Bounded Collection of Feynman Integral Calabi-Yau Geometries

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We define the rigidity of a Feynman integral to be the smallest dimension over which it is nonpolylogarithmic. We prove that massless Feynman integrals in four dimensions have a rigidity bounded by $2(L-1)$ at $L$ loops provided they are in the class that we call marginal: those with $(L+1)D/2$ propagators in (even) $D$ dimensions. We show that marginal Feynman integrals in $D$ dimensions generically involve Calabi-Yau geometries, and we give examples of finite four-dimensional Feynman integrals in massless $\varphi^4$ theory that saturate our predicted bound in rigidity at all loop orders.

Introduction.—One-loop Feynman integrals in generic quantum field theories are known to be polylogarithmic (see, e.g., Refs. [1,2]). More generally, multiple polylogarithms have been found to suffice for sufficiently simple quantities and at sufficiently low multiplicity or low loop orders (see, e.g., Refs. [3–11]). Nevertheless, it has been known for some years that Feynman integrals of worse-than-polylogarithmic complexity are relevant to quantum field theories. Integrals with arbitrarily worse complexity were first observed in massive theories in two dimensions (see, e.g., Refs. [12–17]) but are now known to be important even in massless, integrable theories in four dimensions (see, e.g., Refs. [18–23]).

We define the geometric “rigidity” of a Feynman integral to be its degree of “nonpolylogarithmicity”. More concretely, after eliminating a maximal number of rational integrations, we imagine all the ways in which a Feynman integral can be expressed as a sum of polylogarithms of weight $w$ integrated over a space of higher genus or dimension than a Riemann sphere; maximizing the weight $w$ minimizes the dimension of the space that remains—which we use to define the rigidity of the integral. (Some integrals will consist of a sum of terms with different rigidity; we consider the rigidity of the integral to be the maximum rigidity that contributes.) Thus, polylogarithms have rigidity 0, while the (two-dimensional) massive sunrise integral [12–14,24–36] and the (four-dimensional) kite [17,35–37] and massless double-box integrals [18,22,38,39] have rigidity 1 (as they involve integration over a one-dimensional variety with genus one). Feynman integrals with higher rigidity are also known to exist: The massive $L$-loop sunrise integral in two dimensions (see, e.g., Refs. [14,15,27,40,41]) and the massless $L$-loop traintrack integral in four dimensions [23] both have rigidity $L-1$.

In this Letter, we probe the limits of Feynman-integral rigidity. In particular, we prove that a large class of massless Feynman integrals in four dimensions have a rigidity bounded by $2(L-1)$ at $L$ loops, and we provide explicit examples that saturate this bound. Maximal rigidity is easiest to demonstrate for the case of integrals that we call marginal: those $L$-loop integrals involving exactly $(L+1)D/2$ propagators in (even) $D$ dimensions. Moreover, we show using the Symanzik polynomial formalism that marginal integrals generically involve Calabi-Yau geometries with a dimension equal to the rigidity of the integral. Thus, we describe examples of massless integrals in four dimensions that involve a K3 surface at two loops, CY$_4$’s at three loops, CY$_6$’s at four loops, and so on—all examples exceeding previously known limits of rigidity. Our searches have uncovered a wealth of examples which saturate our predicted bound, some to all loop orders.

All of our results are described in terms of the Symanzik polynomial representation of Feynman integrals. We review this formalism momentarily and use this to motivate the notion of marginality described above—including its relevance to (proving) a Calabi-Yau condition to be qualified below. We show that finite marginal integrals in two dimensions have rigidity $\leq (L-1)$ and that this bound is saturated if (and only if) all propagators are massive—thus reproducing the well-known fact that the $L$-loop massive sunrise integral can be expressed as a logarithm integrated over a $(L-1)$-dimensional (singular) Calabi-Yau manifold [14,15,40,41]. We then generalize this to four dimensions and show that finite marginal integrals involving massless propagators have a bounded rigidity—and we describe a number of examples which saturate this bound. We conclude with some general observations and
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Moreover, trading $F$ marginal integrals especially important for physics: They
dependence.} We call such integrals marginal.

