

Multiphoton Quantum Simulation of the Generalized Hopfield Memory Model

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In the present Letter, we introduce, develop, and investigate a connection between multiphoton quantum interference, a core element of emerging photonic quantum technologies, and Hopfield-like Hamiltonians of classical neural networks, the paradigmatic models for associative memory and machine learning in systems of artificial intelligence. Specifically, we show that combining a system composed of N_{ph} indistinguishable photons in superposition over M field modes, a controlled array of M binary phase shifters, and a linear-optical interferometer, yields output photon statistics described by means of a p -body Hopfield Hamiltonian of M Ising-like neurons ± 1 , with $p = 2N_{\text{ph}}$. We investigate in detail the generalized four-body Hopfield model obtained through this procedure, undergoing a transition from a memory retrieval to a memory black-out regime, i.e., a spin-glass phase, as the amount of stored memory increases. The mapping enables novel routes to the realization and investigation of disordered and complex classical systems via efficient photonic quantum simulators and describes aspects of structured photonic systems in terms of classical spin Hamiltonians.

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In recent years, the study of bosonic quantum interference with photons coherently evolving through linear-optical networks has been a central focus in the development of photonic quantum technologies. This phenomenon underpins a range of applications, including architectures for fault-tolerant photonic quantum computing [1–3], potential near-term quantum computational advantages with specialized algorithms [4], and high-precision sensing [5]. New advances have enabled the generation, interference, and detection of systems with over 100 photons, achieving quantum computation [6,7], quantum simulation of quantum and classical systems [8–10], or memory retrieval [11], at scales competitive with conventional supercomputers. In

this Letter, we present a connection between multiphoton quantum interference in linear optics and associative memory neural networks with multisynaptic connections: the generalized p -body Hopfield model (p HM) [12–18].

The Hopfield model [19] has been a milestone in realizing a statistical physical model of associative memory. It is a full feedback neural network providing pattern recognition, associative memory functions, and error correction, now recognized as a fundamental approach to studying memory storage and retrieval [20–22], as well as machine learning problems [23,24]. It has recently been suggested to introduce generalized HMs with multisynaptic interactions in the structure of deep neural networks to store information or learn prototypes [17,18].

We show that photonic systems composed of [cf. Figs. 1(a)–1(d)] (i) N_{ph} photons in an initial superposition state over M optical modes, (ii) a layer of M phase shifters, and (iii) a linear-optical interferometer implementing a general scattering matrix over the M modes, present photon statistics at their output modes that

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can be described through the Hamiltonian of a $p = 2N_{\text{ph}}$ -body Hopfield model for a neural network of M neurons.

Optical simulation of the 2HM was first demonstrated with an incoherent source [25], while a performance advantage was achieved only with coherent light and large-scale optical manipulation [26], also supporting Hopfield-inspired memory storage and computation [27]. As in the original HM [19], the number P of retrievable memories is a fraction of the number of neurons, $\alpha = P/M < 1$, and cannot grow more than linearly with M .

In the Hopfield model, (see the Supplemental Material [28] and Refs. [12–20,29]) the neuronal activity is represented by a set σ of M Ising spins $\sigma_i = \pm 1$. A Hamiltonian $\mathcal{H}$ can be devised by requiring that the planted memory patterns $\xi^{(\mu)}$ are *attractors* of the σ dynamics, i.e., the minima of $\mathcal{H}[\sigma]$ are realized by those $\sigma \rightarrow \xi^{(\mu)}$. Introducing the memory overlap between the neuronal state σ and the memory pattern $\xi^{(\mu)}$: $m_\mu \equiv \sigma \cdot \xi^{(\mu)} / M$, the Hamiltonian of a multineuron synapsis Hopfield model (p HM), connecting $p > 2$ neurons reads as [12–16]

$$\mathcal{H}[\sigma] = -M \sum_{\mu=1}^P |m_\mu|^p + O(1) = - \sum_{i_1, \dots, i_p} J_{i_1, \dots, i_p} \prod_{\ell=1}^p \sigma_{i_\ell}, \quad (1)$$

with $J_{i_1, \dots, i_p} \propto \sum_{\mu} \prod_{\ell=1}^p \xi_{i_\ell}^{(\mu)} / M^{p-1}$.

Synaptic nonlinearity ($p > 2$) realizes a larger memory capacity ($P_{\text{max}} \sim M^{p-1}$ [20]), improves the recognition and classification capabilities of a network, enabling the “universal approximation” of any function [17], and potentially boosts the next generation of neural networks [18].

A drawback is that simulating the dynamics of fully connected p HM requires a time which scales super extensively with the number of neurons generically, like the average connectivity $\sim M^{p-1}$. However, analog photonic computation would drastically reduce this time.

