

Anyonic Phase Transitions in the 1D Extended Hubbard Model with Fractional Statistics

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We study one-dimensional lattice anyons with extended Hubbard interactions at unit filling using bosonization and numerical simulations. The behavior can be continuously tuned from bosonic to fermionic statistics by adjusting the topological exchange angle θ , which leads to a competition of different instabilities. We present the bosonization theory in the presence of dynamic gauge fields, which predicts a phase diagram of four different gapped phases with distinct dominant correlations. Advanced numerical simulations determine and analyze the exact phase transitions between Mott insulator, charge density wave, dimerized state, and Haldane insulator, all of which meet at a multicritical line in the parameter space of anyonic angle θ , on-site interaction U , and nearest neighbor repulsion V . Superfluid and pair-superfluid phases are stable in a region of small V .

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Introduction—For continuous one-dimensional (1D) systems an exotic particle species interpolating between bosons and fermions was first proposed by Leinaas and Myrheim [1,2]. This concept of anyons has received renewed interest with the implementation by so-called “lattice anyons” [3–17] based on experimental progress to create artificial gauge fields by dynamic manipulations and Floquet driving in ultracold gases [15–25], by Raman assisted coupling [26,27] or by impurities in 1D spin-charge separated systems [28]. The originally proposed continuous 1D anyons [1] correspond to the limit of low filling of interacting lattice anyons [29]. However, for unit filling of one particle per site strong lattice effects give rise to a completely different situation, where several broken symmetry phases are possible, which is the topic of this Letter. This is motivated by the observation that different instabilities have been established for fermions and bosons in the 1D extended Hubbard model at unit filling. In particular, the competition of on-site U and nearest neighbor V interactions leads to a possible dimerized (or bond ordered) phase for fermions [30–33], while for bosons topological string correlations in the form of a Haldane insulator (HI) are observed in an intermediate regime [34–37]. This invites the question for anyons: what is the nature of an intermediate phase as the anyonic exchange phase θ is tuned continuously between bosonic behavior for $\theta = 0$ to

fermionic behavior for $\theta = \pi$? The goal of this Letter is to examine the transitions between all different instabilities and determine the dominant correlations as a function of θ , U , and V at unit filling.

The 1D extended Hubbard model for anyons can be written in standard notation as

$$\hat{H} = \sum_l \left[-J \hat{a}_l^\dagger \hat{a}_{l+1} + \text{H.c.} + \frac{U}{2} \hat{n}_l (\hat{n}_l - 1) + V \hat{n}_l \hat{n}_{l+1} \right]. \quad (1)$$

The anyonic statistics enters via the exchange phase θ in the deformed commutation relations on different sites $l \neq l'$: $\hat{a}_l \hat{a}_{l'}^\dagger - e^{i\theta \text{sgn}(l-l')} \hat{a}_{l'}^\dagger \hat{a}_l = 0$. Using a generalized Jordan-Wigner transformation $\hat{a}_l = \hat{b}_l \exp(i\theta \sum_{j<l} \hat{n}_j)$ [6] the anyonic model can be mapped to a bosonic one $[\hat{b}_{l'}^\dagger, \hat{b}_l] = 0$ for $l \neq l'$ with density-dependent hopping

$$\hat{H}_{\text{kin}} = -J \sum_l \left[\hat{b}_l^\dagger \hat{b}_{l+1} e^{i\theta \hat{n}_l} + \text{H.c.} \right]. \quad (2)$$

The on-site commutator $l = l'$ depends on the physical implementation, which is typically spinless bosons restricted to a maximum of two particles per site [7]. The limit of $\theta = \pi$ corresponds to spinless “pseudo-fermions” with the same on-site behavior but anticommutation relations for $l \neq l'$. We now derive the corresponding low-energy field theory, which will allow for drawing conclusions about the expected phase transitions.

Bosonization of gauge fields—For the constraint of maximally two particles per site at average unit filling,

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an exact representation can be used in terms of spin-1 operators $\hat{S}_l^z = -\hat{n}_l + 1$ which take on eigenvalues $-1, 0, 1$ [38]. The operations of the transformed hopping operator are given in terms of spin operators $\hat{S}_l^+|-1\rangle = \sqrt{2}|0\rangle$ and $\hat{S}_l^+|0\rangle = \sqrt{2}|1\rangle$ as

$$\hat{H}_{\text{kin}} = -\frac{1}{2} \sum_l \left[t(\hat{S}_l^z, \hat{S}_{l+1}^z) \hat{S}_l^- \hat{S}_{l+1}^+ + \text{H.c.} \right], \quad (3)$$

