

Nonlinear Gravitational Memory in the Post-Minkowskian Expansion

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We present the first computation of the nonlinear gravitational memory waveform for the scattering of two compact objects in general relativity at leading order in the post-Minkowskian expansion. We use the scattering-amplitude-based representation of the gravitational waveform, which naturally expresses the nonlinear memory as the contribution of soft gravitons emitted by the gravitational waves themselves. We perform the calculation by applying a multipolar decomposition to the waveform and using the reverse unitarity method to obtain explicit exact-in-velocity predictions. We validate the results by calculating the corresponding velocity-expanded post-Newtonian multipoles, finding perfect agreement. Our results complete the knowledge of the gauge-invariant nonanalytic-in-frequency part of the $\mathcal{O}(G^3)$ multipolar waveform, thus providing a useful benchmark for future calculations of this quantity.

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Introduction—The advent of gravitational-wave (GW) science has sparked the need for precision in gravitational physics. The success of the scientific program of current and future GW experiments hinges on our ability to extract accurate predictions for the dynamics and radiation of compact binary systems from general relativity (GR). This is a nontrivial task, as Einstein's theory is highly nonlinear, and in order to solve it one must resort to numerical methods [1,2] or various analytical approximations schemes, such as the post-Newtonian (PN) [3] and self-force [4] theories. In recent years, a novel synergy between ideas from scattering amplitudes, effective field theory and classical GR has brought in a wealth of results and significantly advanced our ability to make predictions in the post-Minkowskian (PM) expansion [5–35], which applies for weak fields and relativistic velocities, employing Newton's constant G as an expansion parameter.

The main quantity measured in experiments is the waveform of the gravitational wave passing through the detector, $w_{\mu\nu}$. This is simply the radiative part of the gravitational field, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, at distances $r \rightarrow \infty$ far from the emitter, $h_{\mu\nu} \sim (4G/r)w_{\mu\nu}(T, n)$, and a function of the observation time T and a null vector n^μ characterizing

the observer's direction. Following pioneering papers based on classical methods [36–38], the gravitational waveform has been recently connected to scattering-amplitude and diagrammatic methods [16,39–44] through $\mathcal{O}(G^2)$ [43,45–54] leading to new results.

Perhaps the conceptually simplest observable feature of the waveform is the so-called memory of the gravitational wave, which is the net change of the gravitational waveform during the observation process [55–61],

$$\left(\lim_{T \rightarrow \infty} w_{\mu\nu}(T, n) \right) - \left(\lim_{T \rightarrow -\infty} w_{\mu\nu}(T, n) \right) = -F_{\mu\nu}. \quad (1)$$

The memory dictates a permanent displacement of free-falling test masses after the passing of the gravitational wave [56,57] and is related by Fourier transform to a $1/\omega$ pole in the small-frequency limit $\omega \rightarrow 0$ of the spectral waveform [62]. See Refs. [63–69] for prospects on its measurement at interferometers.

Scattering amplitudes also enjoy general factorization properties in the low-frequency limit known as soft theorems, and their leading-order incarnation in the context of gravity is the celebrated Weinberg soft theorem [70,71]. As it turns out, the Weinberg soft factor is the memory effect [62], and similar general results also hold for sub-leading $\log \omega$ and $\omega(\log \omega)^2$ contributions in the low-frequency limit [72,73] (see Refs. [74,75] for an all-order conjecture). Such soft theorems are nonperturbative in G and thus provide all-order constraints for the PM expansion of the waveform. In Refs. [40,41,49,53,76] it was checked

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that amplitude-based calculations are in perfect agreement with such constraints at $\mathcal{O}(G)$ and $\mathcal{O}(G^2)$.

The key idea at the basis of [70,71] is that the leading soft term is sourced by gravitons attached to the external lines of the process, to which the graviton couples in a democratic way owing to the equivalence principle. As already emphasized by Weinberg, the leading soft theorem is completely universal and includes the contributions due to soft gravitons emitted by other, hard, gravitons. In the classical GR context, this contribution is known as “nonlinear memory” [58,60], as opposed to “linear memory” sourced by the massive bodies [55,57].

In this Letter, we show how the nonlinear memory emerges from the amplitude representation of the gravitational waveform for the scattering of two compact objects, at the leading order at which it appears, $\mathcal{O}(G^3)$. Furthermore, we explicitly calculate the associated multipoles in the post-Minkowskian limit for the first time. This completes the predictions given by classical soft theorems for the nonanalytic-in- ω part of the waveform at $\mathcal{O}(G^3)$, which includes the memory and logarithms of ω .