To generalize to the multisynaptic case, though, requires a further stratagem in the photonic system proposed for the energy measurement: the use of photonic coherent superpositions and interferometry. In this Letter, we realize p HMs with even $p = 2N_{\text{ph}}$ by means of coherent superposition of N_{ph} photon state scattering, providing a memory capacity scaling up to $P \sim M^{p/2}$.

The connection between linear-optical quantum systems and statistical physics opens new ways for the investigation of classical p HM with quantum photonic systems, potentially enabling efficient calculations for large p and M . We illustrate this through the simulation of photonic systems implementing a 4HM, showing evidence of the characteristic transition from retrieval to the spin-glass phase emerging from the photons’ statistics.

The system is a linear optical quantum processor, composed of single photons as inputs; a linear interferometer, which can universally be implemented with only phase shifter beam splitters [30,31]; and photo detectors.

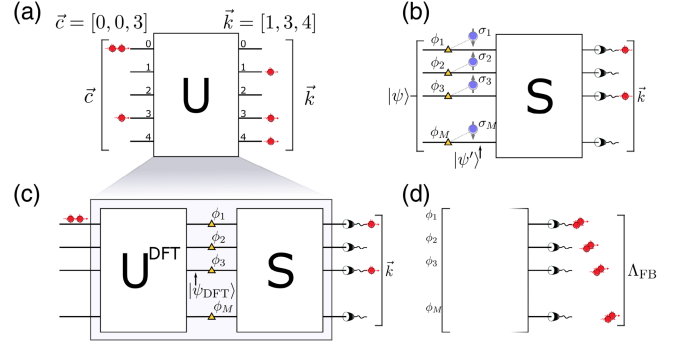


FIG. 1. (a) Standard description of a linear optical transformation from an input configuration $\vec{c}$ to an output configuration $\vec{k}$ through a linear scattering matrix U . An example is depicted for $N_{\text{ph}} = 3$ photons and $M = 5$ modes. (b) Schematic of the mapping of a p HM to a photonic system composed of an input state $|\psi\rangle$, a set of M phase shifters which can have binary values $\phi_i \in \{0, \pi\}$ and map to the M spin states $\sigma_i = \exp(i\phi_i) = \pm 1$, and a scattering matrix S . (c) Full schematic including the initial discrete Fourier transform U^{DFT} for generating the near-uniform input state $|\psi^{\text{DFT}}\rangle$. (d) Simplified scheme with the set of output states Λ_{FB} given by fully bunched configurations with all photons exiting in the same output mode.

Such a platform is, in principle, compatible with scalable quantum computers [1–3] as well as near-term quantum simulators [32].

The advantages of photonic implementations include lightning-fast transformations, inherent parallel operations, and low consumption, all valuable properties for future generations of computing hardware [26,27,33,34]. Here, we propose a new platform for optical computing in which we employ the inherent quantum properties of photon particles to map the classical p HM (see the Supplemental Material [28]).

The mapping stems from the equations governing the N_{ph} -photon current after a linear transformation, such as the transmission through a random medium, characterized by a scattering matrix. We show how feeding a prepared state of N_{ph} single photons to an interferometric network enables the simulation of the dynamics of a corresponding statistical physical $2N_{\text{ph}}$ HM.

A single photon j can be in M modes, labeled by $c_j = 0, \dots, M-1$. The Fock state of the N_{ph} photons is represented by $\vec{c} = \{c_1, \dots, c_{N_{\text{ph}}}\}$. The set of all possible configurations of N_{ph} indistinguishable photons over M modes is denoted by the set C_M , whose cardinality is $|C_M| = \binom{N_{\text{ph}}+M-1}{N_{\text{ph}}} \simeq M^{N_{\text{ph}}} / N_{\text{ph}}!$ for large M .

A set of controllable elements composes the neuronal state, as shown in Fig. 1(b). The first is the initial photonic state: a superposition $|\psi\rangle = \sum_{\vec{c} \in C_M} a_{\vec{c}} |\vec{c}\rangle$ of N_{ph} indistinguishable photons over M modes with amplitudes $a_{\vec{c}}$. State normalization gives $\sum_{\vec{c} \in C_M} |a_{\vec{c}}|^2 = 1$. The state, then, undergoes a set of single-mode linear phase shifters,

providing the opportunity of individual phase control, enabling the setting of an additive phase delay ϕ_c for each mode c . The phases take values 0 or π and transform the state as $|\psi'\rangle = \sum_{\vec{c} \in C_M} a'_c |\vec{c}\rangle$ with new amplitudes a'_c given by

$$a'_c = a_c e^{i \sum_{j=1}^{N_{\text{ph}}} \phi_{c_j}} = a_c \prod_{j=1}^{N_{\text{ph}}} e^{i \phi_{c_j}} = a_c \prod_{j=1}^{N_{\text{ph}}} \sigma_{c_j}, \quad (2)$$

where $\sigma_c = e^{i \phi_c} = \pm 1$, an *Ising spin* variable playing the role of the *neuronal activity* of neuron c .

