

Bondi-Metzner-Sachs Particles

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We construct wave functions for unitary irreducible representations (UIRs) of the Bondi-Metzner-Sachs (BMS) group, i.e., BMS particles, and show that they describe quantum superpositions of (Poincaré) particles propagating on inequivalent gravity vacua. This follows from reconsidering McCarthy's classification of BMS group UIRs through a unique, Lorentz-invariant, but nonlinear, decomposition of supermomenta into hard and soft pieces.

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Introduction—While the Bondi-Metzner-Sachs (BMS) group appeared long ago as the asymptotic symmetry group of flat spacetimes [1–3], the fact that these symmetries act simultaneously on past and future null infinity was only understood recently [4]. The global nature of BMS symmetry was critical in Strominger's demonstration of the BMS invariance of the gravitational S -matrix [4–6].

Previous disregard for these asymptotic symmetries is deeply related to the longstanding problem [7–9] that scattering amplitudes involving massless particles are plagued with infrared (IR) divergences, rendering the S -matrix ill defined. Indeed, as understood in [10] for electrodynamics (see also [11]) and later extended to the gravitational case in [12], IR divergences arise, for conventional states, as a necessity to respect BMS conservation laws.

In the AdS/CFT holographic correspondence, both bulk states and boundary operators organize into unitary irreducible representations (UIRs) of the conformal group. If there exists a holographic dual theory to quantum gravity in asymptotically flat spacetime, the correspondence between boundary and bulk states should similarly be given by an equivalence of UIR representations of the BMS group (see [13–16] for early works).

In a series of pioneering works, McCarthy studied and classified BMS₄ UIRs [17–24] (see also [25–27] and, for BMS₃ UIRs, [28–30]). While McCarthy's results provide a foundational basis, from a physics perspective they raise

just as many questions as they answer. Notably, there are infinitely more BMS representations than usual Poincaré particles. Out of these, only the representations identified by Sachs [31] (for massless particles) and Longhi-Materassi [32] (for massive particles) are well understood. This is because these very specific BMS particles, referred to as “hard” in the following, are in 1:1 correspondence with usual Poincaré particles. However, we are still left with infinitely many BMS particles that require a physical interpretation. The above discussion on IR divergences suggests that these particles should correspond to some form of IR-modified states. The aim of this Letter is to provide such realizations, based on group-theoretical considerations.

Infrared gravitons are associated with two related features [4,33]: the possibility of carrying “soft” BMS charges (or “memory” charges) and the existence of “boundary gravitons” (or “supertranslation Goldstone” modes). The possibility of having nonzero soft charge $\partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}(z, \bar{z})$ is what makes BMS particles infinitely richer than their Poincaré counterparts. At the level of representation theory, this translates [34] into the fact that the supermomentum $\mathcal{P}(z, \bar{z})$ can be uniquely decomposed as

$$\mathcal{P}(z, \bar{z}) = \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}(z, \bar{z}) + P(z, \bar{z}), \quad (1)$$

where the hard contribution $P(z, \bar{z})$ is a (nonlinear) function of the momentum p_μ only: $P(z, \bar{z}) = \omega \delta^{(2)}(z - \zeta)$ if $p_\mu = \omega q_\mu(\zeta, \bar{\zeta})$ is null, $P(z, \bar{z}) = -[m^4/\pi](p \cdot q(z, \bar{z}))^{-3}$ if $p^2 = -m^2 \neq 0$ (see below for our conventions). Boundary gravitons $\partial_z^2 \mathcal{C}(z, \bar{z})$ parametrize “gravity vacua,” i.e., possible—diffeomorphic but inequivalent—backgrounds. As we shall demonstrate by constructing the corresponding wave functions, BMS particles are best thought of as quantum superpositions of Poincaré particles propagating on different gravity vacua.

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The Letter is organized as follows. We start with a review of the gravitational phase space in the next section, where we introduce mode expansions for the asymptotic radiative data as well as the soft graviton mode and supertranslation Goldstone mode. The interpretation of the latter as the mode labeling different gravity vacua is illustrated by the Kirchhoff-d'Adhémar formula in the section on gravity vacua. In the next two sections, BMS wave functions are presented and related to BMS UIRs. We then close the loop by explaining how these states are realized in terms of Strominger's phase space. Finally, in the last section, we summarize the results and discuss some implications of this Letter.