Linear and nonlinear memory waveform—The memory effect is captured by a simple pole in the low-frequency expansion of the spectral waveform $\tilde{w}_{\mu\nu}(\omega n)$ [55,57,58,60,62,72,73],

$$\tilde{w}_{\mu\nu} \sim -\frac{i}{\omega} F_{\mu\nu} + \mathcal{O}(\log \omega), \quad \text{as } \omega \rightarrow 0, \quad (2)$$

and affords the simple expression [70–73]

$$F^{\mu\nu} = \sum_a \frac{p_a^\mu p_a^\nu}{p_a \cdot n}. \quad (3)$$

In the Weinberg factor (3), the sum encompasses *all* hard external states that can emit soft gravitons. For a gravitational two-body scattering, this includes not only the two massive compact objects, with initial momenta $-p_{1,2}^\mu = m_{1,2} v_{1,2}^\mu$ (and final momenta $p_{4,3}^\mu$), but also the dynamically generated gravitons themselves, which build the GW signal. Therefore, it is convenient to split $F^{\mu\nu} = f^{\mu\nu} + \delta F^{\mu\nu}$ in terms of

$$f^{\mu\nu} = \sum_{a=1}^4 \frac{p_a^\mu p_a^\nu}{p_a \cdot n}, \quad \delta F^{\mu\nu} = \int \widetilde{dk} \rho(k) \frac{k^\mu k^\nu}{k \cdot n}, \quad (4)$$

where we introduced a shorthand notation for the integral over the Lorentz-invariant phase space (LIPS) of emitted gravitons (t^μ being a timelike vector specifying the time direction)

$$\int \widetilde{dk} = \int \frac{d^D k}{(2\pi)^{D-1}} \theta(-t \cdot k) \delta(k^2), \quad (5)$$

and $\rho(k)$ denotes the corresponding phase-space distribution,

$$\rho = 8\pi G \tilde{w}_{\mu\nu} \left(\eta^{\mu\rho} \eta^{\nu\sigma} - \frac{1}{D-2} \eta^{\mu\nu} \eta^{\rho\sigma} \right) \tilde{w}_{\rho\sigma}^*. \quad (6)$$

Since $f^{\mu\nu}$ is due to the field generated by the incoming and outgoing massive objects, it is referred to as linear memory, while $\delta F^{\mu\nu}$ is due to soft gravitons emitted by hard gravitons, a phenomenon known as nonlinear memory. Importantly, gauge invariance holds only for the sum of linear and nonlinear memory, $n_\nu F^{\mu\nu} = 0$, thanks to

$$\sum_{a=1}^4 p_a^\mu + \int \widetilde{dk} \rho(k) k^\mu = 0, \quad (7)$$

which is the balance law linking mechanical and radiated energy-momentum.

While the expressions (4) are formally exact, they rely on the knowledge of the incoming and outgoing momenta of the massive objects, as well as on the full gravitational-wave spectrum $\rho(k)$. However, order by order in the PM expansion, we can evaluate them explicitly as functions of the initial velocities $v_{1,2}^\mu$, thus, in particular, of the Lorentz factor $\sigma = -v_1 \cdot v_2 = 1/\sqrt{1-v^2}$ and of the impact parameter b^μ orthogonal to them, $b \cdot v_{1,2} = 0$.

Linear memory—Before moving on to the more interesting case of nonlinear memory, let us recall the PM expansion of the linear memory contribution. Its expression is determined by the total dynamical changes in the momenta of the two particles. These classical impulses have been computed as functions of the initial velocities and impact parameters up to high PM orders. Let us recall their structure up to 3PM, i.e., $\mathcal{O}(G^3)$ [6–9,11,12,14,15,17],

$$p_1^\mu + p_4^\mu = -Q \frac{b_e^\mu}{b_e} - Q_\parallel \check{v}_2^\mu + \mathcal{O}(G^4), \quad (8a)$$

$$p_2^\mu + p_3^\mu = +Q \frac{b_e^\mu}{b_e} - Q_\parallel \check{v}_1^\mu + \mathcal{O}(G^4), \quad (8b)$$

where $\check{v}_{1,2}^\mu = (\sigma v_{2,1}^\mu - v_{1,2}^\mu)/(\sigma^2 - 1)$ and