The state $|\psi'\rangle$ evolves through a scattering medium with associated matrix S and transformation $\mathcal{S}$. The scattering amplitude for a final state $|\vec{k}\rangle$ can be calculated (see the End Matter) using the linearity of $\mathcal{S}$:

$$\langle \vec{k} | \mathcal{S}(|\psi'\rangle) \rangle = \sum_{\vec{c} \in C_M} \frac{a_c}{\sqrt{\mu(\vec{c})\mu(\vec{k})}} \text{Perm}(S_{\vec{k}|\vec{c}}) \prod_{j=1}^{N_{\text{ph}}} \sigma_{c_j}. \quad (3)$$

Finally, we consider the placement of photo detectors on the output modes of the interferometer [see Fig. 1(b)], to measure the output probability for each configuration $\vec{k}$ in a subset $\Lambda \subseteq C_M$ of all possible output configurations. Denoting as $\vec{x}$ and $\vec{y}$ the input configurations in Eq. (3), the joint occurrence probability of detecting photons in any configuration $\vec{k} \in \Lambda$ is simply given by the sum of single scattering probabilities for each $\vec{k}$, cf. Eq. (A4), and the Supplemental Material [28] and Ref. [35]:

$$\text{Pr}(\Lambda|\sigma) = \sum_{\vec{k} \in \Lambda} \text{Pr}_S(\vec{k}|\sigma) = \sum_{\vec{x}, \vec{y} \in C_M} J_\Lambda(\vec{x}, \vec{y}) \prod_{j=1}^{N_{\text{ph}}} \sigma_{x_j} \sigma_{y_j}, \quad (4)$$

where σ is the set of all possible configurations of 0, π phase modulations for all M modes in $\vec{x}$ and $\vec{y}$. The coupling tensor J_Λ is the sum of Hebb-like [19,29] terms:

$$J_\Lambda(\vec{x}, \vec{y}) = \sum_{\vec{k} \in \Lambda} \mathbb{X}_{\vec{x}}^{(\vec{k})} \mathbb{X}_{\vec{y}}^{(\vec{k})*}, \quad \mathbb{X}_{\vec{x}}^{(\vec{k})} \equiv \sum_{\vec{c} \in C_M} \frac{a_{\vec{c}} \text{Perm}(S_{\vec{k}|\vec{c}})}{\sqrt{\mu(\vec{x})\mu(\vec{k})}}. \quad (5)$$

Setting $H[\sigma|\Lambda] = -M \text{Pr}(\Lambda|\sigma)$, Eq. (4) is related to a p HM Hamiltonian, with $p = 2N_{\text{ph}}$, where Λ plays the role of the set of planted memory patterns. The last term in (4) corresponds to a random walk of $|C_M|^2$ steps in the space of input states. As M increases, in order for $\text{Pr}(\Lambda|\sigma)$ to remain of $O(1)$, as it suits to a probability, the J_Λ 's must have a mean square displacement scaling as $|J_\Lambda| \sim 1/|C_M| \sim 1/M^{N_{\text{ph}}}$ for large M . In the two photons case, Eq. (5) is a four index coupling $J_\Lambda(x_1, x_2, y_1, y_2)$ composed of $\vec{k}$ -labeled matrix memory patterns:

$$\mathbb{X}_{x_1, x_2}^{(k_1, k_2)} = a_{x_1, x_2} \frac{S_{k_1, x_1} S_{k_2, x_2} + S_{k_1, x_2} S_{k_2, x_1}}{\sqrt{\mu(x_1, x_2)\mu(k_1, k_2)}}. \quad (6)$$

If the set of output configurations Λ is composed of a single configuration $\vec{k}_0$, then the Hamiltonian (4) is factorized into two N_{ph} -body Hamiltonians [26]. On the other hand, if Λ encompasses all possible $P_{\text{all}} \equiv \binom{N_{\text{ph}}+M-1}{N_{\text{ph}}}$ configurations of N_{ph} indistinguishable photons $\Lambda = C_M$, we will have $\text{Pr}(C_M|\sigma) = 1$, implying a trivial dependence on σ . Therefore, a choice of $P \equiv |\Lambda| < P_{\text{all}} \sim M^{N_{\text{ph}}}$ as $M \gg 1$ will be implemented.

In Eq. (5) the coefficients $a_{\vec{x}}$ factor out in the coupling function. This means, for example, that to have a fully connected neural network, the amplitudes $a_{\vec{x}}$ of each configuration $\vec{x}$ in superposition in the input state $|\psi\rangle$ need to be nonzero. A practical scheme to achieve this regime is shown in Fig. 1(c). We consider the preparation of $|\psi\rangle$ by starting with all the N_{ph} photons in a single input mode of the interferometer, e.g., the zeroth one, and transmitting them through a discrete Fourier transform (DFT) interferometer [36,37], implemented by the unitary transformation $U_{jq}^{\text{DFT}} = e^{2\pi i(jq/M)}/\sqrt{M}$. In free space optics, this configuration corresponds to a source from a single-mode fiber coupled with a collimation lens (see the Supplemental Material [28]). Applying this transformation to the input Fock state with all photons in, e.g., mode zero, prepares a superposition state with amplitude $a_{\vec{x}} = \sqrt{N_{\text{ph}}!/[M^{N_{\text{ph}}}\mu(\vec{x})]}$ for any configuration $\vec{x} \in C_M$ (see the End Matter). This ensures a fully connected structure in the mapped synaptic couplings J .