Gravitational phase space—The mode expansion of a graviton of momentum $p^\mu = (p^0, \vec{p})$ is given by

$$\hat{h}_{\mu\nu}(X) = \kappa \int \frac{d^3 p}{(2\pi)^3 2p^0} \left[\varepsilon_{\mu\nu}^{*\alpha}(\vec{p}) \hat{a}_\alpha(\vec{p}) e^{ip \cdot X} + \text{H.c.} \right], \quad (2)$$

where $X^\mu \in \mathbb{R}^{3,1}$, $\kappa = \sqrt{32\pi G}$, and $\alpha = \pm$ are the two helicities. Using the parametrization $p^\mu = \omega q^\mu(z, \bar{z})$, with the null vector $q^\mu = [1/\sqrt{2}](1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z})$, the polarization tensors can be written as $\varepsilon^{\pm\mu\nu} = \varepsilon^{\pm\mu} \varepsilon^{\pm\nu}$ with $\varepsilon^{+\mu} = \partial_z q^\mu$, $\varepsilon^{-\mu} = \partial_{\bar{z}} q^\mu$; see, e.g., [5]. The commutation relations for annihilation and creation operators, $\hat{a}_\alpha(\omega)$, $\hat{a}_\alpha^\dagger(\omega)$ ($\omega \neq 0$), are then given by

$$[\hat{a}_\alpha(\omega, z), \hat{a}_\alpha^\dagger(\omega', z')] = \frac{2(2\pi)^3}{\omega} \delta(\omega - \omega') \delta(z - z') \delta_{\alpha\alpha'}. \quad (3)$$

In flat Bondi coordinates, where the Minkowski metric is $ds^2 = -2dudr + 2r^2 dz d\bar{z}$, the large r expansion of the graviton mode yields $dx^\mu dx^\nu \hat{h}_{\mu\nu}(X) \sim r \hat{C}_{zz}(u, z, \bar{z}) dz^2 + \text{H.c.} + \mathcal{O}(r^0)$, where the asymptotic shear $\hat{C}_{zz}$ is given by [5]

$$\hat{C}_{zz} = \frac{\kappa}{8i\pi^2} \int_0^\infty d\omega (\hat{a}_+(\omega, z, \bar{z}) e^{-i\omega u} - \hat{a}_-^\dagger(\omega, z, \bar{z}) e^{i\omega u}). \quad (4)$$

A similar expression holds for $\hat{C}_{\bar{z}\bar{z}}$, which encodes the other graviton helicity. As shown in [4,5], the gravitational phase space at null infinity must include, in addition to the radiative modes $\hat{a}_\alpha(\omega)$, $\hat{a}_\alpha^\dagger(\omega)$ ($\omega \neq 0$), the soft mode $\hat{\mathcal{N}}(z, \bar{z})$ and its symplectic partner, the Goldstone boson $\hat{\mathcal{C}}(z, \bar{z})$ of spontaneously broken supertranslation invariance. They can be related to the late and early time behavior of the shear as

$$\begin{aligned} -\partial_z^2 \hat{\mathcal{N}}(z, \bar{z}) &= \frac{4}{\kappa^2} \lim_{u \rightarrow \infty} (\hat{C}_{zz}(u, z, \bar{z}) - \hat{C}_{zz}(-u, z, \bar{z})), \\ -2\partial_z^2 \hat{\mathcal{C}}(z, \bar{z}) &= \frac{1}{2} \lim_{u \rightarrow \infty} (\hat{C}_{zz}(u, z, \bar{z}) + \hat{C}_{zz}(-u, z, \bar{z})). \end{aligned}$$

As emphasized by Ashtekar, e.g., in [33,35], the presence of these modes implies that the Sachs norm for the shear is not finite. How, then, should we think of quantum states in the presence of soft charge? This is one of the questions we shall answer.