$$b_e^\mu = b^\mu - \left(\frac{\check{v}_1^\mu}{2m_1} - \frac{\check{v}_2^\mu}{2m_2} \right) Q b_e \quad (9)$$

is the “eikonal” impact parameter. Explicit expressions for the transverse Q and longitudinal $Q_\parallel$ components of the impulse as functions of σ and b can be found in Supplemental Material, Sec. II [77]. Note that only the longitudinal component enters the balance law (7) at $\mathcal{O}(G^3)$, which holds thanks to [14]

$$P^\mu = \int \widetilde{dk} \rho(k) k^\mu = Q_\parallel (\check{v}_1^\mu + \check{v}_2^\mu) + \mathcal{O}(G^4). \quad (10)$$

Contracting with a fixed-helicity polarization vector ε and expanding for small G one thus finds

$$\varepsilon_\mu f^{\mu\nu} \varepsilon_\nu = Q S_1^{(\varepsilon)} + Q^3 S_3^{(\varepsilon)} + Q_\parallel T_3^{(\varepsilon)} + \mathcal{O}(G^4), \quad (11)$$

in terms of elementary tensor structures $S_{1,2}^{(\varepsilon)}$ and $T_3^{(\varepsilon)}$ whose expressions [74] are collected for completeness in Supplemental Material, Sec. II [77].

Nonlinear memory from scattering amplitudes—The gravitational waveform $\tilde{w}^{\mu\nu}$ can be expressed in terms of scattering amplitudes, which are the T -matrix elements with $S = 1 + iT$, as the impact-parameter Fourier transform of $W^{\mu\nu}/\sqrt{8\pi G}$ with [39,42,45–48,50]

$$W^{\mu\nu} = \text{[Diagram: A shaded circle with two incoming and two outgoing lines, and a wavy line attached to the right side]} - i \sum_X \text{[Diagram: A shaded circle with two incoming and two outgoing lines, and a wavy line attached to the right side, with a vertical line through the center labeled X]} \quad (12)$$

$$W_1^{\mu\nu} \sim \text{[Diagram: A shaded circle with two incoming and two outgoing lines, and a wavy line attached to the right side, with a vertical line through the center labeled N]} + \frac{i}{2} \left(\text{[Diagram: Two shaded circles connected by a wavy line, with two incoming and two outgoing lines]} - \text{[Diagram: Two shaded circles connected by a wavy line, with two incoming and two outgoing lines, with a vertical line through the center labeled N]} \right), \quad (13)$$

where we have dropped terms that do not contribute at leading order as $\omega \rightarrow 0$. At this order, the N -matrix element appearing in (13) is simply the real part of the one-loop $2 \rightarrow 3$ amplitude. Moreover, the complete amplitude-based representation of $W_1^{\mu\nu}$ would involve additional unitarity cuts due to rescattering effects, which, however, can be dropped because they do not contribute to the leading pole for small frequencies. References [49,53,76] checked that the leading $\mathcal{O}(G)$ and to subleading $\mathcal{O}(G^2)$ waveforms thus obtained are in perfect agreement with the corresponding

and in the loop expansion $W^{\mu\nu} = \sum_L W_L^{\mu\nu}$. The first term on the right-hand side of (12) is the $2 \rightarrow 3$ amplitude for the scattering of two scalar objects with masses m_1, m_2 (thick solid lines) and the emission of one graviton (wiggly line) with momentum ωn^μ . The second term is a sum over all unitarity cuts involving the two massive objects plus all possible additional states, represented by X . A key feature of (12) is that, being a classical observable, it ought to possess a well-defined classical limit $Gm_i^2 \gg \hbar$. To make this more explicit, it is convenient to relate it to the N -matrix elements, with $S = e^{iN}$ [80], which are well-behaved in the classical limit, unlike their T -matrix counterpart. Let us write down the first three loop orders of (12) in terms of N -matrix elements and unitarity cuts, with the objective to obtain an amplitude-based expression of the waveform in the small-frequency expansion and compare with the prediction given by the memory effect. At tree level, $W_0^{\mu\nu}$ coincides with the $2 \rightarrow 3$ amplitude [81,82] (and with the N -matrix element). Instead, at one loop one finds [43,76]

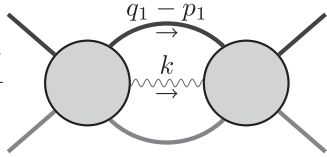
expansion of the term $Q S_1^{(\varepsilon)}$ in (11). Therefore, they agree with the general prediction given by the soft theorem and are completely exhausted by linear memory.

