

Inverse Problem in Effective Field Theory

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We show that the tree-level spectrum of heavy particles can be directly extracted from the Wilson coefficients of the corresponding effective field theory at low energies. This procedure is exact when the number of resonances is finite and otherwise approximate. Our results are derived from a new class of analytic dispersion relations that depend nonlinearly on the scattering amplitude and apply to an exceedingly broad class of theories and kinematics.

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Introduction—The space of effective field theories (EFTs) is infinite. This vast freedom is parametrized by an unending sequence of Wilson coefficients encoding the strengths of all possible low-energy interactions. At the same time, our actual world is dictated by some fixed set of fundamental laws, so physics at the very shortest distances must be specified by a single set of equations. Given knowledge of this complete description, a mechanical but laborious calculation then fixes the precise values of the Wilson coefficients in the low-energy EFT. A natural question then arises: Can this mapping be inverted? We dub this challenge the *EFT inverse problem*.

In this Letter, we show how the ultraviolet spectrum of particles is subtly encrypted within the Wilson coefficients of the EFT. We then present a simple algorithm for reconstructing the full dynamics of a tree-level ultraviolet completion—spectrum, scattering amplitudes, and all—directly from those Wilson coefficients.

To be very concrete, let us consider an arbitrary tree-level quantum field theory (QFT) composed of a finite number of particles. The corresponding amplitudes are rational functions of kinematic invariants whose simple poles describe the full spectrum of the theory into the ultraviolet. The low-energy expansion of these functions define the Wilson coefficients of the corresponding EFT. Since any rational function is fixed by a finite number of evaluations, a finite number of Wilson coefficients in the EFT should encode the very same data. Here, we derive a universal algorithm that achieves precisely this feat, distilling the ultraviolet

structure directly from infrared. (i) Expand the log derivative of the EFT amplitude:

$$\frac{d}{ds} \log A(s) = \sum_k c_k s^k. \quad (1)$$

(ii) Build a matrix from these EFT coefficients:

$$\mathbf{c}_r = \begin{pmatrix} c_r & c_{r+1} & c_{r+2} & \cdots \\ c_{r+1} & c_{r+2} & c_{r+3} & \\ c_{r+2} & c_{r+3} & c_{r+4} & \\ \vdots & & & \ddots \end{pmatrix}. \quad (2)$$

(iii) Count the poles and zeros using the rank:

$$\text{rank}(\mathbf{c}_r) = d = d_{\text{poles}} + d_{\text{zeros}}. \quad (3)$$

(iv) Determine the locations of these poles and zeros:

$$\det \left(\mathbf{c}_{r+1}^{(d)} - \lambda \mathbf{c}_r^{(d)} \right) = 0. \quad (4)$$

Here, s is the squared center-of-mass energy, d counts poles and zeros of $A(s)$, and $\mathbf{c}_r^{(d)}$ is the $d \times d$ principal submatrix of the $\infty \times \infty$ matrix $\mathbf{c}_r$.

Steps (i) and (ii) collate the Wilson coefficients of the EFT into the Hankel matrix $\mathbf{c}_r$. While this object is formally infinite, step (iii) indicates that it actually has finite rank equal to d . This offers a wonderfully practical litmus test for deducing d explicitly. In particular, simply tabulate more and more Wilson coefficients, building up the $\ell \times \ell$ leading principal submatrix $\mathbf{c}_r^{(\ell)}$ for incrementally increasing values of ℓ . At each value of ℓ , compute the

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determinant of $\mathbf{c}_r^{(\ell)}$. Once that determinant vanishes, we know that $\ell = d + 1$ and we have determined the number of poles and zeros. Afterward, solve the simple characteristic equation in step (iv) for λ , which as we will show satisfies

$$1/\lambda = \text{locations of poles and zeros}, \quad (5)$$

thereby dictating the spectrum as well as the entire functional form of the amplitude deep into the ultraviolet. Of course, there is a long history of studying the Wilson coefficients of EFTs [1–3] in terms of the mathematics of the moment problem [4]. The new element here is that our dispersion relation maps the poles and zeros of the amplitude to this moment sequence directly. All dependence on the residues has been sequestered into the locations of the zeros, and the determination of the spectrum has been transformed into finding the solutions of the generalized eigenvalue problem in Eq. (4).