The soft and Goldstone modes commute with $\hat{a}_\alpha(\omega)$, $\hat{a}_\alpha^\dagger(\omega)$ ($\omega \neq 0$) and the only nonvanishing commutator is [5]

$$[\partial_z^2 \hat{\mathcal{N}}(z, \bar{z}), \partial_w^2 \hat{\mathcal{C}}(w, \bar{w})] = -i\delta^{(2)}(z - w). \quad (5)$$

Under the action of the group $\text{BMS}_4 \simeq \text{SL}(2, \mathbb{C}) \ltimes C^\infty(S^2)$, the hard operators transform as

$$\hat{a}_\pm^\dagger(\omega', z', \bar{z}') = \left(\frac{\partial z'}{\partial z} \right)^{\mp 1} \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \right)^{\pm 1} e^{-i\omega T(z, \bar{z})} \hat{a}_\pm(\omega, z, \bar{z}), \quad (6)$$

where $z' = (az + b/cz + d)$ and $T(z, \bar{z}) \in C^\infty(S^2)$ are the corresponding Möbius transformation and supertranslations, respectively. Soft operators both transform as $\text{SL}(2, \mathbb{C})$ primaries of weights $(-\frac{1}{2}, -\frac{1}{2})$,

$$\begin{aligned} \hat{\mathcal{N}}'(z', \bar{z}') &= \left(\frac{\partial z'}{\partial z} \right)^{-\frac{1}{2}} \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \right)^{-\frac{1}{2}} \hat{\mathcal{N}}(z, \bar{z}), \\ \hat{\mathcal{C}}'(z', \bar{z}') &= \left(\frac{\partial z'}{\partial z} \right)^{-\frac{1}{2}} \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \right)^{-\frac{1}{2}} \left[\hat{\mathcal{C}}(z, \bar{z}) + T(z, \bar{z}) \right], \end{aligned} \quad (7)$$

while the action of supertranslations induces a “shift” of the Goldstone mode. As a result, Minkowski spacetime transforms under BMS supertranslations as

$$\begin{aligned} 2(ds^2)' &= -2dudr + 2r^2 dz d\bar{z} \\ &\quad - 2r(\partial_z^2 T dz^2 + \partial_{\bar{z}}^2 T d\bar{z}^2) + \mathcal{O}(r^0). \end{aligned} \quad (8)$$

The space of all possible Minkowski spacetimes obtained in this way will be denoted $\mathbb{V}$ and will be referred to as the space of gravity vacua.

Gravity vacua—The mode $\partial_z^2 \hat{\mathcal{C}}$, labeling the different gravity vacua, and the radiative data $\hat{a}_\alpha(\omega)$ ($\omega \neq 0$) play a very different role in the theory: the field $\partial_z^2 \hat{\mathcal{C}}$ defines the background, denoted $\mathbb{M}_{\mathcal{C}}$, on which the radiative data propagate.

Following Newman [36], the field $\partial_z^2 \hat{\mathcal{C}}$ can indeed be understood as the piece of data needed to reconstruct Minkowski spacetime in a holographic manner from null infinity $\mathcal{I}$. The key idea here is to consider the space $\Gamma[\mathcal{I}]$ of cuts of null infinity, i.e., the space of sections

$$U: \begin{cases} S^2 & \rightarrow \mathcal{I} \\ (z, \bar{z}) & \mapsto (u = U(z, \bar{z}), z, \bar{z}). \end{cases} \quad (9)$$

There are infinitely many of these sections $U \in \Gamma[\mathcal{I}]$. Picking a shear that is pure gauge $C_{\bar{z}\bar{z}} = -2\partial_z^2 \hat{\mathcal{C}}$, one

can, however, consistently define Minkowski spacetime $\mathbb{M}_C \subset \Gamma[\mathcal{S}]$ as the subspace of real solutions to the “good cut” equation [37],

$$U \in \mathbb{M}_C \Leftrightarrow \partial_{\bar{z}}^2 U = \frac{1}{2} C_{\bar{z}\bar{z}}. \quad (10)$$

A generic solution must be, indeed, of the form

$$U_X(z, \bar{z}) = -\mathcal{C}(z, \bar{z}) - q_\mu(z, \bar{z})X^\mu. \quad (11)$$

It is, however, clear that, through this procedure, one can construct infinitely many different copies $\mathbb{M}_C \in \mathbb{V}$ of Minkowski spacetime, altogether forming the space of gravity vacua $\mathbb{V}$. Each of these gravity vacua can be related to the reference flat spacetime $\mathbb{M}_{C=0}$ via a supertranslation.