Generalizing this exercise two loops is straightforward, although of course it involves more terms and unitarity cuts. Once again we expand (12) in terms of amplitude and their cuts, isolate the N -matrix element for convenience and drop rescattering diagrams because they do not contribute for small frequencies. We find that the terms contributing at leading order as $\omega \rightarrow 0$ take the form

$$\begin{aligned} W_2^{\mu\nu} \sim & \text{[Diagram: A shaded circle with two incoming and two outgoing lines, and a wavy line attached to the right side, with two vertical lines through the center labeled N]} + \frac{i}{2} \left(\text{[Diagram: Two shaded circles connected by a wavy line, with two incoming and two outgoing lines]} - \text{[Diagram: Two shaded circles connected by a wavy line, with two incoming and two outgoing lines, with a vertical line through the center labeled N]} \right) \\ & + \frac{i}{2} \left(\text{[Diagram: A shaded circle with two incoming and two outgoing lines, and a wavy line attached to the right side, with a vertical line through the center labeled N]} + \text{[Diagram: A shaded circle with two incoming and two outgoing lines, and a wavy line attached to the right side, with a vertical line through the center labeled N]} - \text{[Diagram: A shaded circle with two incoming and two outgoing lines, and a wavy line attached to the right side, with a vertical line through the center labeled N]} - \text{[Diagram: A shaded circle with two incoming and two outgoing lines, and a wavy line attached to the right side, with a vertical line through the center labeled N]} \right) \\ & - \frac{1}{6} \left(\text{[Diagram: Three shaded circles connected by wavy lines, with two incoming and two outgoing lines]} - 2 \text{[Diagram: Three shaded circles connected by wavy lines, with two incoming and two outgoing lines]} + \text{[Diagram: Three shaded circles connected by wavy lines, with two incoming and two outgoing lines]} \right) \end{aligned} \quad (14)$$

Since we are focusing on the limit in which the emitted graviton is soft, $\omega \rightarrow 0$, we can simplify all the elementary building blocks by applying Weinberg's soft theorem [70], according to which an amplitude involving a soft emission is given by attaching the soft graviton to all external legs of the hard process. Doing so, we have checked that (14) agrees with the subsubleading $\mathcal{O}(G^3)$ expansion of the general expression in (2) and (4) given by the soft theorem. More in detail, the N -matrix element in the first line of (14) plus the second and third lines match the expansion of the linear memory terms $QS_1^{\mu\nu} + Q^3S_3^{\mu\nu}$ in (11). Notice that, like at one loop, also $\tilde{w}_2$ given by (14) is expressed in terms of the vectors v_1^μ , v_2^μ , and b^μ characterizing the initial state. Another point to keep in mind to simplify the comparison is that the N -matrix element for the $2 \rightarrow 2$ process is directly related to the 3PM radial action [13,19], whose J derivative gives the 3PM deflection angle.

The novelty is represented here by the appearance of the nonlinear memory contribution $\delta F^{\mu\nu}$. This emerges from the cuts within the round parentheses in the first line of (14). To show this, we apply once again the Weinberg soft theorem to the emitted graviton, while the graviton appearing in the intermediate state is generically hard. In this way, $\delta F^{\mu\nu}$ emerges from the insertion of the soft graviton on the hard one, as physically expected, while $Q_{||}T_3^{\mu\nu}$ arises from the insertions on the massive legs. To summarize, these cuts evaluate to the combination $\delta F^{\mu\nu} + Q_{||}T_3^{\mu\nu}$, which is gauge invariant as expected for unitarity cuts [see Eq. (S.20) in Supplemental Material]. It is also convenient to use the reverse unitarity representation of the nonlinear memory,

$$\delta F^{\mu\nu} = \text{FT}_4 \int_{q_1} \int_k \frac{k^\mu k^\nu}{k \cdot n} \text{ (diagram) }, \quad (15)$$