where the effective hopping is $t(-1, 0) = 2Je^{i\theta}$, $t(0, 1) = J$, $t(-1, 1) = \sqrt{2}Je^{i\theta}$, and $t(0, 0) = \sqrt{2}J$. A similar parametrization for interacting bosons has been used before [35] with the simplification $t = J$. Here we keep the full expression for t to avoid an artificial spin-flip symmetry and implement the θ -dependence, which can be written in terms of spin-operators as

$$t(\hat{S}_l^z, \hat{S}_{l+1}^z) = J \left[1 + d(\theta) \hat{S}_l^z \right] \left[1 + d(0) (\hat{S}_{l+1}^z - 1) \right], \quad (4)$$

with $d(\theta) = 1 - \sqrt{2}e^{i\theta}$. Following Refs. [35,39], we represent the spin-1 degrees of freedom by triplet states of two spin- $\frac{1}{2}$ operators, $\hat{S}_l^\pm = \hat{S}_{1,l}^\pm + \hat{S}_{2,l}^\pm$ and $\hat{S}_l^z = \hat{S}_{1,l}^z + \hat{S}_{2,l}^z$, for each site. This results in a spin- $\frac{1}{2}$ two-leg ladder model:

$$\begin{aligned} \hat{H}_{\text{kin}} = & -\frac{1}{2} \sum_l \left[\tilde{t}(\hat{S}_{2,l}^z, \hat{S}_{2,l+1}^z) \hat{S}_{1,l}^- \hat{S}_{1,l+1}^+ \right. \\ & + \tilde{t}(\hat{S}_{2,l}^z, \hat{S}_{1,l+1}^z) \hat{S}_{1,l}^- \hat{S}_{2,l+1}^+ \\ & \left. + \text{all terms with } 1 \leftrightarrow 2 + \text{H.c.} \right], \end{aligned} \quad (5)$$

where $1 \leftrightarrow 2$ denotes the interchange of chain indices 1 and 2 and $\tilde{t}(\hat{S}_{i,l}^z, \hat{S}_{j,l+1}^z) \equiv t(\hat{S}_{i,l}^z - 1/2, \hat{S}_{j,l+1}^z + 1/2)$ for $i, j = 1, 2$. Note that hopping in chain 1 depends on the spin operators $\hat{S}_{2,l}^z$ in chain 2, and vice versa. In that sense we may speak of a lattice gauge function [40]

$$\tilde{t}(\hat{S}_{i,l}^z, \hat{S}_{j,l+1}^z) = \tilde{J} + J_z \hat{S}_{i,l}^z + \bar{J}_z \hat{S}_{j,l+1}^z + J_{zz} \hat{S}_{i,l}^z \hat{S}_{j,l+1}^z, \quad (6)$$

with $\tilde{J} = (J/4)(1 + \sqrt{2})(1 + \sqrt{2}e^{i\theta})$ and higher spin interaction constants $J_z = (J/2)(1 + \sqrt{2})(1 - \sqrt{2}e^{i\theta})$, $\bar{J}_z = (J/2)(1 - \sqrt{2})(1 + \sqrt{2}e^{i\theta})$, and $J_{zz} = J(1 - \sqrt{2})(1 - \sqrt{2}e^{i\theta})$.

For low energies, the Hamiltonian of the coupled spin chains is expressed in terms of dual bosonic fields Θ_n and Φ_n for each chain $n = 1, 2$, which are defined for a continuum variable normalized such that $[\Phi_n(x), \Theta_{n'}(x')] = (i/2)\delta_{n,n'}\text{sgn}(x - x')$ [35,41]. More details are given in Supplemental Material [40]. The resulting Hamiltonian is

$$\begin{aligned} \hat{H} \simeq \int dx & \left[\sum_{\nu=+,-} \frac{u_\nu}{2} \left(K_\nu (\partial_x \Theta_\nu)^2 + K_\nu^{-1} (\partial_x \Phi_\nu)^2 \right) \right. \\ & + \Delta (\partial_x \Theta_+ \partial_x \Phi_+ + \partial_x \Theta_- \partial_x \Phi_-) \\ & + g_1 \cos \sqrt{4\pi} \Phi_+ + g_2 \cos \sqrt{4\pi} \Phi_- \\ & \left. + g_3 \cos \sqrt{4\pi} \Theta_- + g_4 \cos \sqrt{4\pi} \Phi_+ \cos \sqrt{4\pi} \Phi_- \right], \end{aligned} \quad (7)$$

which we discuss below. Here $\Phi_\pm = \Phi_1 \pm \Phi_2$ and $\Theta_\pm = (\Theta_1 \pm \Theta_2)/2$ are the fields for the total (+) and relative (−) densities and phases of the two spin-1/2 chains, respectively. Terms which only affect conserved quantum numbers have been omitted. The parameters in the Hamiltonian can be determined in a weak-coupling expansion for small $U, V, J_z, \bar{J}_z$, and J_{zz} :