Once a background $\mathbb{M}_C$ has been chosen, one can propagate the radiative data. This is best illustrated by the Kirchhoff-d’Adhémar formula, cf. [38] or Chap. 5.12 of [39], which allows one to reconstruct the bulk field $h_{\mu\nu}(X)$ from the joint data of the boundary value C_{zz} at $\mathcal{S}$ and the gravity vacuum $\mathbb{M}_C$,

$$h_{\mu\nu}(X, \mathcal{C}) = -\frac{1}{2\pi} \int_{u=U_X} d^2 z \varepsilon_{\mu\nu}^{*+}(z, \bar{z}) \partial_u C_{zz}(u, z, \bar{z}) + \text{H.c.} \quad (12)$$

Equivalently, making use of (4) and (11),

$$\begin{aligned} h_{\mu\nu}(X, \mathcal{C}) &= \frac{\kappa}{16\pi^3} \int \omega d\omega d^2 z \left[\varepsilon_{\mu\nu}^{*\alpha}(z, \bar{z}) a_\alpha(\omega, z, \bar{z}) e^{i\omega(q \cdot X + C)} + \text{H.c.} \right], \end{aligned} \quad (13)$$

which, when $\mathcal{C} = 0$ and upon quantization, is seen to give back formula (2). Note that going from (2) to (13) amounts to replacing the creation and annihilation operators by the dressed operators of [40]. The above is the wave function of a hard (massless) BMS particle. This is the simplest instance of a BMS particle.

BMS wave functions—Equation (13) suggests to write a general (massless) BMS wave function as

$$|\Psi\rangle = \int_{\mathbb{V}} \mathcal{D}\mathcal{C} \int_{\mathbb{M}_C} d^4 X \Psi(X; \partial^2 \mathcal{C}) |X; \partial^2 \mathcal{C}\rangle, \quad (14)$$

with $\partial_\mu \partial^\mu \Psi = 0$ at fixed $\mathcal{C}$. Here $|X; \partial^2 \mathcal{C}\rangle$ is an eigenstate of both position and gravity vacuum

$$\hat{X}^\mu |X; \partial^2 \mathcal{C}\rangle = X^\mu |X; \partial^2 \mathcal{C}\rangle, \quad \partial_z^2 \hat{\mathcal{C}} |X; \partial^2 \mathcal{C}\rangle = \partial_z^2 \mathcal{C} |X; \partial^2 \mathcal{C}\rangle$$

and the first integral is a path integral over the space of gravity vacua. The above form will ultimately be justified by the fact that all BMS UIRs can be realized in this way

(see the next section). We will also later show how to relate it to Strominger’s gravitational phase space (3)–(5). Let us here only highlight some essential features.

Keeping $\mathcal{C}$ fixed gives the wave function of a field in a given gravity vacuum $\mathbb{M}_C$,

$$\int_{\mathbb{M}_C} d^4 X \Psi(X; \partial^2 \mathcal{C}) |X; \partial^2 \mathcal{C}\rangle. \quad (15)$$

Therefore, the generic state (14) is a quantum superposition of such wave functions in all possible gravity vacua $\mathbb{M}_C \in \mathbb{V}$.