For our present purposes, however, we note that, for
dimensions (that is, power counting consistent with one-loop
momenta results in the following “Symanzik” representation of a loop integral:

$$I = \int_{x_i \geq 0} [d^{E-1}x_i] \frac{U^{E-(L+1)D/2}}{\mathcal{S}^{E-LD/2}}. \quad (1)$$

Here, $[d^{E-1}x_i]$ is the integration measure over $E$ homo-
genous variables $x_i$ on $\mathbb{R}^{E-1}$, where we have dropped
some conventionally included numerical prefactors. The first and second Symanzik polynomials, $U$ and $\mathcal{S}$ in (1), are
defined in terms of the Feynman graph. Specifically, letting $e_i$ denote the $i$th edge,
$$U := \sum_{\{T\} \in \Xi_k} \left( \prod_{e_i \in T} x_i \right);$$
$$\mathcal{S} := \left[ \sum_{\{T_1,T_2\} \in \Xi_k} s_{T_1} \left( \prod_{e_i \notin T_1 \cup T_2} x_i \right) \right] + U \sum_{e_i} x_i m_i^2. \quad (2)$$

Here, $m_i$ denotes the mass of the $i$th propagator, $\Xi_k$ denotes the set of spanning $k$-forests of the graph (a spanning
$k$-forest of a graph is a subgraph involving all vertices and
having $k$ connected components, each of which are trees),
and $s_T$ denotes an ordinary Mandelstam—the square of the
sum of the external momenta flowing into the tree $T$.

Note that $U$ and $\mathcal{S}$ are homogenous polynomials and
that $\mathcal{S}$ is linear in at least one variable provided there
is a massless propagator in the graph. Moreover, it is easy
to see that the integral (1) simplifies considerably when
the exponent of the $U$ polynomial vanishes—that is, when
$E = (L+1)D/2$. [An analogous simplification arises when
$E = LD/2$; this case was studied by Brown in Ref. [43] (see also Refs. [44–47]) with similar conclusions. For our present purposes, however, we note that, for
dimension $D > 2$, such integrals necessarily involve less-than-the-conjecturally-maximal “weight”—as (1)
would describe a rational integral of dimension $< LD/2$. Moreover, trading $\mathcal{S}$ for $U$ eliminates all kinematic
dependence.] We call such integrals marginal.

Before moving on, we should describe why we consider
marginal integrals especially important for physics: They
are capable of having $D$-gon power counting in $D$ dimen-
sions (that is, power counting consistent with one-loop
$D$-gon integrals). As such, they represent important
irreducible elements of any unitarity basis of loop inte-
grands (and are thus relevant to the scattering amplitudes of
all quantum field theories in $D$ dimensions). Another
important aspect of marginal integrals—at least in two
and four dimensions—is that these integrals can be shown
to have maximal “transcendental weight” (or weight). It is
not actually known how to make the notion of weight
precise beyond polylogarithms (see Ref. [48] for recent
progress in the elliptic case), but here we use it as a proxy
for the minimal dimension of the algebraic differential form
over which it can be expressed as an integral. We note that,
outside of marginal integrals in four dimensions, it can be
extremely hard to find a 2$L$-fold rational integral repre-
sentation of a generic Feynman integral. In the cases where
it has proven possible, more elaborate methods than
described here have been utilized [22,23,49,50].


transcendental weight

Calabi-Yau geometry of loop integrals.—In a large and
growing number of examples, the (nonpolylogarithmic part
of the) geometry relevant to individual Feynman integrals
has been found to be Calabi-Yau. Examples include the
massive $L$-loop sunrise integrals in two dimensions (see,
e.g., Refs. [14,15,40,41]) and the $L$-loop massless train-
track integrals [22,23], which are known at low loops (and
conjectured at high loops) to involve polylogarithms
integrated over $(L - 1)$-dimensional Calabi-Yau manifolds.
We are unsure of the extent to which this property holds
more generally, but it turns out it is fairly easy to prove that,
upon exhausting all rational and polylogarithmic integra-
tions, marginal integrals in any number of dimensions are
naturally defined over (possibly singular) Calabi-Yau
manifolds.

The argument is fairly straightforward (and essentially
the same as Brown describes in Ref. [43] for Feynman
integrals with $E = 2L$ in $D = 4$). For a marginal Feynman
integral, the integrand in (1) becomes $1/\mathcal{S}^{D/2}$. Recall that
$$\mathcal{S}$$ is a homogeneous polynomial (of total degree $L + 1$) in
$\mathbb{R}^{E-1}$ and that it is of degree $\leq 2$ in each homogeneous
coordinate. If $\mathcal{S}$ were linear in any variable, we could
integrate over it to obtain another rational function or a
polylogarithmic function. If any factor in the denominator
were still linear, we could partial-fraction and integrate over
it. We may continue in this manner until we have terms
whose denominators are irreducibly quadratic or higher in
all remaining parameters.