$$K_+ = 2 \left[1 + \frac{6V + U - 4\text{Re}J_{zz}/\pi}{\text{Re}(\pi\tilde{J} - J_{zz}/2\pi)} \right]^{-1/2}, \quad (8)$$

$$K_- = 2 \left[1 + \frac{2V - U - 4\text{Re}J_{zz}/\pi}{\text{Re}(\pi\tilde{J} - J_{zz}/2\pi)} \right]^{-1/2}, \quad (9)$$

$$u_\nu = 2a\text{Re}(\tilde{J} - J_{zz}/2\pi^2)/K_\nu, \quad (10)$$

$$\Delta = -\frac{a}{\pi} \text{Im}(J_z + \bar{J}_z) = \frac{2Ja}{\pi} \sin \theta, \quad (11)$$

$$\frac{g_1}{a} = \frac{1}{2} (2V - U) + \frac{2\text{Re}(J_{zz})}{\pi}, \quad (12)$$

$$\frac{g_2}{a} = \frac{1}{2} (U - 2V) + \frac{2\text{Re}(J_{zz})}{\pi}, \quad (13)$$

$$\frac{g_3}{a} = -\text{Re}(\tilde{J})\pi, \quad (14)$$

$$\frac{g_4}{a} = V - \frac{\text{Re}(J_{zz})}{\pi}, \quad (15)$$

where a is the lattice constant. Those expressions follow from a first order expansion in the interaction parameters and an operator product expansion [42,43] of the θ -dependent hopping terms as discussed in Supplemental Material [40]. However, the mapping from bosons to spin operators already implies a finite value $J_{zz} \neq 0$ even for $\theta = U = V = 0$, so the exact weak-coupling limit cannot be realized in the phase diagram. Hence, the actual coupling constants quantitatively differ from the formulas given above, but the operator content and the qualitative behavior with increasing θ, V and U are robust.

For $\theta = 0$ and $\tilde{t} = J$ we recover the structure and phases for bosons as discussed in Ref. [35]. For $\theta > 0$ we notice the appearance of a characteristic anyonic current-density interaction Δ on the second line in the Hamiltonian in Eq. (7). This term is present for any filling and leads to

different left- and right-moving velocities [29]. While the corresponding time-dependent correlation functions now show chiral behavior, remarkably the static mode expansions of the fields remain unaffected [29], so scaling dimensions and the renormalization behavior are not changed by Δ .

Phase transitions—Next, we turn to the cosine interactions g_1, g_2, g_3, g_4 on the last two lines of the Hamiltonian in Eq. (7), all of which may be relevant and lead to gapped phases, depending on K_ν [35]. Notably, however, the $\sin \sqrt{4\pi}\Phi_+$ term is absent, even though spatial inversion symmetry $\mathcal{I}$ is broken [35]. This is due to the fact that a modified inversion symmetry $\tilde{\mathcal{I}}$ is still obeyed as discussed in Ref. [8], which forbids this operator (see Supplemental Material [40] for a discussion of symmetries).

In the (+) sector a pinning of the field Φ_+ may occur due to the g_1 interaction, which is relevant for $K_+ < 2$. This case is discussed in the following, while $K_+ > 2$ corresponds to superfluid phases with irrelevant g_1 , which is considered in End Matter Appendix B. For relevant $g_1 < 0$ the renormalization flow fixes the value of $\Phi_+ = 0$, while it becomes $\Phi_+ = \sqrt{\pi}/2$ for $g_1 > 0$, with a phase transition at $g_1 = 0$, characterized by a free field of conformal charge $c = 1$ [35]. The relative quantum numbers in the (−) channel will also be pinned with long-range correlations. Here, two interactions $g_2 < 0$ and $g_3 < 0$ compete with scaling dimensions of K_- and $1/K_-$, respectively. The renormalization flow for $K_- < 1$ will lead to a minimum at $\Phi_- = 0$, while $K_- > 1$ is characterized by $\Theta_- = 0$. The phase transition for $K_- = 1$ is predicted to be in the Ising universality class with $c = 1/2$ as long as $g_2 \approx g_3$ [35,44]. For the model at hand a sign change of g_2 is only possible where it is not the leading instability. The two phase transitions are plotted schematically in Fig. 1 by using the conditions $g_1 = 0$ and $K_- = 1$ to determine a critical U^c as a function of $-\text{Re}J_{zz} \propto 1 - \sqrt{2} \cos \theta$ for a fixed value of V . From Eqs. (8)–(15) we get the weak-coupling limit