as well as the analogous ones for $\mathbf{Q}_1^\alpha = -Q_{||}\tilde{v}_2^\alpha$ (respectively, $\mathbf{Q}_2^\alpha = -Q_{||}\tilde{v}_1^\alpha$), in which $k^\mu k^\nu / k \cdot n$ is replaced by a factor $q_1^\alpha - \frac{1}{2}q^\alpha$ (respectively, $-q_1^\alpha + \frac{1}{2}q^\alpha - k^\alpha$). This representation follows from (4) by turning the integral of the product of Fourier transforms appearing in ρ into the Fourier transform (FT) of a cut convolution (see, e.g., Ref. [83] for more details). In (15), we use the shorthand $\int_q = \int [d^D q / (2\pi)^D]$, the FT

$$\text{FT}_4[f] = 4\pi^2 \int_q e^{ibq} \delta(2p_1 \cdot q) \delta(2p_2 \cdot q) f(q), \quad (16)$$

and the contraction between the two tensor structures as in (6) is implicit. Additionally, all exposed lines are on shell, so that the integrand includes the LIPS measure $d\text{LIPS}(q_1, k)$, which in the near-forward limit reads

$$d\text{LIPS}(q_1, k) = (2\pi)^3 \delta(2p_1 \cdot q_1) \delta[2p_2 \cdot (q_1 + k)] \theta(k^0) \delta(k^2). \quad (17)$$

The integral appearing on the right-hand side of (15) can be interpreted as the three-particle cut of a two-loop integral and thus calculated from the corresponding discontinuity. Evaluating (15) requires discussing families of two-loop master integrals that, in addition to the elastic case analyzed in [16–19,21,84], involve the extra factor $(k \cdot n)^{-1}$.

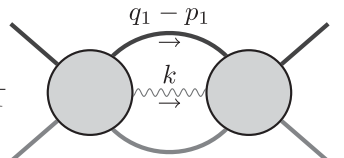
Multipoles from reverse unitarity—Generically, the nonlinear memory is a function of three dimensionless variables: two angles denoting the orientation in the celestial sphere and the relative speed of the incoming massive objects. Instead of working at fixed n^μ , we first perform the multipole decomposition of $\delta F^{\mu\nu}$ in a reference frame characterized by the time direction t^μ . Introducing the spin-weighted spherical harmonics $Y_{\pm 2}^{\ell m}$ (see Supplemental Material Sec. I for more details [77]), we thus consider

$$\delta F^{\ell m} = \oint Y_{\pm 2}^{\ell m*} \delta F_{\mu\nu} \epsilon_\pm^\mu \epsilon_\pm^\nu d\Omega. \quad (18)$$

We find that the result for the both helicities is the same, i.e., the magnetic-type or V multipoles vanish, and reads (see Refs. [3,61,85,86] for an equivalent formula in terms of Cartesian multipoles)

$$\delta F^{\ell m} = -\frac{2\pi}{\ell(\ell-1)} \mathcal{N}_2^{\ell m} Y_{\mu_1 \dots \mu_\ell}^{\ell m*} \text{FT}_4[\mathbb{I}^{\mu_1 \dots \mu_\ell}], \quad (19)$$

where $Y_{\mu_1 \dots \mu_\ell}^{\ell m}$ denotes the standard basis of symmetric trace-free tensors, $\mathcal{N}_2^{\ell m}$ is a normalization factor [see Eq. (S.20) in Supplemental Material, Sec. I [77]], and

$$\mathbb{I}^{\mu_1 \dots \mu_\ell} = \int_{q_1} \int_k \frac{k^{\mu_1} \dots k^{\mu_\ell}}{(-t \cdot k)^{\ell-1}} \text{ (diagram) } \quad (20)$$


where $-t \cdot k$ in the denominator is the frequency of the graviton with four-momentum k^μ as measured in the chosen reference frame.

To calculate the nonlinear memory multipoles, we choose to align the axes of our reference frame as follows:

$$t^\mu = v_1^\mu, \quad \hat{x}^\mu = \frac{b^\mu}{b}, \quad \hat{y}^\mu = \tilde{v}_2^\mu \sqrt{\sigma^2 - 1}, \quad (21)$$

so that the $Y_{\mu_1 \dots \mu_\ell}^{\ell m}$ are purely spatial and can be expressed in terms of the characteristic vectors of the problems. For instance, we have

TABLE I. Functions specifying the quadrupolar nonlinear memory waveform.