The linchpin of this procedure is a new class of dispersion relations that are *nonlinear* in the scattering amplitude. Our only assumptions are that the amplitude is meromorphic and that its logarithm is polynomially bounded, so our construction is exceedingly insensitive to ultraviolet behavior. It can thus be applied where conventional methods fail, for instance, in the kinematic regime of *hard scattering* or in theories that are *non-polynomially bounded* in energy. As outlined above, our approach is maximally powerful for QFTs with finite numbers of degrees of freedom but also generalizes to theories with infinite towers of particles, like string theory.

Dispersion relation—The central object in our construction is the quotient

$$Q(s) = \frac{d}{ds} \log A(s) = \frac{A'(s)}{A(s)}, \quad (6)$$

where $A(s)$ is the four-point tree-level amplitude, suppressing all other kinematic dependences.

Clearly, $Q(s)$ exhibits poles when $A'(s)$ is singular or $A(s)$ is vanishing. At the locus of a pole in $A(s)$ at $s = \mu$, the amplitude and its derivative satisfy $\lim_{s \rightarrow \mu} A(s) = -R/(s - \mu) + \dots$ and $\lim_{s \rightarrow \mu} A'(s) = R/(s - \mu)^2 + \dots$, and similarly at the locus of a zero at $s = \rho$, $\lim_{s \rightarrow \rho} A(s) = S(s - \rho) + \dots$ and $\lim_{s \rightarrow \rho} A'(s) = S + \dots$. Observe that the residue R and slope S each exactly cancel, so [5]

$$Q(s) = -\sum_{\text{poles}} \frac{1}{s - \mu} + \sum_{\text{zeros}} \frac{1}{s - \rho} + \dots, \quad (7)$$

where the ellipses denote terms that are entire in s . Something a bit miraculous has happened [6]: All information except the locations of the poles and zeros has evaporated from singular components of $Q(s)$.

The simple structure of Eq. (7) motivates us to consider a dispersion relation for $Q(s)$. Importantly, $Q(s)$ is calculable within the low-energy expansion of the EFT:

$$Q(s) = \frac{d}{ds} \log A(s) = \sum_k c_k s^k \quad (8)$$

whose series coefficients c_k will play a crucial role for us. While the Wilson coefficients one would measure experimentally are those in the expansion of $A(s)$ rather than $Q(s)$, the coefficients of these expansions are, of course, bijectively related, with a closed form given in Appendix A; we will therefore view the c_k themselves as equivalent to Wilson coefficients. Here, the lower limit on k can be negative if there are long-range forces. We can then relate c_k to the ultraviolet dynamics via a contour integral about the origin of the complex s plane:

$$\begin{aligned} c_k &= \frac{1}{2\pi i} \oint_{s=0} \frac{ds}{s^{1+k}} Q(s) \\ &= -\frac{1}{2\pi i} \oint_{s \neq 0} \frac{ds}{s^{1+k}} \left(-\sum_{\text{poles}} \frac{1}{s - \mu} + \sum_{\text{zeros}} \frac{1}{s - \rho} \right), \end{aligned} \quad (9)$$

where k defines the degree of the subtraction. Evaluating the contour integral on the right-hand side yields a generalized version of Cauchy's argument principle [8]:

$$c_k = \sum_{\text{poles}} \frac{1}{\mu^{1+k}} - \sum_{\text{zeros}} \frac{1}{\rho^{1+k}}. \quad (10)$$

Physically, this expression implies our central result that the Wilson coefficients of the EFT directly encode the locations of poles and zeros of the amplitude [9].

In the above equations, we have dropped boundary terms at complex infinity for s . This is justified if

$$\lim_{s \rightarrow \infty} \frac{Q(s)}{s^k} = 0 \quad (11)$$

for our chosen value of k . We emphasize that this condition is typically *exceedingly weak*, since the log of the amplitude in all known cases—including all QFTs and string theories—grows at worst as a polynomial in s . This criterion fails only if the high-energy behavior of the amplitude is *doubly exponential* in s .