Let us suppose that we have done the above, resulting in a
form of the integral which is a polylogarithm integrated over
a sum of rational forms whose denominators are each
irreducibly quadratic or higher in all $m$ remaining variables.
Suppose the forms involve polynomials no worse than
quadratic; then any parameter of such an integral can be
chosen for one further polylogarithmic integration—but at
the cost of introducing a discriminant $Q(x_i)$ of this quadratic
with respect to the remaining $(m - 1)$ variables. It is easy to
see that $Q(x_i)$ is homogeneous, with degree $2(m - 1)$. Thus,
the hypersurface $y^2 = Q(x_i)$ defines an algebraic variety in

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massive case

It is fairly well known [14, 15, 40] that generally, we are presently not able to show that this will hold true more requires blowing up these singularities [51]. It is possible to identifying it with a smooth Calabi-Yau manifold further earlier. Since the hypersurface will typically be singular, a Calabi-Yau in all cases we are aware of [52, 53], but we are presently not able to show that this will hold true more generally.

Exempli gratia: Massive sunrise integrals in $D = 2$.—In two dimensions, the only (tadpole-free) marginal integrals are the so-called “sunrise” (or “banana”) graphs depicted in Fig. 1. It is fairly well known [14, 15, 40] that—in the fully massive case—these integrals have a rigidity of ($L - 1$). That is, these integrals may be represented as (onefold) logarithms integrated over Calabi-Yau manifolds of dimension ($L - 1$). For the sake of illustration, let us see how we can understand this fact from the discussion above.

The Symanzik representation of the $L$-loop sunrise integral is fairly trivial to derive. For the $U$ polynomial, there are 1-forests associated with (not cutting) each edge, and for $G$, there is a single 2-forest which cuts every edge. Letting $s$ be the squared momentum of the external line, we see that we may write $I = \int 1/G$ with

$$U := \left( \prod_{i} x_i \right) \sum_{i} \frac{1}{x_i},$$

$$G := \left( \prod_{i} x_i \right) \left[ s + \left( \sum_{i} \frac{1}{x_i} \right) \sum_{i} x_i m_i^2 \right].$$ \hspace{1cm} (3)

Provided $m_i^2 \neq 0$ for each $i$, it is easy to see that $G$ is an irreducible quadratic in each ($a$-parameter) $x_i$. Thus, following the discussion above, we may easily integrate any $x_j$—resulting in a logarithm—but at the cost of introducing the square root associated with the discriminant of $G$. Suppose that $x_j$ were integrated out; it is not hard to see that the discriminant of $G$ with respect to $x_j$ is

Integrating over $x_j$ results in an integral over the remaining $L$ parameters of the form $1/\sqrt{Q_j}$ times a logarithm. Notice that $Q_j$ is homogeneous of degree $2L$—irreducibly quartic in each remaining parameter. Describing the (singular) hypersurface as $y^2 = Q_j$ allows us to embed this as a homogeneous hypersurface in $\mathbb{P}^{[m-1]:1}\cdots:1$. As such, the integral is over a Calabi-Yau manifold of dimension ($L - 1$). It is worth noting that if even a single mass had vanished, then $G$ would have been linear in at least one variable, rendering the rigidity of the sunrise integral strictly less than ($L - 1$).

Maximally rigid, massless integrals in $D = 4$.—Consider now a finite marginal $L$-loop Feynman integral in four dimensions involving $E = 2(L + 1)$ massless propagators. Such a Feynman integral is represented in the Symanzik formalism (1) by

$$I = \int_{x_i \geq 0} \left[ d^{2L+1} x_i \right] \frac{1}{G^2}.$$ \hspace{1cm} (5)

Because the integral is massless, $G$ is linear in every variable. As such, we may integrate out any one parameter $x_j$. Writing $G = G_0^{(j)} + x_j G_1^{(j)}$, this results in

FIG. 3. Massless, finite four-dimensional Feynman integrals with rigidity $2(L - 1)$ for even $L \geq 2$. 