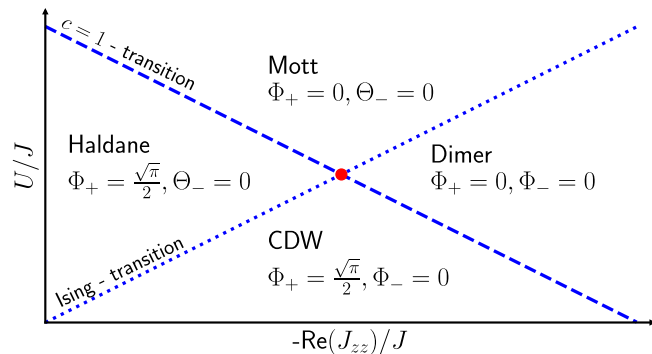


FIG. 1. Schematic phase diagram resulting from bosonization in Eqs. (16) and (17) as a function of U and $-\text{Re}J_{zz} \propto 1 - \sqrt{2} \cos \theta$ for fixed V and $\tilde{J}$. At one point (red) four gapped phases meet, each of which is characterized by the minimum values of bosonic fields.

$$U_{c=1}^c(\theta) \sim 2V + \frac{4\text{Re}[J_{zz}(\theta)]}{\pi} \quad (16)$$

$$U_{\text{Ising}}^c(\theta) \sim 2V - \frac{5\text{Re}[J_{zz}(\theta)]}{2\pi} - 3\pi\text{Re}[\tilde{J}(\theta)]. \quad (17)$$

For $\theta = 0$ the corresponding long-range ($|i - j| \rightarrow \infty$) order parameters have been discussed in Ref. [35] in terms of densities $\delta\hat{n}_i = \hat{n}_i - 1$ and fields $\Phi_\pm$. For $U > U_{c=1}^c > U_{\text{Ising}}^c$ the pinning $\Phi_+ = \Theta_- = 0$ corresponds to the Mott insulator (MI) with parity order parameter

$$\mathcal{O}_{\text{MOTT}}^2 = \langle e^{i\pi \sum_{i < k < j} \delta\hat{n}_k} \rangle \sim (\cos \sqrt{\pi}\Phi_+)^2, \quad (18)$$

while $U_{c=1}^c > U_{\text{Ising}}^c > U$ is a charge density wave (CDW) with alternating charge order $\Phi_+ = \sqrt{\pi}/2$ and fixed relative densities $\Phi_- = 0$ characterized by long-range density correlations

$$\mathcal{O}_{\text{CDW}}^2 = (-1)^{|i-j|} \langle \delta\hat{n}_i \delta\hat{n}_j \rangle \sim (\sin \sqrt{\pi}\Phi_+ \cos \sqrt{\pi}\Phi_-)^2. \quad (19)$$

Alternatively, a local order parameter $\mathcal{O}_{\text{CDW}}$ can be defined as the even-odd density difference, which manifests the breaking of the $\mathbb{Z}_2$ symmetry with respect to bond-centered inversion [40]. In the intermediate case, $U_{c=1}^c > U > U_{\text{Ising}}^c$, a Haldane insulating phase has been identified with alternating local charge $\Phi_+ = \sqrt{\pi}/2$ but fluctuating relative densities $\Theta_- = 0$ and topological string order parameter

$$\mathcal{O}_{\text{HI}}^2 = \langle \delta\hat{n}_i e^{i\pi \sum_{i \leq k < j} \delta\hat{n}_k} \delta\hat{n}_j \rangle \sim (\sin \sqrt{\pi}\Phi_+)^2. \quad (20)$$

For larger θ , anyons now open another possibility for the intermediate phase, as U_{Ising}^c increases and $U_{c=1}^c$ decreases with θ , such that $U_{c=1}^c < U < U_{\text{Ising}}^c$, which is not possible for bosons. In this case the order parameter is given by alternating energies on even and odd bonds as

$$\mathcal{O}_{\text{DIMER}} = \langle \hat{H}_{\text{even}}^{\text{bond}} - \hat{H}_{\text{odd}}^{\text{bond}} \rangle \sim \cos \sqrt{\pi}\Phi_+ \cos \sqrt{\pi}\Phi_-, \quad (21)$$

corresponding to uniform local charges $\Phi_+ = 0$ with relative quantum numbers entangled on every second bond $\Phi_- = 0$. For the Fermionic Hubbard model the dimer phase is also known as a bond ordered wave [31]. However, it is not trivial that such a phase is now also observed for $\theta = \pi$ corresponding to “pseudo-fermions” where the on-site commutator is bosonic with at most two particles per site. The Fermionic Hubbard model has a local Hilbert state of four states per site, while the model in Eq. (1) is restricted to three states, where the relative densities in the (−) channel play the role of fermionic spins.