Field theory examples—It will be illuminating to explore the implications of Eq. (10) in various QFTs.

To begin, let us consider a toy model describing the exchange of a single particle of mass squared μ in the s channel. The corresponding amplitude is

$$A(s) = -\frac{R}{s - \mu}. \quad (12)$$

Since the quotient $Q(s)$ goes to zero at large s , we can consider any $k \geq 0$. From Eq. (10), we have $c_k = 1/\mu^{1+k}$, which confirms the obvious expectation that the Wilson coefficients are infinitely redundant and just reconstruct the geometric series induced by the s channel exchange.

For a theory with two exchanges in the s channel, the story is essentially the same. The amplitude is

$$A(s) = -\frac{R_1}{s - \mu_1} - \frac{R_2}{s - \mu_2}, \quad (13)$$

and Eq. (10) confirms that

$$c_k = \frac{1}{\mu_1^{1+k}} + \frac{1}{\mu_2^{1+k}} - \frac{1}{\rho^{1+k}}, \quad (14)$$

where the single zero is $\rho = (R_1\mu_2 + R_2\mu_1)/(R_1 + R_2)$.

In general, the amplitude for any tree-level, polynomially bounded QFT takes the form

$$A(s) \propto \prod_{n=1}^{d_{\text{zeros}}} (s - \rho_n) / \prod_{n=1}^{d_{\text{poles}}} (s - \mu_n). \quad (15)$$

Here, all kinematic invariants other than s have been fixed in such a way that the resulting amplitude is meromorphic in s . This would occur, for instance, at fixed t . By inserting arbitrary values of μ_n and ρ_n into Eq. (15), computing $Q(s)$, and expanding as in Eq. (8), we precisely confirm Eq. (10).

String theory examples—Next, let us confirm our result in Eq. (10) for the case of the Veneziano amplitude for the scattering of open strings [10]. In the context of the superstring, this amplitude is

$$A(s, t) = -\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)}, \quad (16)$$

where for simplicity we have stripped away the gauge-invariant prefactor that encodes the external polarization data. It will be instructive to study our results for various choices of kinematics.

Fixed t : Consider the Regge limit of fixed t and large s . Inserting Eq. (16) into Eq. (8), we obtain

$$\begin{aligned} Q(s) &= \frac{d}{ds} \log A(s, t) \\ &= -\frac{1}{s} + \gamma + \psi(1-t) + \left(\frac{\pi^2}{6} - \psi^{(1)}(1-t) \right) s \\ &\quad + \left(\zeta(3) + \frac{1}{2} \psi^{(2)}(1-t) \right) s^2 + \dots \end{aligned} \quad (17)$$

Since the Veneziano amplitude in Eq. (16) has poles at $s = n$ and zeros at $s = 1 + n - t$, we find that the terms in Eq. (17) agree exactly with Eq. (10):

$$\begin{aligned} c_k^{(t)}(t) &= \sum_{n=1}^{\infty} \frac{1}{n^{1+k}} - \sum_{n=0}^{\infty} \frac{1}{(1+n-t)^{1+k}} \\ &= \zeta(1+k) - \zeta(1+k, 1-t) \end{aligned} \quad (18)$$

after analytic resummation, since the Hurwitz zeta function satisfies $\zeta(1+k, 1-t) = (-1)^{k+1} \psi^{(k)}(1-t)/k!$ [11].

Fixed t/s : Conventional dispersion relations fail for *hard scattering* in the context of higher-spin towers because the corresponding amplitudes grow as [12]

$$\lim_{s, t \rightarrow +\infty} A(s, t) \sim e^{B(s, t)}, \quad (19)$$

where we have defined

$$B(s, t) = (s+t) \log(s+t) - s \log s - t \log t + \dots, \quad (20)$$

which is familiar from string amplitudes. This exponential blowup stymies conventional dispersion relations by introducing uncontrolled boundary contributions.