The Fourier transform in X ,

$$|\omega, z, \bar{z}; \partial^2 \mathcal{C}\rangle := \int_{\mathbb{M}_C} d^4 X e^{i\omega(q(z, \bar{z}) \cdot X + \mathcal{C}(z, \bar{z}))} |X; \partial^2 \mathcal{C}\rangle, \quad (16)$$

yields eigenstates of momentum in a given gravity vacuum, and (14) can then be rewritten as

$$|\Psi\rangle = \int_{\mathbb{V}} \mathcal{D}\mathcal{C} \int \omega d\omega d^2 z \Psi(\omega, z, \bar{z}; \partial^2 \mathcal{C}) |\omega, z, \bar{z}; \partial^2 \mathcal{C}\rangle, \quad (17)$$

where

$$\Psi(X, \partial^2 \mathcal{C}) = \int \omega d\omega d^2 z e^{i\omega(q(z, \bar{z}) \cdot X + \mathcal{C}(z, \bar{z}))} \Psi(\omega, z, \bar{z}; \partial^2 \mathcal{C}).$$

In the hard case (13), this last equation gives

$$\Psi^{\text{hard}}(\omega, z, \bar{z}; \partial^2 \mathcal{C}) = \begin{cases} a^\dagger(\omega, z, \bar{z}) & \omega > 0, \\ a(\omega, z, \bar{z}) & \omega < 0. \end{cases} \quad (18)$$

BMS particles—In usual Poincaré representation theory (*à la* Wigner), a particle is given by a function $\Psi(P)$ in momentum space $(\mathbb{R}^{3,1})^*$, defined as the dual space to the space of translations. More precisely, Poincaré particles (i.e., UIRs of the Poincaré group) are described by wave functions that only have support on a given orbit of the Lorentz group (e.g., the null cone for massless particles), the spin being given by a choice of UIR for the little group ℓ_P .

A BMS particle [17–24] is described by a wave function $\Psi(\mathcal{P})$ in supermomentum space [i.e., $(C^\infty(S^2))^*$, the dual space to the space of supertranslations],

$$|\Psi\rangle = \int \mathcal{D}\mathcal{P} \Psi(\mathcal{P}) |\mathcal{P}\rangle. \quad (19)$$

As the notation suggests, $|\mathcal{P}\rangle$ is here an eigenstate of the supermomentum operator $\hat{\mathcal{P}}(z, \bar{z})$, which is the generator of supertranslations. Since supertranslations have weights $(-\frac{1}{2}, -\frac{1}{2})$, supermomenta must have weights $(\frac{3}{2}, \frac{3}{2})$, the duality pairing being given by

$$\langle \mathcal{P}, T \rangle = \int d^2 z \mathcal{P}(z, \bar{z}) T(z, \bar{z}). \quad (20)$$

The space of states of the form (19) always forms a BMS representation: an element of the BMS group $(M, T) \in \text{SL}(2, \mathbb{C}) \ltimes C^\infty(S^2)$ acts as

$$|\Psi\rangle \mapsto \int \mathcal{D}\mathcal{P} e^{i\langle \mathcal{P}, T \rangle} \Psi(\mathcal{P} \cdot M) |\mathcal{P}\rangle, \quad (21)$$

where $\mathcal{P} \cdot M$ indicates that Möbius transformations act from the right on supermomenta. The UIRs of the BMS group are then given by functions that only have support on a given orbit $\mathcal{O}_{\mathcal{P}}$ of the Lorentz group. These orbits generalize the mass shell of usual Poincaré particles; the BMS equivalent of spin being given by a choice of UIR for the BMS little group $\ell_{\mathcal{P}} \subset \text{SL}(2, \mathbb{C})$, defined as the stabilizer of a supermomenta $\mathcal{P} \in \mathcal{O}_{\mathcal{P}}$ in the orbit. The heart of the seminal work of McCarthy [17–24] was to classify all possible little groups appearing in this way.

The existence of a Lorentz-invariant projection from supermomenta $\mathcal{P}$ to momenta p_μ ,

$$\mathcal{P}(z, \bar{z}) \mapsto \pi_\mu(\mathcal{P}) = \int d^2 z q_\mu(z, \bar{z}) \mathcal{P}(z, \bar{z}), \quad (22)$$

means that one can talk about the Poincaré little group $\ell_{\pi(\mathcal{P})}$ of a BMS particle and, in particular, define its mass square $m^2 = \pi_\mu(\mathcal{P}) \pi^\mu(\mathcal{P})$. It follows from (22) that the BMS little group is necessarily contained in the Poincaré little group

$$\ell_{\mathcal{P}} \subseteq \ell_{\pi(\mathcal{P})} \subseteq \text{SL}(2, \mathbb{C}). \quad (23)$$