Numerical results—The density matrix renormalization group (DMRG) algorithm [45,46] and the infinite DMRG (iDMRG) [47] with $M = 100$ –800 kept states were used to calculate the fidelity susceptibility, the entanglement

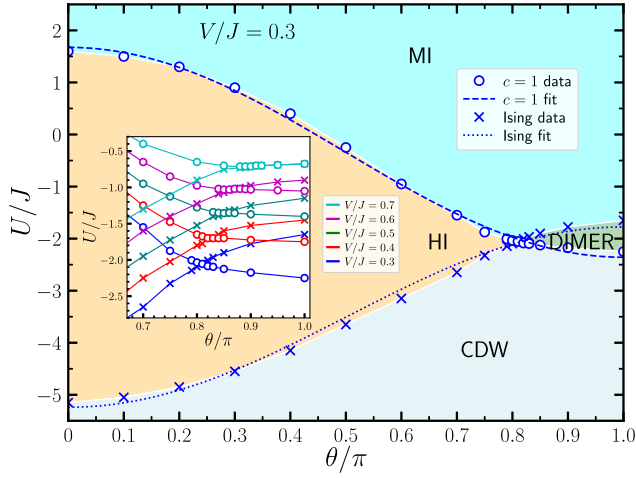


FIG. 2. Phase diagram as a function of θ and U/J for $V/J = 0.3$ obtained by fidelity susceptibility from iDMRG with bond dimension $M = 400$. Circles denote the $c = 1$ transition, and crosses show the Ising transition. The dashed and dotted lines are fits to Eqs. (16) and (17), respectively (see text). Inset: behavior near the points, where all four phases meet for different V/J .

entropy, order parameters, and the correlation length, as well as level spectroscopy with finite-size scaling. The fidelity susceptibility provides a robust sharp signal and has been used to obtain the phase boundaries between the four gapped phases in Figs. 2 and 3, which agree with maxima of the other quantities using an accuracy of parameter spacing $0.1J$. See End Matter Appendix A for a more detailed discussion of the numerical approach.

The bosonization results in Eqs. (16) and (17) predict a $\cos \theta$ dependence of the phase transition lines of the form $(U_{c=1}^c(\theta) - U_{c=1}^c(\pi/2))/J = a \cos \theta$ and $(U_{\text{Ising}}^c(\theta) - U_{\text{Ising}}^c(\pi/2))/J = b \cos \theta$, where the weak-coupling expressions evaluate to $a \sim 0.74$ and $b \sim -8.5$ with opposite signs. Due to finite coupling values those prefactors renormalize, but the $\cos$ behavior appears to be robust, as can be seen by the fits of the phase transition lines in Fig. 2, which are given by $U_{c=1}^c/J = -0.34 + 2.02 \cos \theta$ and $U_{\text{Ising}}^c/J = -3.51 - 1.73 \cos \theta$.

The inset of Fig. 2 shows the behavior near the points where all four phases meet for different V . For each fixed V , the phase transition lines cross, which would signal the appearance of a four-critical point between the MI, HI, CDW, and dimer phases. According to bosonization the multicritical point would have central charge $c = 3/2$ [48]. All data is consistent with four phases meeting exactly at one multicritical line in the U - V - θ parameter space.

More detailed phase diagrams in the U - V -parameter space for different θ are presented in Fig. 3. The width of the HI phase has considerably decreased for $\theta = \pi/2$, and for $\theta = 3\pi/4$ it is almost gone. For $\theta = \pi$ the HI phase is completely replaced by a dimer phase consistent with the prediction by bosonization. For large V the intermediate phase always merges into a first order transition between Mott and CDW phases. The dimer phase extends to negative U over a larger region than for the Fermionic Hubbard model [31].

An interesting aspect for small values of V is the appearance of a superfluid (SF) for bosons and a pair-superfluid (PSF) phase for both bosons and pseudo-fermions in the insets of Fig. 3. For $V = 0$ the behavior was analyzed numerically in Ref. [7], which can now be explained in terms of our bosonization approach at unit filling. In particular, superfluid behavior requires an irrelevant coupling g_1 in the (+) sector $K_+ > 2$, i.e., $U + 6V - 4\text{Re}(J_{zz})/\pi < 0$ from the weak coupling limit in Eq. (8). Accordingly, superfluid behavior is possible even for positive regions of U at $V = 0$ due to the fact that $J_{zz}(\theta = 0) > 0$, which would not be the case if an artificial spin-flip symmetry was assumed as in previous works. With increasing θ the critical value of U decreases since $\text{Re}(J_{zz}) \propto \sqrt{2} \cos \theta - 1$, which is exactly observed numerically in Ref. [7]. The SF-PSF transition is discussed in End Matter Appendix B, where it is shown that it corresponds to $K_- = 1$. As can be seen in the inset of Fig. 3 it is therefore a continuation of the Ising transition discussed above, where it was predicted that the $K_- = 1$ transition occurs at larger values of U with increasing θ . Hence, with larger θ eventually the decreasing $K_+ = 2$ and the increasing $K_- = 1$ lines cross, which marks a multicritical point where the

