L$. Such interactions with general β^{IJKL} were proposed in Ref. [11], where the $\mathcal{N} = 2$ case reproduces the bimetric theory [2]. However, the only ghost-free cases known thus far are the pairwise bimetric interactions [12]. Given the interactions in Eq. (27) with arbitrary β^{IJKL} , a direct analysis that identifies the ghost-free cases is thus far not known. But the construction presented here generates a class of genuinely multivierbein ghost-free interactions with up to four different vierbeins in one vertex. This easily extends to D space-time dimensions, in which case the vertices would contain up to D different fields. Note that these vertices do not contain closed loops of purely pairwise interactions, which are not ghost-free [12], even though they may appear ghost-free in a certain decoupling limit [23].

The class of theories obtained here can be further generalized. First, note that the interactions (1) cannot simply be added to a theory involving the previously known ghost-free interactions of the $\mathcal{N}$ vierbeins $(e_I)_\mu^a$. Such a setup would correspond to forbidden loop couplings in the parent theory with vierbein u_μ^a . However, compatible interactions can be constructed by extending the parent theory by additional allowed pairwise couplings, as will be discussed in Ref. [24]. The new terms couple any of the vierbeins e_I to an additional set of $\mathcal{N}'$ vierbeins v_K and include the following: (1) One of the e_I can interact with the v_K through the standard pairwise interactions; that is, the potential can have the form $V \sim \det(\sum_I e_I) + \sum_K V_{\text{bi}}(e_I, v_K)$. (2) One of the e_I can interact with the v_K through a determinant interaction; i.e., the potential could be $V \sim \det(\sum_I e_I) + \det(e_I + \sum_K v_K)$.

It is straightforward to introduce a standard coupling to matter via any of the dynamical vierbeins $(e_I)_\mu^a$, as this will not influence the procedure of integrating out the non-dynamical vierbein u_μ^a . On the other hand, if u_μ^a couples to matter, then the multiple spin-2 theory for the $(e_I)_\mu^a$ will have matter interactions that are heavily modified [25]. At low energies these reduce to the matter coupling suggested in Refs. [26,27].

The interactions for $(\mathcal{N} + 1)$ vierbeins that we started with in Eq. (4) were not of the most general ghost-free form. It would be interesting to extend our setup to more general interactions and obtain possibly ghost-free multiple spin-2 theories that are different from the one studied here. It is not obvious that the algebraic equations for the nondynamical vierbein can be solved covariantly for more general parameter choices in the action. In any case, if the linear relation in Eq. (7) is lost, the resulting interactions may not have the general form in Eq. (27). It is important to find out if such ghost-free multivierbein interactions, beyond the classes discussed here, could exist.

Our couplings are the first instance of ghost-free spin-2 interactions where the vierbein formulation admits more general configurations than the associated, more restrictive metric formulation. Hence, it is interesting to directly investigate the associated multimetric theories, without recourse to the vierbein formulation. One expects that the extra restrictions on the metrics, already encoded in the multimetric interactions [3], would lead to better causal properties for the theory.

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