FIG. 2. Examples of (a) two-, (b) three-, (c) four-, and (d) five-loop marginal integrals with maximal rigidity; (e) a marginal integral with submaximal rigidity.

$$Q_j := \left( \prod_{i \neq j} x_i^2 \right) \left[ s + m_j^2 + \left( \sum_{i \neq j} \frac{1}{x_i} \right) \sum_{i \neq j} x_i m_i^2 \right]^2 - 4m_j^2 \left( \sum_{i \neq j} \frac{1}{x_i} \right) \sum_{i \neq j} x_i m_i^2 \right].$$ \hspace{1cm} (4)

FIG. 1. The marginal, $L$-loop sunrise integral in two dimensions. It has (maximal) rigidity ($L - 1$) iff all legs are massive.
FIG. 4. Massless, finite four-dimensional Feynman integrals with rigidity $2(L - 1)$ for odd $L \geq 1$.

\[ I = \int_{x_i \geq 0} [d^{2L} x_i] \frac{1}{\mathfrak{G}_0^{(j)} \mathfrak{G}_1^{(j)}}. \]  

(6)

Because $\mathfrak{G}$ was linear in each variable, so are each of the factors in the denominator of (6); thus, we may always perform another integration resulting in a logarithm, but without introducing any nonrational prefactors. More specifically, suppose we choose to integrate $x_k$; then we may write $\mathfrak{G}_i^{(j)} = \mathfrak{G}_i^{(j,k)} + x_k \mathfrak{G}_i^{(j,k)}$ and conclude that

\[ I = \int_{x_i \geq 0} [d^{2L-1} x_i] \frac{\log(\mathfrak{G}_{0,0}^{(j)} \mathfrak{G}_{1,1}^{(j,k)}) - \log(\mathfrak{G}_{0,1}^{(j,k)} \mathfrak{G}_{1,0}^{(j,k)})}{\mathfrak{G}_{0,0}^{(j,k)} \mathfrak{G}_{1,1}^{(j,k)} - \mathfrak{G}_{0,1}^{(j,k)} \mathfrak{G}_{1,0}^{(j,k)}}. \]  

(7)

At this stage we may already conclude that the rigidity of these integrals is $\leq 2(L - 1)$: The denominator of (7) is at most quadratic in each remaining variable, so we can always perform at least one further polylogarithmic integration. In order for an integral to have rigidity exactly $2(L - 1)$, it must then be the case that, for every choice of the first two integrations, (i) the denominator of (7) is quadratic in all remaining variables, and (ii) each quadratic’s discriminant is an irreducible quartic or cubic (that is, has no repeated roots) in the remaining $2(L - 1)$ variables.

It is worth clarifying the role of condition (ii) above. For an integral satisfying condition (i), performing the third integration results in an integral over the square root of the discriminant of the denominator of (7). If the polynomial in this square root has repeated roots in any of its variables, we may factor out a perfect square, yielding a linear factor outside of the square root. The remaining polynomial inside the square root is then quadratic, allowing it to be rationalized by a change of variables (see, e.g., Refs. [54,55]). This in turn will allow another polylogarithmic integration to be carried out (at least in principle), indicating submaximal rigidity.

Although criteria (i) and (ii) are fairly stringent, it turns out that many marginal, four-dimensional integrals saturate this bound in rigidity. Explicit two- through five-loop finite integrals with maximal rigidity (and one example of a three-loop integral with submaximal rigidity) are shown in Fig. 2. The first four of these are special cases of the all-loop sequences shown in Figs. 3–5. For these sequences, we have explicitly checked that they saturate the bound on rigidity through 11 loops, and we expect this to hold for all loop orders.

Notice that each of the infinite families of examples depends on a fixed number of external legs. In particular, those shown in Fig. 5 can be naturally thought of as “twisted” counterparts to the familiar ladder sequence of integrals [56–63]. Unlike the other families considered, the three-loop member of this sequence turns out to have nonmaximal rigidity; however, it appears to be maximal at all higher loops. It would be worthwhile to investigate whether these diagrams can be resummed analytically, as was done for the (nontwisted) ladders in Refs. [56–60], and the six-point pentaladders of Ref. [64].