In contrast, our new dispersion relations are perfectly amenable to hard scattering. Given the fixed ratio

$$z = t/s, \quad (21)$$

we have $B(s, z) = [(1+z) \log(1+z) - z \log z]s + \dots$, corresponding to a constant quotient

$$\lim_{s \rightarrow \infty} Q(s) = (1+z) \log(1+z) - z \log z. \quad (22)$$

Thus, our dispersion relations are applicable for $k \geq 1$. Expanding the Veneziano amplitude at fixed z , we obtain

$$\begin{aligned} Q(s) &= \frac{d}{ds} \log A(s, zs) \\ &= -\frac{2}{s} - \frac{\pi^2 z}{3} s - 3\zeta(3)z(1+z)s^2 + \dots \end{aligned} \quad (23)$$

With the choice in Eq. (21), the Veneziano amplitude in Eq. (16) has poles at $s = n$ and n/z as well as zeros at $s = (1+n)/(1+z)$, giving us the prediction of Eq. (10):

$$\begin{aligned} c_k &= \sum_{n=1}^{\infty} \frac{1}{n^{1+k}} + \sum_{n=1}^{\infty} \frac{1}{\left(\frac{n}{z}\right)^{1+k}} - \sum_{n=0}^{\infty} \frac{1}{\left(\frac{1+n}{1+z}\right)^{1+k}} \\ &= \zeta(1+k)(1+z)^{1+k} - (1+z)^{1+k}. \end{aligned} \quad (24)$$

Comparing with Eq. (23), we find complete agreement.

Fixed u : The case of fixed u is especially interesting. At fixed u , the string amplitude asymptotes to

$$\lim_{s \rightarrow \infty} A(s, u) \sim s^{u-1}. \quad (25)$$

For general fixed u , we compute the quotient

$$\begin{aligned}
 Q(s) &= \frac{d}{ds} \log A(s, -s - u) \\
 &= -\frac{1}{s} + \gamma + \psi(u) + \left(\frac{\pi^2}{6} + \psi^{(1)}(u) \right) s \\
 &\quad + \left(\zeta(3) + \frac{1}{2} \psi^{(2)}(u) \right) s^2 + \dots
 \end{aligned} \tag{26}$$

At fixed u , the Veneziano amplitude has poles at $s = n$ and $s = -n - u$. While the Veneziano amplitude has “hidden zeros” at negative integer u that generalize at higher multiplicity [13], there are otherwise no zeros in s at fixed u . Again computing the prediction of Eq. (10),

$$\begin{aligned}
 c_k^{(u)}(u) &= \sum_{n=1}^{\infty} \frac{1}{n^{1+k}} + \sum_{n=0}^{\infty} \frac{1}{(-n-u)^{1+k}} \\
 &= \zeta(1+k) + (-1)^{k+1} \zeta(1+k, u),
 \end{aligned} \tag{27}$$

we find agreement with Eq. (26). Interestingly, for $u = 0$, the Wilson coefficients c_k are zero for even k and non-negative for odd k , irrespective of unitarity.

As is well known, the EFT coefficients in the expansion of the Veneziano amplitude in Eq. (16) are multiple zeta values expressed as an exponential sum over simple zeta values [14,15]. Our dispersion relation for the quotient $Q(s)$ provides an alternative understanding of this behavior.

Spectroscopy—The above examples illustrate how the poles and zeros of the amplitude exactly fix the coefficients c_k . Here, we will see that the converse is also true: The Wilson coefficients can be manipulated to extract the spectrum of poles and zeros.