$f_1^{22} = (1/960)(-4200\sigma^8 + 92\,820\sigma^7 - 124\,980\sigma^6 - 133\,050\sigma^5$ $+ 106\,492\sigma^4 + 489\,510\sigma^3 - 627\,644\sigma^2 + 200\,025\sigma + 5212)$
$f_2^{22} = \frac{1}{8}(\sigma^2 - 1)^2(35\sigma^4 + 420\sigma^3 - 510\sigma^2 + 292\sigma + 67)$
$f_3^{22} = (1/64)(-280\sigma^9 - 2880\sigma^8 + 2380\sigma^7 + 9072\sigma^6 - 4996\sigma^5$ $- 11\,790\sigma^4 + 3460\sigma^3 + 7008\sigma^2 - 564\sigma - 1689)$
$f_1^{20} = (1/2880)(-4200\sigma^8 - 176\,568\sigma^7 + 199\,020\sigma^6 + 545\,268\sigma^5$ $- 782\,780\sigma^4 - 424\,458\sigma^3 + 757\,396\sigma^2 + 52\,923\sigma - 175\,196)$
$f_2^{20} = (1/24)(\sigma^2 - 1)(35\sigma^6 - 1230\sigma^5 + 385\sigma^4 + 2188\sigma^3$ $- 3419\sigma^2 + 1538\sigma + 119)$
$f_3^{20} = (1/192)(-280\sigma^9 + 10\,320\sigma^8 - 5060\sigma^7 - 29\,256\sigma^6$ $+ 14\,516\sigma^5 + 37\,476\sigma^4 - 10\,748\sigma^3 - 24\,348\sigma^2$ $+ 1956\sigma + 5997)$

$$Y_{i_1 i_2}^{2(\pm 2)} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} (\hat{x} \pm i\hat{y})_{i_1} (\hat{x} \pm i\hat{y})_{i_2}, \quad (22a)$$

$$Y_{i_1 i_2}^{20} = \frac{3}{4} \sqrt{\frac{5}{\pi}} \left(\hat{z}_{i_1} \hat{z}_{i_2} - \frac{1}{3} \delta_{i_1 i_2} \right), \quad (22b)$$

for the structures relevant to the explicit results displayed in Table I. Thus, the resulting integrals in Eq. (20) fall in the families described in [16–19,21,84], with the addition of extra powers of the propagator $-u_1 \cdot k$. Using integration-by-parts (IBP) identities, these can be easily reduced to a set of master integrals, which are the same as those in [84] plus an additional integral in the ladder sector. Computing the IBP reduction of higher multipoles, while systematic, becomes computationally intensive because the tensor rank of the numerator increases linearly with the multipole number ℓ . Finally, we calculate the master integrals using the by-now-familiar method of canonical differential equations in the soft region [16,17,84] (more details will be provided in [87]).

Results—The resulting multipoles of the leading-order nonlinear memory take the form

$$\delta F^{\ell m} = \frac{G^3 \pi^2 m_1^2 m_2^2}{b^3 (\sigma^2 - 1)^{\frac{3}{2}}} i^\ell \mathcal{N}_2^{\ell m} \mathcal{F}^{\ell m}(\sigma) + \mathcal{O}(G^4), \quad (23)$$

where we introduced the function of velocity

$$\mathcal{F}^{\ell m}(\sigma) = \frac{f_1^{\ell m} + f_2^{\ell m} \log\left(\frac{\sigma+1}{2}\right) + f_3^{\ell m} \frac{\text{arccosh } \sigma}{\sqrt{\sigma^2 - 1}}}{(\sigma^2 - 1)^{\frac{\ell}{2}}}, \quad (24)$$

which depends on the multipolar numbers. The coefficients $f_i^{\ell m}$ are real polynomials in σ , which are provided in Table I for the quadrupolar ($\ell = 2$) waveform and in Supplemental Material Sec. IV [77] up to $\ell = 5$. We note that $\delta F^{\ell m} = 0$ if $\ell + m$ is odd and, as a consequence of the symmetry of

$\rho(k)$ under $b \mapsto -b$ at leading order, $\delta F^{\ell(-m)} = (-)^{\ell} \delta F^{\ell m}$. We emphasize that these results and the ones in Supplemental Material Sec. IV hold in the rest frame of particle 1 [see Eq. (21)] and thus only coincide with those calculated in the center-of-mass frame to leading order in the mass ratio $m_2/m_1 \ll 1$.