In this Letter, we will only discuss massless BMS representations, i.e.,

$$\pi_\mu(\mathcal{P}) = \omega q_\mu(\zeta, \bar{\zeta}). \quad (24)$$

See [34] for a discussion on the massive case (which does not alter significantly the picture described here). Of crucial importance is the possibility to always decompose a massless supermomentum as

$$\mathcal{P}(z, \bar{z}) = \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}(z, \bar{z}) + \omega \delta^{(2)}(z - \zeta). \quad (25)$$

We wish to emphasize that this decomposition is unique and Lorentz invariant. However, it is *not* stable under addition, due to the nonlinear dependence in the momentum (24) of the hard contribution $P(z, \bar{z}) = \omega \delta^{(2)}(z - \zeta)$. The decomposition (25) means that, introducing the notation $|\omega, \zeta, \bar{\zeta}; \partial^2 \mathcal{N}\rangle$ for the eigenstates of $\hat{\mathcal{P}}$ of eigenvalue (25), one can always write a massless BMS particle as

$$|\Psi\rangle = \int \omega d\omega d^2 \zeta \int \mathcal{D}\mathcal{N} \Psi(\omega, \zeta, \bar{\zeta}; \partial^2 \mathcal{N}) |\omega, \zeta, \bar{\zeta}; \partial^2 \mathcal{N}\rangle, \quad (26)$$

where the wave function $\Psi(\omega, \zeta, \bar{\zeta}; \partial^2 \mathcal{N})$ must only have support on a given orbit $\mathcal{O}_{\mathcal{P}}$ of the Lorentz group inside the space of supermomenta. To be more explicit, factorize elements of the Lorentz group as

$$M(p, W) = L(p)W \in \text{SL}(2, \mathbb{C}), \quad (27)$$

where $p \mapsto L(p)$ is an injective map from the null cone to $\text{SL}(2, \mathbb{C})$ and $W \in \ell_p \subset \text{SL}(2, \mathbb{C})$. Let $\mathcal{N}^0(z, \bar{z})$ be a reference soft charge defining the orbit of the BMS particle and define $\mathcal{N}_p := \mathcal{N}^0 \cdot L(p)^{-1}$. One can then write [41]

$$|\Psi\rangle = \int d^3 p \int_{\ell_p / \ell_{\mathcal{P}}} dW \Psi(p; (\partial^2 \mathcal{N}_p) \cdot W) |p; (\partial^2 \mathcal{N}_p) \cdot W\rangle. \quad (28)$$

More generally, BMS particles are always realized as *finite-dimensional integrals* of dimension

$$\dim(\mathcal{O}_{\mathcal{P}}) = 6 - \dim(\ell_{\mathcal{P}}). \quad (29)$$

As explained in [34], for hard representations $\dim(\ell_{\mathcal{P}}) = \dim(\ell_{\pi(\mathcal{P})}) = 3$ and one recovers the usual mass-shell dimension 3. However, a generic orbit will, in fact, have no nontrivial stabilizer and thus will have support on a six-dimensional submanifold in the space of supermomenta. All intermediate dimensions are also allowed and correspond to nontrivial BMS little groups (see, e.g., [21–24]).

Since supermomenta $\mathcal{P}$ are dual to supertranslations $\mathcal{T}$, one can define the (infinite-dimensional) Fourier transform

$$|\mathcal{T}\rangle = \int \mathcal{D}\mathcal{P} e^{-i\langle \mathcal{P}, \mathcal{T} \rangle} |\mathcal{P}\rangle \quad (30)$$

and rewrite BMS particles as

$$|\Psi\rangle = \int \mathcal{D}\mathcal{T} \Psi(\mathcal{T}) |\mathcal{T}\rangle. \quad (31)$$