Consider a general QFT amplitude, which is polynomially bounded and exhibits a finite number of poles and zeros. Our results imply that the c_k are given as moment sums over the poles and zeros of $A(s)$ as in Eq. (10). It will be convenient to repackage this sum as

$$c_k = \sum_{n=1}^d \sigma_n \lambda_n^{1+k}. \tag{28}$$

Here, $d = d_{\text{poles}} + d_{\text{zeros}}$, and $\sigma_n = +1$ or -1 when $1/\lambda_n$ is a pole or a zero, respectively. We consider the case where the λ_n are all distinct, though this can be generalized [16]. The Hankel matrix in Eq. (2) becomes

$$[\mathbf{c}_r]_{ij} = c_{r+i+j} = \sum_{n=1}^d \sigma_n \lambda_n^{1+r+i+j}. \tag{29}$$

Formally, the i and j indices extend infinitely, since there is an infinite sequence of Wilson coefficients in the EFT. By inspection, we see that $\mathbf{c}_r$ factorizes into

$$\mathbf{c}_r = \lambda \cdot \mathbf{d}_r \cdot \lambda^T, \tag{30}$$

where we have defined the $\infty \times d$ Vandermonde matrix $[\lambda]_{\text{in}} = \lambda_n^i$ and the $d \times d$ diagonal matrix

$$[\mathbf{d}_r]_{nm} = \sigma_n \lambda_n^{1+r} \delta_{nm}. \tag{31}$$

Since the λ_n are by definition distinct, λ is left-invertible. The Vandermonde decomposition in Eq. (30) preserves rank, so $\mathbf{c}_r$ has rank d , confirming Eq. (3).

A corollary of this logic is that the $\ell \times \ell$ leading principal submatrix satisfies $\text{rank}(\mathbf{c}_r^{(\ell)}) = \min(\ell, d)$ which reduces to Eq. (3) when ℓ is infinite. This observation implies that the determinant of the matrix is zero beyond a critical value:

$$\det(\mathbf{c}_r^{(\ell)}) = 0 \quad \text{for } \ell > d. \tag{32}$$

Then d is determined by populating the matrix $\mathbf{c}_r^{(\ell)}$ until ℓ is sufficiently large that the determinant vanishes.

To enumerate these quantities, we solve the characteristic equation in Eq. (4) for λ . The argument of the determinant in Eq. (4) is the $d \times d$ submatrix of $\mathbf{c}_{r+1} - \lambda \mathbf{c}_r = \lambda \cdot (\mathbf{d}_{r+1} - \lambda \mathbf{d}_r) \cdot \lambda^T$. Rewriting the diagonal matrix in the center as $[\mathbf{d}_{r+1} - \lambda \mathbf{d}_r]_{nm} = \sigma_n \lambda_n^{1+r} (\lambda_n - \lambda) \delta_{nm}$, we see that zeros occur precisely when $\lambda = \lambda_n$. The solutions of Eq. (4) thus define the entire sequence λ_n . Inverting the Vandermonde matrix, we have $\mathbf{d}_r = \lambda^{-1} \cdot \mathbf{c}_r \cdot (\lambda^T)^{-1}$. For odd r , we see from Eq. (31) that the eigenvalues of this matrix precisely dictate σ_n , which flags whether $1/\lambda_n$ is a pole or a zero. We thus find that

$$\sigma_n = \text{sgn}([\mathbf{d}_r]_{nn}) \quad \text{for odd } r. \tag{33}$$

Since we can always choose odd r , this final datum allows us to determine the exact positions of all poles and zeros.

In summary, to extract the complete spectrum of poles and zeros from the EFT coefficients, solve for λ in Eq. (4) to extract all λ_n . Then insert λ_n into the definition of λ to define the matrix $\mathbf{d}_r$, which determines the sign of σ_n from Eq. (33) and specifies $1/\lambda_n$ as a pole or a zero. While this procedure takes as input the Wilson coefficient expansion of $Q(s)$, the very same algorithm applies to the coefficients of $A(s)$, in which case λ_n and σ_n in Eq. (31) are the mass and residue at each pole.