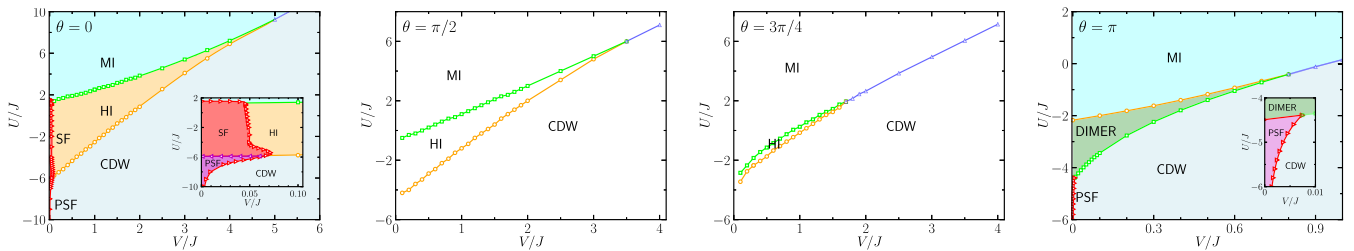


FIG. 3. Phase diagram as a function of U/J and V/J showing the fate of the intermediate phase at $\theta = 0$, $\theta = \pi/2$, $\theta = 3\pi/4$, and $\theta = \pi$ (from left to right), numerically determined by the maxima of the fidelity susceptibility using iDMRG. Green ($c = 1$) and yellow (Ising) denote continuous transition lines, and blue is first order. Insets: for $\theta = 0$ and $\theta = \pi$, close-up of the superfluid and pair-superfluid behavior determined by level spectroscopy (red lines, see End Matter Appendix B).

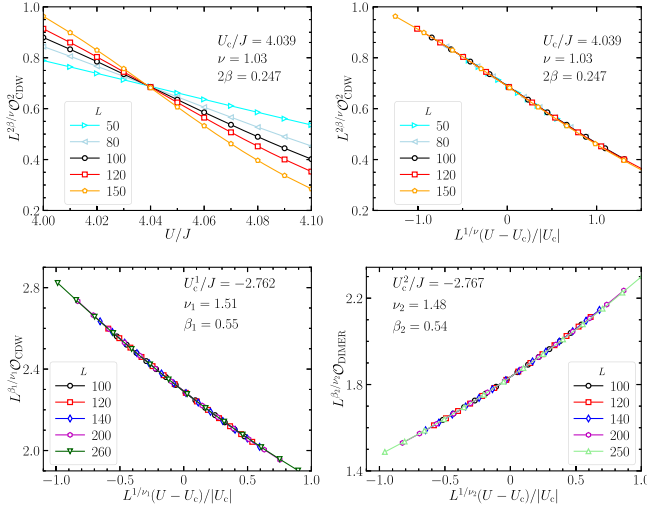


FIG. 4. Finite size L scaling analysis of the order parameters from DMRG. Upper: data collapse of $\mathcal{O}_{\text{CDW}}^2$ along the HI-CDW transition at $V/J = 3$, $\theta = 0$. Lower: data collapse of $\mathcal{O}_{\text{CDW}}$ (density difference) (lower left) and $\mathcal{O}_{\text{DIMER}}$ (lower right) along the CDW-dimer transition at $V/J = 0.2$, $\theta = \pi$.

SF phase is replaced by the dimer phase at $\theta \sim 0.8\pi$ for $V = 0$ as first shown in Ref. [7]. More details on the superfluid phases are discussed in End Matter Appendix B.

Classification of transitions—For the determination of universality classes the scaling of the order parameters is analyzed, which is numerically more involved (see Supplemental Material [40]). This is illustrated in Fig. 4 (upper) for the transition between the HI and CDW phases. The data collapse gives an accurate determination of critical exponents $\nu = 1.03$ and $2\beta = 0.247$. This is in agreement with the prediction of an Ising transition with $c = 1/2$ [8,35], which we confirmed by entanglement entropy scaling [49] in the insets of Fig. 5 in End Matter Appendix A.