We now focus on the $N_{\text{ph}} = 2$ case, mapping into a 4HM; cf. Eq. (6). This equivalence can be exploited to extract the energy of a 4HM by the analog measurement of the occurrence probability, Eq. (4).

In a single experimental iteration $n-1 \rightarrow n$, one of the modes' phases is flipped, and the new two-photon current is measured. The change is straightforwardly accepted in the simulated dynamics if $\mathcal{H}$ decreases [$\text{Pr}(\Lambda|\vec{\sigma})$ increases]. If we introduce synaptic noise, though, in the form of a statistical mechanical *temperature* T , the change can also be accepted if the energy cost momentarily *increases*, according to the Markov chain Monte Carlo (MC) Metropolis algorithm [38] (see also the Supplemental Material [28] about influence of measurement noise, other experimental sources of nonideality, and related Refs. [7,39–46]).

Defining $\Delta\mathcal{H} \equiv \mathcal{H}[\sigma(n)] - \mathcal{H}[\sigma(n-1)]$ for a phase change,

if $\Delta\mathcal{H} < 0$ change accepted is with Prob. 1

if $\Delta\mathcal{H} > 0$ change is accepted with Prob. $e^{-\Delta\mathcal{H}/T}$. (7)

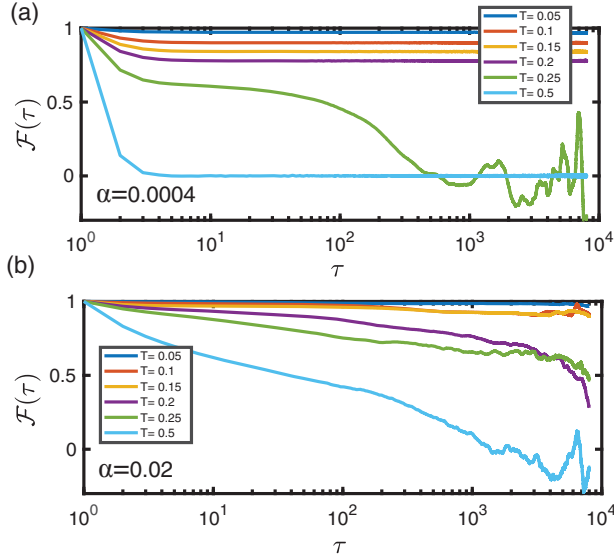


FIG. 2. Panels (a) and (b) report $\mathcal{F}(\tau)$ for various temperatures (see legend), for $\alpha = 0.0004$ (retrieval regime) and $\alpha = 0.02$ (spin-glass phase) respectively.

At each MC step t (sequence of M single phase shifts) $\sigma(t)$ is recorded. Then we compute the self-correlation function $F_{\text{self}}(\tau) = \langle \sigma(t) \cdot \sigma(t + \tau) \rangle$ (see the End Matter, Supplemental Material [28], and references therein: [47–49]). The $F_{\text{self}}(\tau)$ for various temperatures and two different values of the storage capacity ratio $\alpha \equiv P/M^{N_{\text{ph}}} = P/M^2$ are reported in Fig. 2 for $M = 50$. [50].

For $\alpha = 0.0004$, corresponding to a single two-photon output channel k_1, k_2 , Fig. 2(a) shows that $F_{\text{self}}(\tau)$ tends to a plateau at T low enough, thus displaying the relaxation to the unique fixed point, i.e., to the retrieval of a memory

$\mathbb{X}_{x_1, x_2}^{(k_1, k_2)}$. As the synaptic noise increases, at some critical T the σ configuration decorrelates to zero (the faster, the higher is T), and the retrieval is eventually lost.

As α increases [Fig. 2(b)] $F_{\text{self}}(\tau)$ decreases to zero at lower and lower T , and memory retrieval becomes progressively harder. This is due to the onset of spurious excited states [20], attracting the dynamics though not corresponding to the scattering patterns embedded in the network; cf. Eq. (6). The latter are the global minima of the $\mathcal{H}[\sigma|\Lambda]$ landscape, whereas the former are excited states corresponding to local minima. Their existence implies that a plateau of $F_{\text{self}}(\tau)$ may not correspond to retrieval if the initial configuration $\sigma(0)$ lies within the *basin of attraction* of a spurious state. Eventually, when α is larger than a critical value, the proliferation of spurious states leads to memory blackout (spin glass) [51].