The above procedure works beautifully if we have access to a sufficient number of Wilson coefficients. Since the Hankel matrix $\mathbf{c}_r$ has rank d , we need all Wilson coefficients in the leading $d \times d$ principal submatrix $\mathbf{c}_r^{(d)}$. But what if we have insufficient data? Concretely, consider the case where we are privy only to the Wilson coefficients in the leading $\ell \times \ell$ principal submatrix $\mathbf{c}_r^{(\ell)}$ for $\ell < d$. Notably, this case subsumes that of string theory, whose infinite spectrum implies $d = \infty$. The naturally truncated version of Eq. (4) is then [17]

$$\det(\mathbf{c}_{r+1}^{(\ell)} - \lambda^{(\ell)} \mathbf{c}_r^{(\ell)}) = 0 \quad \text{for } \ell < d. \quad (34)$$

As shown in Appendix B, while $1/\lambda$ labels the poles and zeros of $Q(s)$, the quantity $1/\lambda^{(\ell)}$ labels the poles and zeros of the best-fit rational polynomial for $Q(s)$, known as a Padé approximant. Even for an infinite spectrum, the convergence properties of Padé approximants [18] imply the convergence of $1/\lambda^{(\ell)}$ to the exact locations of the poles and zeros as $\ell \rightarrow \infty$. See Fig. 1 for the approximate string spectrum extracted from the Wilson coefficients of the Veneziano amplitude in Eq. (16) for $t = -s$.

Double copy—Conventional dispersion relations are linear in the amplitude, so they act simply on *sums* of amplitudes. In contrast, our new dispersion relations are logarithmic, so they play well with *products*. A context where such products arise is the double copy [19]. In particular, the Kawai-Lewellen-Tye (KLT) relations [20] recast the Virasoro-Shapiro amplitude [21,22] of closed strings as

$$\begin{aligned} A^{(\text{VS})}(s, t) &= -\frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \\ &= \frac{\sin \pi s}{\pi} A^{(\text{V})}(s, t) A^{(\text{V})}(s, u), \end{aligned} \quad (35)$$

where $A^{(\text{V})}(s, t)$ is the Veneziano amplitude defined in Eq. (16). Inserting Eq. (35) into our dispersion relation, we discover a subtle link between the Wilson coefficients of the open and closed string:

$$c_k^{(\text{VS})}(t) = c_k^{(t)}(t) + c_k^{(u)}(t) + c_k^{(\text{KLT})}, \quad (36)$$

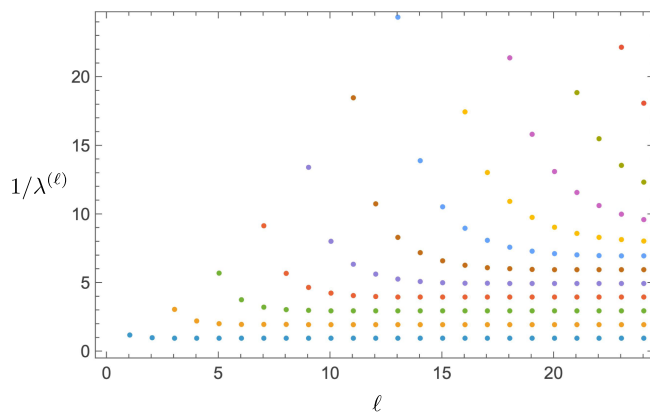


FIG. 1. The spectrum of string theory is extracted directly from its low-energy EFT. Given the Wilson coefficients c_k in Eq. (27) for $u = 0$, construct the $\ell \times \ell$ principal submatrix $\mathbf{c}_r^{(\ell)}$, whose entries are $[\mathbf{c}_r^{(\ell)}]_{ij} = c_{r+i+j}$. Solving $\det(\mathbf{c}_{r+1}^{(\ell)} - \lambda^{(\ell)} \mathbf{c}_r^{(\ell)}) = 0$ for $r = 1$, then plot $1/\lambda^{(\ell)}$ as a function of ℓ . The spectrum approaches that of the string for large ℓ .

where $c_k^{(t)}$ and $c_k^{(u)}$ are defined in Eqs. (18) and (27). Here, $c_k^{(\text{KLT})}$ are new terms that arise in the dispersion relation because of zeros in the KLT kernel and can be given in terms of the Bernoulli numbers:

$$\begin{aligned} \frac{d}{ds} \log \sin \pi s &= \frac{1}{s} + \sum_{k \text{ odd}} \frac{(-1)^{(k+1)/2} (2\pi)^{k+1}}{(k+1)!} B_{k+1} s^k \\ &= \frac{1}{s} + \sum_{k=0}^{\infty} c_k^{(\text{KLT})} s^k. \end{aligned} \quad (37)$$

We have verified Eq. (36) using the Wilson coefficients for the Veneziano and Virasoro-Shapiro amplitudes.