More interesting is the transition from CDW to dimer phase. Both phases show a spontaneously broken translation symmetry but different broken inversion symmetries: bond centered and site centered, respectively. Recent literature has extensively discussed continuous transitions between two phases with different broken $\mathbb{Z}_2$ symmetries as interesting examples [48,50–59] beyond the Landau-Ginzburg paradigm [60] in connection with deconfined quantum criticality [57–59]. While in two-dimensional systems numerical evidence for a continuous transition is hard to obtain [61,62], the one-dimensional system now offers more controlled methods. For the anyon model we can confirm that the CDW-dimer transition also falls in this remarkable category. It is continuous at $V/J = 0.2$, $\theta = \pi$ for a $c = 1$ transition as shown in Fig. 4 (lower). From bosonization this can be understood because the sign change of g_1 causes the $c = 1$ transition and at the same time a change of the $\mathbb{Z}_2$ symmetry of the respective order parameter, interchanging the role of $\cos \sqrt{\pi}\Phi_+$ and

$\sin \sqrt{\pi}\Phi_+$ in Eqs. (19) and (21). See Supplemental Material [40] for a discussion of symmetries.

For the extended Bose Hubbard model, the HI to Mott transition is known to be $c = 1$ [36,37,63]. It is weaker for small V [8] and merges to a first order Mott-CDW transition for large V as shown in Fig. 3. We observe no significant change in the nature of the transition for finite θ . A coupling in the form of the operator $\sin \sqrt{4\pi}\Phi_+$ is therefore always absent, which would otherwise lead to a crossover behavior instead of a sharp $c = 1$ line [35].

Finally, all transitions between the (pair) superfluid phases and the gapped phases are of Berezinskii-Kosterlitz-Thouless (BKT) type (red lines in the insets of Fig. 3). As discussed in End Matter Appendix B, the corresponding transition lines were established using level spectroscopy [64–70] with exact diagonalization for finite-size systems extrapolated to the thermodynamic limit.

Conclusions—By derivation of the effective field theory and large scale numerical DMRG calculations we have obtained the phase diagram as a function of θ , U , and V of the extended anyonic Hubbard model including negative on-site interactions. The phase diagram consists of four gapped and two superfluid phases. By tuning the exchange angle θ from 0 to π the behavior changes continuously from bosonic to fermionic. In the process the Haldane insulator phase disappears at an interesting multicritical point where it is replaced by the dimer phase. The transition between dimer and CDW phases provides an example of a continuous $c = 1$ transition between two distinct broken $\mathbb{Z}_2$ symmetries beyond the Landau-Ginzburg paradigm.

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End Matter

Appendix A: Numerical data—For the numerical determination of the phase transition lines, we discuss the fidelity susceptibility, the entanglement entropy, and the correlation length here. Order parameters are discussed in Supplemental Material [40].

For determining the continuous transitions between gapped phases we use iDMRG [47] to calculate the fidelity F and its susceptibility χ_F [71,72]. The fidelity is defined as the absolute value of the overlap between a quantum state $|\Psi(\lambda)\rangle$ with a state with a slightly shifted parameter $|\Psi(\lambda + \delta\lambda)\rangle$, $F(\delta\lambda) = |\langle\Psi(\lambda)|\Psi(\lambda + \delta\lambda)\rangle|$, from which the fidelity susceptibility is derived:

$$\chi_F = -\frac{2 \ln(F)}{\delta\lambda^2}. \quad (\text{A1})$$

Here, λ is the parameter across which the phase transition is to be located, and $\delta\lambda$ is typically fixed around $0.05J$ to define the desired accuracy in parameter space.

Another useful quantity is the entanglement entropy,

$$S(\hat{\rho}_L) = -\text{tr} \hat{\rho}_L \ln(\hat{\rho}_L), \quad (\text{A2})$$

of the reduced density matrix $\hat{\rho}_L = \text{tr}_R \hat{\rho}$ for the left subsystem, where the right part has been traced out [73–76]. At a gap-to-gap continuous phase transition, the entanglement entropy diverges logarithmically in the correlation length ξ when approaching a critical point [49] $S(\hat{\rho}_L) \propto c \ln(\xi)$ with c being the central charge of the underlying conformal field theory. For finite size systems the correlation length at the continuous transition is limited by the system size L . At the critical point, $S(\hat{\rho}_L)$ then diverges as

$$S(\hat{\rho}_L) = \frac{c}{6} \ln(L) \quad (\text{A3})$$

for open boundary conditions [49]. With open boundary DMRG the entanglement entropy is alternating along the chain due to edge effects. The average of the larger S^{str} and

smaller S^{weak} values determines c in the thermodynamic limit shown in the insets of Fig. 5.