To explore the full phase space through numerical simulations on a classical computer, we consider a configuration of the photonic system which is advantageous for classical simulations [see Fig. 1(d)]: the case where photons are fully bunched together in any of the output modes ($\Lambda = \{(0, 0), (1, 1), \dots, (M-1, M-1)\} \equiv \Lambda_{\text{FB}}$). In this case, the output probability [cf. Eqs. (4) and (5)] for a $\Lambda \subseteq \Lambda_{\text{FB}}$ reduces to (see the End Matter)

$$\Pr(\Lambda|\sigma) = \frac{1}{M^{N_{\text{ph}}}} \sum_{k \in [0, M-1]} \left| \sum_{j=0}^{M-1} S_{k,j} \sigma_j \right|^{2N_{\text{ph}}}, \quad (8)$$

with indexes k running on a subset of optical modes and j running through all modes. This approach significantly speeds up classical simulation while still maintaining the whole structure of the mapped 4HM. According to Eq. (8), we have access to a storage capacity $\alpha \leq 1/M$, e.g.,

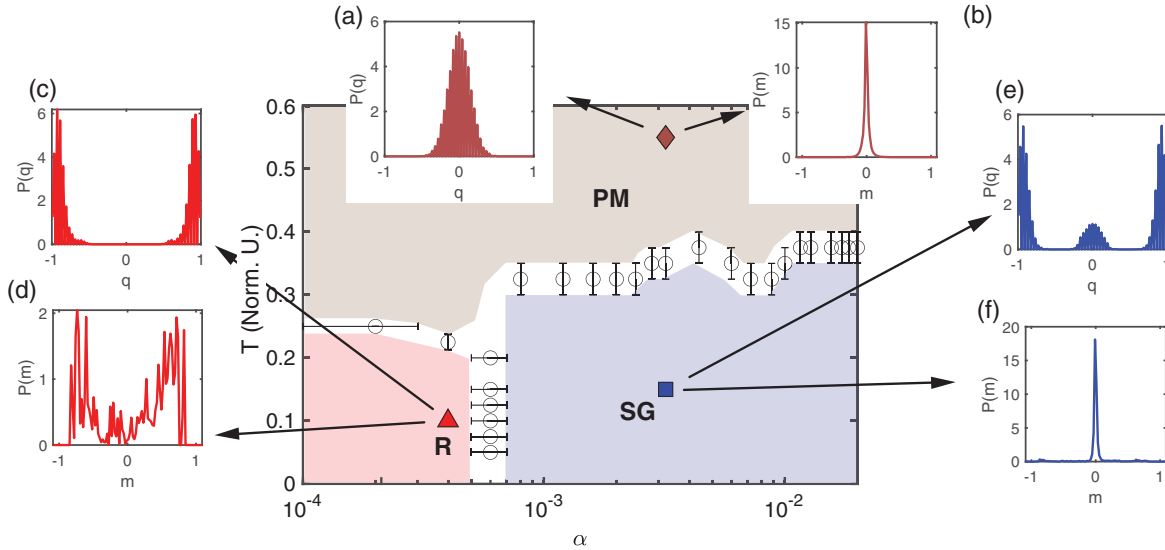


FIG. 3. Phase diagram of the four-Hopfield model realized with a quantum interferometer of two photons on $M = 50$ modes. The insets report $P(q)$ and $P(m)$ for the paramagnetic phase [(a),(b) $\alpha = 0.0032$; $T = 0.55$], retrieval phase [(c),(d) $\alpha = 0.0004$; $T = 0.1$], and spin glass phase [(e),(f) $\alpha = 0.0032$; $T = 0.15$].

$\alpha \leq 0.02$ for $M = 50$, enough to explore all possible regimes, from memory retrieval to blackout. Results of the simulations performed with the exchange Monte Carlo method [52] on $M = 50$ spins, are presented in Fig. 3 (simulation details reported in the End Matter). We identify three phases: retrieval, spin glass, and paramagnet. The retrieval (R) phase is characterized by large values of the memory overlap $\hat{m}_{\vec{k}}$ [cf. Eqs. (1) or (D3) in the End Matter], describing the degree of similarity of the retrieved state σ to the stored memory represented by permanents of the scattering matrix. In Fig. 3(d), the probability distribution of the $\hat{m}_{\vec{k}}$ for a single configuration of the output two-photon current displays two peaks for relatively large $|m|$, indicating the (noisy) alignment with the only pattern stored in the 4HM.