Discussion and conclusions.—At any fixed loop order and spacetime dimension (or, equivalently, fixed multiplicity), the scope of Feynman integral complexity is bounded by the finiteness in extent of the relevant loop integrands. In this Letter, we have identified several infinite classes of four-dimensional Feynman integrals—relevant to a wide range of quantum field theories—that involve more complicated geometries at each loop order than all previously known examples. Nevertheless, the extent of relevant geometries is dramatically more restrictive than even that of Calabi-Yau manifolds. Indeed, there is a real sense in which the fact that these geometries are Calabi-Yau is beside the point: The manifolds relevant to loop integration are much rarer and much more special than this.

It is not clear whether the geometry of the (CY) manifold relevant to a given Feynman integral is unique—or if it depends on the order of integrations, for example. In the case which is best understood, the elliptic double box [22] (which has rigidity 1), different integration pathways result in different parameterizations of the “same” elliptic curve. (By this, we mean that all pathways give curves with equal moduli.) It would be extremely interesting to know whether or not a similar statement were true for the integrals discussed here: Do the hypersurfaces obtained via different integration pathways encode the same manifold? We expect so; but if this were not the case, it would imply very interesting identities among period integrals over different manifolds (reminiscent of mirror symmetry, perhaps).

Whether or not the geometry of the hypersurface is uniquely fixed by the graph, our analysis above shows that—at least for the maximally rigid examples discussed—each can be written as a weight-two polylogarithm integrated over
some $2(L-1)$-dimensional algebraic variety. As such, it may be tempting to view this underlying geometry as fixed and develop technology for iterated integrals over these geometries (as has been done with considerable success in the case of the elliptic multiple polylogarithms [48,65–69]). However, it is easy to construct examples which involve sums over terms with different geometries (or even different degrees of rigidity). (This can be done by simply adding a propagator to any of the examples discussed here. It is not hard to do this in such a way that the resulting integral includes multiple, distinct copies of the original integral as contact terms.)

Conjectures.—Having proven an upper bound on rigidity for the class of marginal Feynman integrals, we conjecture that the same bound holds for all $L$-loop integrals in four dimensions. This conjecture rests on two observations: first, that polylogarithmic integrals are expected to have transcendental weight of at most $2L$ at $L$ loops, and second, that one-loop integrals are always polylogarithmic. Together these suggest that an $L$-loop integral in four dimensions can have only $2L-2$ irreducibly rigid integrations.

It is natural to wonder if a similar statement to our bounded rigidity holds for higher (even) numbers of dimensions. That is, is the rigidity of any $D$-dimensional loop integral bounded by $(L-1)D/2$? A possible route to proving such a property might involve Tarasov’s dimension-shift relations [70].

We have seen that, in two and four dimensions, marginal Feynman integrals exist that saturate the rigidity bounds described above. Beyond one loop for $D > 2$, marginality implies nonplanarity for integrals with $D$-gon power counting. It is therefore natural to wonder whether stricter rigidity bounds exist for four-dimensional planar integrals. In the examples we have considered, an upper bound on the rigidity of a Feynman graph is given by the number of propagators that are “shared” between different loops. Thus, we might compare the three-loop traintrack integral [23] with the three-loop diagram shown in Fig. 6. The traintrack has only two shared propagators and rigidity 2, while the diagram in Fig. 6 has three shared propagators and can be shown to have rigidity 3 [71]. Both graphs have lower rigidity than the three-loop diagram [Fig. 2(b)], which saturates our predicted bound having rigidity 4. We expect similar hierarchies to hold at higher loops.

Finally, we should note that the notion of geometric rigidity is rather coarse and in need of considerable refinement and elaboration. For example, an integral over an elliptic curve is clearly simpler than one over a higher-genus surface of the same dimension, and the product of two elliptic curves is clearly less complicated than a generic K3 surface. (Moreover, there are interesting distinctions to be drawn among integrals sharing the same underlying geometry; see, e.g., Refs. [72,73].) Our notion of rigidity does not allow us to distinguish such cases. An extremely important question going forward would be to develop a better understanding of the (ir)reducibility of the algebraic varieties relevant to Feynman integrals. We hope the examples discussed here may help inspire the development of better tools for understanding (and, ultimately, evaluating) such integrals.

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FIG. 6. The two most rigid three-loop integrals in the planar limit: (a) the traintrack with rigidity 2 and (b) a diagram with rigidity 3.

References