Discussion—This Letter offers numerous avenues for future inquiry. A natural question regards the inclusion of spin. For simplicity, we have focused entirely on scattering processes with external scalars only, but extending to general spin should be straightforward. In particular, one can strip off all helicity weights at will and apply our methodology to the multiplicative remainders. For example, given a four-dimensional scattering amplitude of fermions, $A(s, t) = \langle 12 | [34] f(s, t)$, we would derive dispersion relations for $f(s, t)$. While this approach implicitly drops helicity-dependent zeros, another alternative would be to instead manipulate the analytic continuation of the *squared amplitude*, $|A(s, t)|^2 = s^2 f(s, t)^2$, which carries no helicity weight whatsoever.

Our results should generalize to tree-level amplitudes at higher multiplicity. In particular, Eq. (10) applies equally well to higher-point amplitudes, provided we define the variable s in the dispersion relation as some higher-point kinematic invariant. We have verified by explicit calculation that the analogue of Eq. (10) holds for five-point amplitudes in planar ϕ^3 theory and six-point amplitudes in the nonlinear sigma model.

On the other hand, a clear limitation of our current framework is the necessity of the tree-level approximation. Since the mechanics of our dispersion relations depend so crucially on the meromorphic structure of tree amplitudes, the extension to loop level is far from clear. Typical loop amplitudes exhibit logarithms and polylogarithms, with rare exceptions such as the rational loop amplitudes of Yang-Mills theory [23] and self-dual gravity [24], which preserve meromorphicity. We leave this important question to future work.

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Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- $$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz = d_{\text{zeros}} - d_{\text{poles}},$$
- where $f(z)$ is meromorphic and γ is a closed curve encircling d_{zeros} zeros and d_{poles} poles of $f(z)$.
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End Matter

Appendix A: From $A(s)$ to $Q(s)$ —The series expansion of $Q(s) = d \log A(s)/ds$ encodes that exact same information as the series expansion for $A(s)$. However, any experimental probe will reconstruct the

low-energy expansion of $A(s)$ rather than $Q(s)$. For completeness, we now derive a simple formula relating the series expansions of these functions. In particular, let us define the EFT expansion of the amplitude:

$$A(s) = \sum_{j=0}^{\infty} b_j s^j, \quad (\text{A1})$$

for convenience considering the case without a massless pole, since the generalization is straightforward. By explicit calculation, we find that the expansion of $Q(s)$ can be written in terms of the expansion of $A(s)$ as

$$\begin{aligned} c_0 &= \frac{b_1}{b_0} \\ c_1 &= -\frac{b_1^2 - 2b_0 b_2}{b_0^2} \\ c_2 &= \frac{b_1^3 - 3b_0 b_1 b_2 + 3b_0^2 b_3}{b_0^3} \\ c_3 &= -\frac{b_1^4 - 4b_0 b_1^2 b_2 + 2b_0^2 b_2^2 + 4b_0^2 b_1 b_3 - 4b_0^3 b_4}{b_0^4}, \end{aligned} \quad (\text{A2})$$

and so on. The numerical coefficients appearing in these equations coincide with those of the multivariable Faber partition polynomials. In fact, one can find an explicit relation between b_j and c_k , given by

$$c_k = -(k+1) \sum_{\substack{m_i \in \mathbb{Z}_{\geq 0} \\ \sum_{i=1}^{k+1} i m_i = k+1}} \Gamma\left(\sum_{i=1}^{k+1} m_i\right) \prod_{j=1}^{k+1} \frac{(-b_j/b_0)^{m_j}}{m_j!}, \quad (\text{A3})$$

where the right-hand side sums over partitions of $k+1$. Importantly, any c_k depends on b_j only for $j \leq k+1$. Reconstruction of the spectrum of poles and zeros requires knowledge of some leading set of the c_k , which translates into knowledge of some leading set of the b_j and so might, in principle, be accessed experimentally.