The Parisi spin-glass overlap distribution [cf. Eq. (D4) in the End Matter and the Supplemental Material [28] and Refs. [53,54]] is shown in Fig. 3(c), displaying two symmetric peaks representing all replicas aligned or counteraligned to the unique pattern. As the number $|\Lambda|$ of permanent patterns $\mathbb{X}^{(\vec{k})}$ [cf. Eq. (6)] increases, at low T the system undergoes a phase transition to the spin-glass (SG) phase, characterized by all memory overlaps being negligible [Fig. 3(f), memory blackout] and a nontrivial $P(q)$ with both strong ($|q| \rightarrow 1$) and weak ($q \sim 0$) correlations between replicas [Fig. 3(e)]. As the temperature increases, a paramagnetic (PM) phase arises both from the retrieval and the spin-glass phase. This phase is characterized by a complete independence of all spins (neurons) due to the high temperature or high synaptic noise and manifested in $P(q) \sim P(m)$ strongly peaked around zero [Figs. 3(a) and 3(b)].

In summary, the proposed approach highlights a connection between a structured quantum photonic interferometer and a p -body Hopfield model. We numerically investigated the mapping with $p = 4$, characterizing the phase diagram of the system with respect to α and T , identifying a retrieval phase in the low α -low T region and a spin-glass phase in the high α -low T region.

The proposed quantum simulator reproduces spin systems on fully connected graphs: future research can explore networks with local or sparse couplings. Indeed, local interactions can be engineered by implementing scattering matrices that mix only neighboring spins. Photonic states with nonuniform amplitudes in Eqs. (5) and (6) may also enable diluted or distance-dependent interactions.

Moreover, our mapping opens up the possibility of simulating p HM with photonic quantum simulators more efficiently than current methods for conventional computers. Bulk optics photonics enables one to exert control on systems with $M = 10^6$ optical modes at kHz speed (digital micromirror devices; see the Supplemental Material [28]) realizing p HMs with millions of spins enabling a superextensive memory storage capability [55,56]

surpassing the size of pairwise models simulated on classical computers [57–60].

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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End Matter

Multiphoton detection probability—For a general photonic system as shown in Fig. 1(a), the scattering amplitude of an input photon configuration $\vec{c}$ to an output configuration $\vec{k}$, when evolved through the transformation $\mathcal{U}$ described by the scattering matrix U is given by [35]

$$\langle \vec{k} | \mathcal{U}(|\vec{c}\rangle) \rangle = \frac{1}{\sqrt{\mu(\vec{c})\mu(\vec{k})}} \text{Perm}(U_{\vec{k}|\vec{c}}). \quad (\text{A1})$$

Here,

$$U_{\vec{k}|\vec{c}} \equiv \begin{pmatrix} U_{x_1, k_1} & U_{x_1, k_2} & \cdots & U_{x_1, k_{N_{\text{ph}}}} \\ U_{x_2, k_1} & U_{x_2, k_2} & \cdots & U_{x_2, k_{N_{\text{ph}}}} \\ \vdots & \vdots & & \vdots \\ U_{x_{N_{\text{ph}}}, k_1} & U_{x_{N_{\text{ph}}}, k_2} & \cdots & U_{x_{N_{\text{ph}}}, k_{N_{\text{ph}}}} \end{pmatrix} \quad (\text{A2})$$

is the $N_{\text{ph}} \times N_{\text{ph}}$ submatrix obtained by taking the row (column) indices of U that correspond to the elements of $\vec{x}(\vec{k})$; $\text{Perm}(U_{\vec{k}|\vec{c}})$ is the permanent function of $U_{\vec{k}|\vec{c}}$, and

$$\mu(\vec{c}) \equiv \prod_{j=0}^{M-1} n_j!, \quad n_j = 0, 1, \dots, N_{\text{ph}}, \quad (\text{A3})$$

where n_j is the occupation number of the mode j and $\sum_{j=0}^{M-1} n_j = N_{\text{ph}}$. For example, considering $N_{\text{ph}} = 3$, if all three photons are in different modes then $\mu(\vec{c}) = 1$, whereas if $c_1 = c_2 \neq c_3$ then $\mu(\vec{c}) = 2$ [see Fig. 1(a)], and if $c_1 = c_2 = c_3$ then $\mu(\vec{c}) = 3$. The transformation $\mathcal{U}$ is a linear function of the input photonic configuration and returns the output state given, in general, by a superposition of various output configurations $\vec{k}$.

If the system is prepared in an input configuration $\vec{c}$, the probability of detecting a configuration $\vec{k}$ at the output of the transformation is, thus, given by

$$\text{Pr}_{\mathcal{U}}(\vec{k}|\vec{c}) = |\langle \vec{k} | \mathcal{U}(|\vec{c}\rangle) \rangle|^2 = \frac{|\text{Perm}(U_{\vec{k}|\vec{c}})|^2}{\mu(\vec{c})\mu(\vec{k})}. \quad (\text{A4})$$

Discrete Fourier transform and input state—In order to map the outcome probability in a $2N_{\text{ph}}$ -Hopfield Hamiltonian with M degrees of freedom (activity of M neurons), we need to distribute N_{ph} photons over M modes. To do so, we start with a state in which all the