Once again, while the wave function in BMS space $\Psi(\mathcal{T})$ appears as a complicated functional on the (infinite-dimensional) space of supertranslations, let us emphasize that, once Fourier transformed to supermomentum space, the wave function $\Psi(\mathcal{P})$ of a BMS particle only lives in a submanifold of dimension (29). Finally, since a supertranslation $\mathcal{T}$ can always be decomposed as

$$\mathcal{T}(z, \bar{z}) = \mathcal{C}(z, \bar{z}) + q_\mu(z, \bar{z}) X^\mu \quad (32)$$

where $\mathcal{C} = \mathcal{T}|_{l \geq 2}$ only contains higher spherical harmonics of $\mathcal{T}$ and $q \cdot X = \mathcal{T}|_{l=0,1}$ the lower ones, one can write

$\Psi(X, \partial^2 \mathcal{C}) := \Psi(\mathcal{T})$ and recover the expression (14) for a BMS state. Note that, while the decomposition (32) is not Lorentz invariant, the projection $\mathcal{T} \mapsto \partial_z^2 \mathcal{C}$ is Lorentz invariant. Thus, it makes sense to talk of the restriction of a BMS wave function (14) to a fixed gravity vacuum $\partial_z^2 \mathcal{C}$ as in (15). Similarly, one can restrict a BMS UIR to the subspace of wave functions in a fixed gravity vacuum. Such a subspace is not BMS invariant, but it is invariant under the corresponding Poincaré group.

To conclude the comparison with the previous gravity vacua section, we note that eigenstates of momentum in a given gravity vacuum (16) are related to supermomentum eigenstates as

$$|\omega, z, \bar{z}; \partial^2 \mathcal{N}\rangle = \int \mathcal{D}\mathcal{C} e^{i \int d^2 w \partial_w^2 \mathcal{N} \partial_{\bar{w}}^2 \mathcal{C}} |\omega, z, \bar{z}; \partial^2 \mathcal{C}\rangle. \quad (33)$$

First quantization of gravity vacua—In this final section, we relate the BMS particle states (14) to Strominger’s gravitational phase space (3)–(5). Let us first recall that the supermomentum operator is realized as [5]

$$\hat{P}(z, \bar{z}) = \partial_z^2 \partial_{\bar{z}}^2 \hat{\mathcal{N}} + \frac{1}{16\pi^3} \int_0^\infty d\omega \omega^2 (\hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_-). \quad (34)$$

The above split between soft and hard parts was also studied in [42], where it was checked that each piece transforms separately in the coadjoint representation of the BMS algebra [43].

Now, the fact that the Goldstone current is shifted under the action of supertranslations (7) suggests to think of $\partial_z^2 \hat{\mathcal{C}}(z, \bar{z})$ as a “position operator” in the space of gravity vacua $\mathbb{V}$. The commutator (5) is then seen to be the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$ in the space of gravity vacua. It follows from the commutation relations (3)–(5) and the expression (34) for the supermomentum operator that supermomentum eigenstates of eigenvalue (25), $|\mathcal{P}\rangle = |\omega, \zeta, \bar{\zeta}; \partial^2 \mathcal{N}\rangle$, are obtained as [44]

$$|\omega, \zeta, \bar{\zeta}; \partial^2 \mathcal{N}\rangle = e^{i(\partial^2 \bar{\mathcal{T}} \cdot \mathcal{N}, \hat{\mathcal{C}})} \hat{a}^\dagger(\omega, \zeta, \bar{\zeta}) |0\rangle, \quad (35)$$

where $|0\rangle$ denotes the BMS vacuum which, by definition, satisfies

$$\hat{a}_\alpha(\omega, z, \bar{z}) |0\rangle = 0, \quad \partial_z^2 \hat{\mathcal{N}}(z, \bar{z}) |0\rangle = 0. \quad (36)$$

BMS wave functions (14) are then constructed from (28) and the successive Fourier transforms (16) and (33). As we saw, they should be thought of as quantum superpositions of usual Poincaré particles in different gravity vacua. Among all possible states that can be constructed in this way, states of the form

$$|p\rangle_{\mathcal{C}} = \int_{\ell_p / \ell_p} dL e^{i(\partial^2 \bar{\mathcal{T}} \cdot \mathcal{N} \cdot L, \mathcal{C})} |p; (\partial^2 \mathcal{N}) \cdot L\rangle \quad (37)$$

stand out as BMS particles of momentum p_μ and “localized” in a gravity vacuum $\partial_z^2 \mathcal{C}$. As opposed to the eigenstates (16) of $\partial_z^2 \hat{\mathcal{C}}$, the above states belong to a UIR of BMS and are thus genuine particles. Finally, note that, for hard particles, $\mathcal{N} = 0$, $\ell_p = \ell_p$, and the $\mathcal{C}$ dependence drops out so that, by construction, such representations cannot discriminate a particular vacuum.