Appendix B: Padé approximant spectroscopy—Our construction is closely related to the methodology of Padé approximants [18,25,26]. Let us define the paradiagonal Padé approximant of order $[(\ell+r-1)/\ell]$ to the function $Q(s)$ as $P(s) = N(s)/D(s)$ for polynomials $N(s)$ and $D(s)$ of degree $\ell+r-1$ and ℓ , respectively. We then have by definition $D(s)Q(s) - N(s) = O(s^{2\ell+r})$. For convenience, let us assume that $Q(s)$ is regular at $s=0$; the analogous construction with a massless pole is straightforward. Writing

$$\hat{Q}(s) = \frac{Q(s) - \sum_{k=0}^{r-1} c_k s^k}{s^r} = \sum_{k=0}^{\infty} c_{k+r} s^k, \quad (\text{B1})$$

the Padé approximant to $\hat{Q}(s)$ of the order of $[(\ell-1)/\ell]$ has the same denominator $D(s)$, since

$$\hat{D}(s)Q(s) - \hat{D}(s) \sum_{k=0}^{r-1} c_k s^k - s^r \hat{N}(s) = O(s^{2\ell+r}) \quad (\text{B2})$$

implies that $\hat{D}(s) = D(s)$. Writing the expansion $\hat{D}(s) = \sum_{j=0}^{\ell} d_j s^j$ and matching the coefficient of $s^{\ell}, s^{\ell+1}, \dots, s^{2\ell-1}$ in $D(s)\hat{Q}(s) = \hat{N}(s) + O(s^{2\ell})$ gives the Padé condition

$$\sum_{j=0}^{\ell} c_{r+i+\ell-j} d_j = 0, \quad i = 0, \dots, \ell-1. \quad (\text{B3})$$

Let $1/y$ be a zero of $\hat{D}(s)$. Then, $\hat{D}(s)/(1-ys)$ is a polynomial whose expansion we can write as $\sum_{j=0}^{\ell-1} e_{\ell-1-j} s^j$. Equating the two expansions of $\hat{D}(s)$, we have

$$\begin{aligned} d_0 &= e_{\ell-1}, & d_{\ell} &= -y e_0, \\ d_j &= e_{\ell-1-j} - y e_{\ell-j}, & 1 \leq j \leq \ell-1. \end{aligned} \quad (\text{B4})$$

Substituting these relations into Eq. (B3), we have $\sum_{j=0}^{\ell-1} (c_{r+1+i+j} - y c_{r+i+j}) e_j = 0$, or, in matrix form,

$$[(\mathbf{c}_{r+1}^{(\ell)} - y \mathbf{c}_r^{(\ell)}) \mathbf{e}]_i = 0, \quad (\text{B5})$$

so $\mathbf{e}$ is an eigenvector of $\mathbf{c}_{r+1}^{(\ell)} - y \mathbf{c}_r^{(\ell)}$ with zero eigenvalue, which implies

$$\det(\mathbf{c}_{r+1}^{(\ell)} - y \mathbf{c}_r^{(\ell)}) = 0. \quad (\text{B6})$$

Conversely, suppose we have a solution y of Eq. (B5) for some eigenvector $\mathbf{e}$. We can construct a polynomial $\tilde{D}(s) = (1-ys) \sum_{j=0}^{\ell-1} e_{\ell-1-j} s^j = \sum_{j=0}^{\ell} \tilde{d}_j s^j$. By inspection, we see that the algebraic relations between $\tilde{d}_j$ and e_j are precisely those satisfied by d_j in Eq. (B4), so $d_j = \tilde{d}_j$ and $\tilde{D}(s) = \hat{D}(s)$. Since $1/y$ is a zero of $\tilde{D}(s)$ by construction, we have $\hat{D}(1/y) = 0$ as well. Thus, the solutions $\lambda^{(\ell)}$ of the truncated Hankel matrix problem in Eq. (34) are precisely the inverses of the poles of the Padé approximant to $Q(s)$ of order $[(\ell+r-1)/\ell]$.