N_{ph} photons are all in the mode 0:

$$|\vec{0}\rangle = \frac{(\hat{a}_1^\dagger)^{N_{\text{ph}}}}{\sqrt{N_{\text{ph}}!}} |0\rangle = |N_{\text{ph}} \dots 0\rangle, \quad (\text{B1})$$

and we apply a DFT unitary transformation, whose matrix elements are defined as

$$U_{kl}^{\text{DFT}} = \frac{1}{\sqrt{M}} e^{-2\pi i \frac{kl}{M}}, \quad k, l \in [0, M-1], \quad (\text{B2})$$

and M is the size of the matrix that coincides with the total number of available photonic modes

$$\hat{a}_i \xrightarrow{U^{\text{DFT}}} \sum_j U_{ij}^{\text{DFT}} \hat{a}_j, \quad (\text{B3})$$

and all the photons in the source state are in mode 0. Using the submatrix from Eq. (A2) the coefficients of the state after the application of the DFT are

$$\begin{aligned} a_{\vec{x}} &= \langle \vec{x} | \mathcal{U}(|\vec{0}\rangle) \rangle = \frac{1}{\sqrt{N_{\text{ph}}! \mu(\vec{x})}} \text{Perm}(U_{\vec{x}|\vec{0}}^{\text{DFT}}) \\ &= \frac{1}{\sqrt{N_{\text{ph}}! \mu(\vec{x})}} \text{Perm}\left(\left[\begin{array}{c|c|c|c} U_{\vec{x}|0}^{\text{DFT}} & U_{\vec{x}|1}^{\text{DFT}} & \dots & U_{\vec{x}|M-1}^{\text{DFT}} \end{array} \right]\right) \\ &= \frac{1}{\sqrt{N_{\text{ph}}! \mu(\vec{x})}} \sum_{\sigma \in \pi(\vec{x})} \prod_{i \in \vec{x}} U_{i,0} = \sqrt{\frac{N_{\text{ph}}!}{\mu(\vec{x})}} \prod_{i \in \vec{x}} U_{i,0}^{\text{DFT}} \\ &= \sqrt{\frac{N_{\text{ph}}!}{M^{N_{\text{ph}}} \mu(\vec{x})}}. \end{aligned} \quad (\text{B4})$$

Hence, the outcome state is

$$|\Psi\rangle = \sum_{\vec{x}} a_{\vec{x}} |\vec{x}\rangle \quad (\text{B5})$$

which is the desired result. At this stage, we apply the controlled-phase shift and the scattering medium S to obtain the output probability given in Eq. (4).

Derivation of $\text{Pr}_S(\vec{k}|\vec{c})$ —The first derivation of the form of $\text{Pr}_S(\vec{k}|\vec{c})$ in the context of linear optics was obtained in Ref. [35]. The evolution of (indistinguishable) photons through a scattering medium is described by a unitary matrix S , under which the creation operators transform as

$$\hat{a}_i^\dagger \rightarrow \sum_{j=1}^M S_{ji} \hat{a}_j^\dagger. \quad (\text{C1})$$

The generic input state of N_{ph} photons is

$$|\Psi\rangle = \sum_{\vec{c}} a_{\vec{c}} |\vec{c}\rangle = \sum_{\vec{c}} \frac{1}{\sqrt{\mu(\vec{c})}} a_{c_1, \dots, c_{N_{\text{ph}}}} \hat{a}_{c_1}^\dagger \dots \hat{a}_{c_{N_{\text{ph}}}}^\dagger |\mathbf{0}\rangle,$$

where $|\mathbf{0}\rangle = |0\rangle^{\otimes N_{\text{ph}}}$ and $\sum_{\vec{c}}(\dots) \equiv \prod_{j=1}^{N_{\text{ph}}} \sum_{c_j=0}^{M-1}(\dots)$. By applying the transformation (C1), each term $|\vec{c}\rangle$ in the superposition transforms as

$$S(|\vec{c}\rangle) = \frac{1}{\sqrt{\mu(\vec{c})}} \sum_{\vec{q}} S_{q_1 c_1} \dots S_{q_{N_{\text{ph}}} c_{N_{\text{ph}}}} \hat{a}_{q_1}^\dagger \dots \hat{a}_{q_{N_{\text{ph}}}}^\dagger |\mathbf{0}\rangle, \quad (\text{C2})$$

with $\mu(\vec{c})$ given by Eq. (A3). Therefore, the transition amplitude to the state

$$|\vec{k}\rangle = \frac{1}{\sqrt{\mu(\vec{k})}} \hat{a}_{k_1}^\dagger \dots \hat{a}_{k_{N_{\text{ph}}}}^\dagger |\mathbf{0}\rangle$$

can be explicitly computed as $\langle \vec{k} | S(\vec{c}) \rangle$. By using the bosonic canonical commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \quad (\text{C3})$$