For completeness, let us also provide here the explicit multipole decomposition for the “longitudinal” component $T_3^{\mu\nu}$ of the linear memory, which appears in the gauge-invariant combination $\delta F^{\mu\nu} + Q_{\parallel} T_3^{\mu\nu}$ together with the nonlinear memory,

$$Q_{\parallel} T_3^{\ell m} = \frac{\partial a^{\ell m}(\sigma)}{\partial \sigma} Q_{\parallel}, \quad (25)$$

where (recall that $\sigma = -v_1 \cdot v_2 = 1/\sqrt{1 - v^2}$)

$$a^{\ell m}(\sigma) = i^m \pi \sigma v^{\ell} \ell! a_{\ell m 2}^0 F_1\left(\frac{\ell-1}{2}, \frac{\ell}{2}, \ell + \frac{3}{2}, v^2\right). \quad (26)$$

See Eq. (S.23) in Supplemental Material Sec. III [77] for an analogous formula that computes all linear memory multipoles.

General results exist beyond the $1/\omega$ pole and also fix the $\log \omega$ and $\omega(\log \omega)^2$ terms in the small-frequency expansion [72,88]. It was observed that nonlinear contributions, due to gravitons, to such terms admit a simple rewriting [73] in terms of the total initial and final four-momentum of the massive objects. In particular, this makes it manifest that nonlinear effects in $\log \omega$ and $\omega(\log \omega)^2$ only appear from $\mathcal{O}(G^4)$. Moreover, recently a conjecture for the resummation of all leading logs of the form $\omega^{n-1}(\log \omega)^n$ for $n \geq 3$ was proposed based on checks in the small-velocity limit [74] and small-mass-ratio limit [75]. It is straightforward to compute the associated multipoles through $\mathcal{O}(G^3)$ in a generic frame (see Supplemental Material Sec. III for more details [77]). Finally, very recent results also encompass multipoles due to “UV logs” such as $\omega \log \omega$ [75,89–91].

Our results for the nonlinear memory multipoles thus complete this picture by providing the missing nonanalytic terms of the waveform at $\mathcal{O}(G^3)$.

Checks—As an independent way of calculating the nonlinear memory waveform, in the small-velocity expansion, one can employ the explicit expression for the tree-level and one-loop waveform calculated in Refs. [52,53,92] and substitute them in (6). The resulting angular integrals are then elementary when performed order by order in the velocity. Based on [92], we have employed this strategy to obtain the nonlinear memory waveform $\delta F^{\mu\nu}$ at $\mathcal{O}(G^3)$ up to $\mathcal{O}(v^{19})$ for small v , i.e., 9PN relative to the leading order, which scales like $\mathcal{O}(v)$. After performing the multipole decomposition (18), we find complete agreement between those results and the small-velocity expansion of the

present results. Moreover, we find agreement with the previous results of Refs. [59,93], which provide the leading result for small v to $\mathcal{O}(G^3)$ and to all orders in $Gm/(v^2b)$, respectively.

It is worth mentioning that, in the scattering scenario considered here, the nonlinear memory contribution $\delta F^{\mu\nu}$ appears at $\mathcal{O}(G^3v)$, i.e., 2.5 orders further suppressed in the PN expansion compared to the leading quadrupole sitting in $f^{\mu\nu}$ [59].

Conclusions—In this Letter, we have analyzed the nonlinear memory waveform for the scattering of compact objects in GR showing how it emerges from scattering amplitudes. For the first time, using the method of reverse unitarity [14,94–96], we have computed the multipoles of the nonlinear memory at leading order in the post-Minkowskian expansion, i.e., at $\mathcal{O}(G^3)$ and all orders in velocity. By taking this step, we have completed our knowledge of the part of the $\mathcal{O}(G^3)$ multipolar waveform which is nonanalytic in frequency, which should serve as a useful check for future calculations of this quantity.

There is, nonetheless, more to say about nonlinear memory. As emphasized in Refs. [97,98], the tree-level six-point amplitude is parametrically suppressed in the classical limit. While our findings are consistent with this result of Refs. [97,98], when the six-point amplitude enters the cuts in Eq. (14), simple power counting shows that it yields a classical contribution, as reflected in our final result. At the same time, the role played by the six-point amplitude in our calculation reveals that, contrary to previous expectations [97–99], classical gravitational radiation is not completely described by a (nonsqueezed) coherent state. This, for instance, implies that incorporating the nonlinear memory in the eikonal framework will require going beyond the proposals in [98,99] (see also Refs. [100,101]). Furthermore, this suggests there might exist classical correlations between gravitational-wave signals at different detectors, arising from the nonlinear nature of GR, which could serve as a novel test of Einstein’s theory.

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