Discussion—In the present Letter, we showed that BMS particles, i.e., BMS UIRs, can always be realized in terms of wave functions of the form (31). These can naturally be interpreted as quantum superpositions of usual particles, each of them propagating on a different gravity vacuum. The (familiar) hard representation (13) stands out by propagating the same particle in all possible gravity vacua, see (18). The essence of our construction was to reconsider McCarthy’s results in light of the hard (soft) decomposition of supermomenta (1) suggested by recent developments [6]; see [34] for more details. Given the close similarity of asymptotic symmetries [45], the present construction carries through almost identically for QED.

The physical picture emerging from the present Letter fits very naturally with Refs. [46–48] suggesting to think of S -matrix observables as living “over the moduli space of vacua.” It is also in line with the reinterpretation [10] of Faddeev-Kulish (FK) states [49], which led to the generalized states of [12]: as already emphasized in the massive case in [50], hard massless particles, whose supermomenta are of the form $\omega \delta^{(2)}(z - \zeta)$, are not enough to ensure conservation of supermomentum. In the language of [10], this means that hard states cannot ensure the conservation of BMS charges, which is the reason for IR divergences. Reference [49] then amounts to constructing states of supermomenta $\omega \delta^{(2)}(z - \zeta) - (\omega/2\pi) \partial_z^2 (\bar{z} - \bar{\zeta}/z - \zeta)$. Keeping in mind that these are weighted distributions, these supermomenta do not vanish, and one can show that, for such states, conservation of momentum implies conservation of supermomentum (see Supplemental Material [51]). From this perspective, the FK construction is rather unnatural and it is simpler to just require the total conservation of BMS charges in order to obtain IR-finite S -matrix elements [10,12].

It follows that if an IR-finite unitary S -matrix for massless particles can be defined, then it will have to be BMS invariant. Therefore, the asymptotic one-particle states must belong to UIRs of the BMS group and the multiparticle states must be suitable tensor products thereof. In the present Letter, we gave a physical realization of the corresponding one-particle states and showed that Strominger’s phase space can be understood as a quantization scheme where gravity vacua are first quantized, while the remaining hard degrees of freedom are second quantized. It should be clear from our presentation that a complete first quantization can be obtained by considering functions on the (infinite-dimensional) homogeneous space $\text{BMS}_4/\text{SO}(3,1)$. However, it is also clear that one should

rather consider a complete second quantization as the starting point for an extension of the massless S -matrix. This is where, we believe, our Letter relates and contrasts with the recent works [52–54], which consider a Hilbert space including supermomentum eigenstates. It appears that the latter does not coincide with the Fock space of BMS particles. The perspective put forward in this Letter is rather to stick to Fock quantization of BMS particles, i.e., asymptotic multiparticle states constructed as tensor products $|\omega_1, \zeta_1, \bar{\zeta}_1; \partial^2 \mathcal{N}_1\rangle \otimes \cdots \otimes |\omega_n, \zeta_n, \bar{\zeta}_n; \partial^2 \mathcal{N}_n\rangle$ of free one-particle states spanning BMS UIRs. S -matrix elements of such BMS particles are structurally different from usual ones, due to the extra degrees of freedom involved. As an initial step in constructing such amplitudes, and investigating their potential to resolve infrared divergences, one should identify the equivalent of Mandelstam variables in this context. Additionally, one should develop appropriate limiting procedures (see [34] for early suggestions) to establish connections with the conventional S -matrix and observable cross sections. We will return to these questions in upcoming works.

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