with some algebra we obtain

$$\langle 0 | \hat{a}_{k_1} \dots \hat{a}_{k_{N_{\text{ph}}}} \hat{a}_{q_1}^\dagger \dots \hat{a}_{q_{N_{\text{ph}}}}^\dagger | 0 \rangle = \sum_{\{q\}} \delta_{k_1, q_1} \dots \delta_{k_{N_{\text{ph}}}, q_{N_{\text{ph}}}},$$

where the sum runs over all the permutations of the indices $q_1, \dots, q_{N_{\text{ph}}}$. Therefore, substituting the last expression in Eq. (C2), the only terms that survive are the ones with $\vec{q} = \vec{k}$ and all their permutations, leading to

$$\langle \vec{k} | S(\vec{c}) \rangle = \frac{1}{\sqrt{\mu(\vec{c}) \mu(\vec{k})}} \text{Perm}(S_{\vec{k}|\vec{c}}), \quad (\text{C4})$$

that is, Eq. (A1), and the transition probability is given by the squared modulus of the amplitude; cf. Eq. (A4).

Efficient probability calculation for fully bunched model— $\text{Pr}_S(\vec{k}|\vec{c})$ can be efficiently calculated digitally if the simplified model shown in Fig. 1(d) is employed. Here Λ_{FB} is composed of all the configurations where output photons are bunched together (i.e., are in the same output mode) (see the Supplemental Material for a full derivation [28]).

Defining the set $\Lambda_{\text{FB}} = \{\vec{k}_0, \dots, \vec{k}_{M-1}\}$ composed of all configurations with bunched photons on any of the modes, when considering the sum over more output photonic configurations of this kind, composing a subset $\Lambda \subseteq \Lambda_{\text{FB}}$ we obtain

$$\text{Pr}(\Lambda|\sigma) = \frac{1}{M^{N_{\text{ph}}}} \sum_{\mu \in [0, M-1]} \left| \sum_{j=0}^{M-1} S_{\mu, j} \sigma_j \right|^{2N_{\text{ph}}}, \quad (\text{D1})$$

where by $\mu \in [0, M-1]$ we mean that the index μ runs through a subset of all possible modes in $[0, M-1]$.

Definition of F_{self} , $\hat{m}$, and q_{ab} : In order to analyze the spin correlations during time, we define the function

$$\begin{aligned} F(\tau) &= \sum_{i=1}^M \langle \sigma_i(t) \sigma_i(t+\tau) \rangle_t \\ &\equiv \frac{1}{N_{\text{MCs}} \cdot N} \sum_{i=1}^M \sum_{t=0}^{N_{\text{MCs}}} \sigma_i(t) \sigma_i(t+\tau). \end{aligned} \quad (\text{D2})$$

In order to check the possible retrieval of the memory $\mathbb{X}^{(\vec{k})}$, proportional to the permanent of the scattering matrix, cf. Eqs. (5) and (6), the memory overlap of the input photonic mode phases $\sigma = \{\sigma_1, \dots, \sigma_M\}$ with such a matrix pattern is the following bilinear form:

$$\begin{aligned} m_{\vec{k}} &\equiv \frac{1}{M} \sigma^T \mathbb{X}^{(\vec{k})} \sigma = \frac{1}{M} \sum_{x_1, x_2}^{1, M} \sigma_{x_1} \mathbb{X}_{x_1, x_2}^{(k_1, k_2)} \sigma_{x_2} \\ &\propto \frac{1}{M} \sum_{x_1, x_2}^{1, M} \sigma_{x_1} [S_{k_1, x_1} S_{k_2, x_2} + S_{k_1, x_2} S_{k_2, x_1}] \sigma_{x_2}. \end{aligned} \quad (\text{D3})$$

In order to identify and characterize the spin-glass phase, by defining $\sigma^{(a)}$ as the M spin configuration of replica a , the replica overlap is introduced as

$$q_{ab} \equiv \frac{1}{M} \sum_{j=0}^{M-1} \sigma_j^{(a)} \sigma_j^{(b)}. \quad (\text{D4})$$

Further information and discussion on these quantities are reported in the Supplemental Material [28].

Numerical simulations—The simulations are implemented using the exchange Monte Carlo (EMC) method [52] with simultaneous system dynamics at different temperatures running in parallel over multiple CPUs. All results are obtained considering a system size of $M = 50$ spins, a total of 2×10^5 simulated Monte Carlo steps for system thermalization, and with exchanges between system dynamics at different temperatures attempted every 200 Monte Carlo steps. The scattering matrix S is taken at random from the Haar random distribution, and all EMC simulations are repeated for 20 different disordered samples, each corresponding to a separately generated S matrix. For each scattering sample, a total of 36 replicas are simulated for each value of α and T to extract the order parameters. In particular, the two order parameters considered are standard parameters for spin-glass and memory retrieval systems: the overlap between simulated replicas q_{ab} and the memory overlap $\hat{m}_{\vec{k}}$ (see above and the Supplemental Material [28]).