The correlation length ξ shows a pronounced maximum near phase transitions. Instead of calculating the density-density correlations, a more convenient way is to analyze the two largest eigenvalues Λ_0 and Λ_1 of the transfer matrix using iDMRG in the sector with the same particle number. The density-density correlation length is then given by $\xi = 2/\ln(\Lambda_0/\Lambda_1)$, and the phase transition occurs when ξ diverges, i.e., when $\Lambda_1 \rightarrow \Lambda_0$.

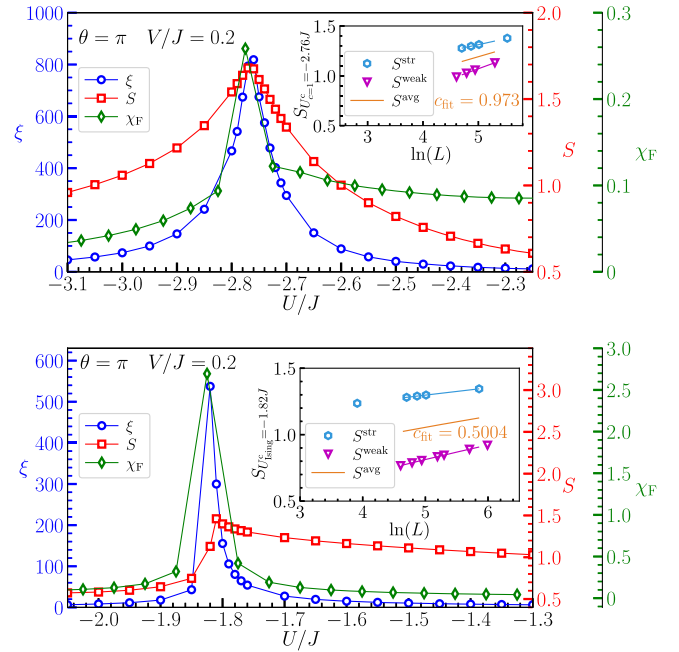


FIG. 5. Correlation length ξ , entanglement entropy S , and fidelity susceptibility χ_F as a function of U at $V = 0.2J$ and $\theta = \pi$ from iDMRG ($M = 800$), showing a $c = 1$ transition for CDW-dimer at $U_{c=1}^c = -2.76J$ (upper) and an Ising transition for dimer-Mott at $U_{\text{Ising}}^c = -1.82J$ (lower). Insets: entanglement entropy scaling from finite size DMRG at the transition points as a function of $\ln(L)$ according to Eq. (A3).

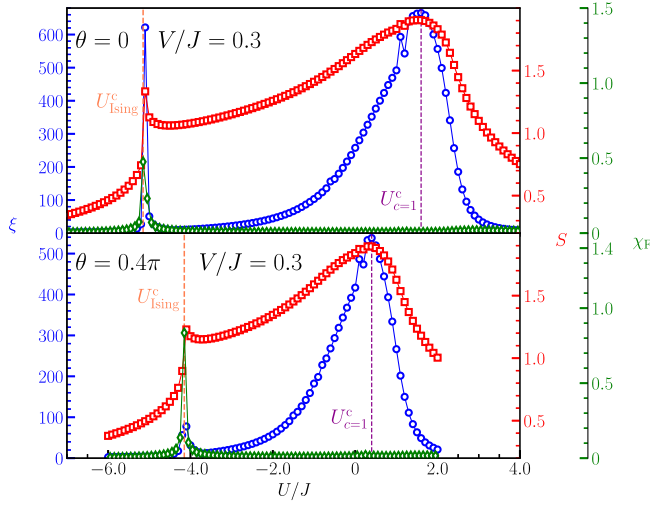


FIG. 6. iDMRG data for correlation length ξ , entanglement entropy S , and fidelity susceptibility χ_F for $V = 0.3J$ as a function of U . For $\theta = 0$ (top), the CDW to HI transition is at $U_{\text{Ising}}^c = -5.15J$, and the HI to Mott transition is at $U_{c=1}^c = 1.6J$. For $\theta = 0.4\pi$ (bottom), $U_{\text{Ising}}^c = -4.15J$, $U_{c=1}^c = 0.4J$.

In Fig. 5 we compare all three quantities at $\theta = \pi$ to illustrate the behavior for the CDW to dimer transition (upper) and the dimer to Mott transition (lower). All quantities have sharp maxima at the phase transition line, which agree within the parameter spacing of the simulations. In addition, the scaling in Eq. (A3) can be used to characterize the nature of the transition as shown in the insets, which give $c = 0.5004$ and $c = 0.973$ consistent with Ising and $c = 1$ transitions, respectively.

In Fig. 6 the analogous analysis is made for $\theta = 0$ (upper) and $\theta = 0.4\pi$ (lower), which show the CDW to HI and the HI to Mott transitions. In the latter case the maxima

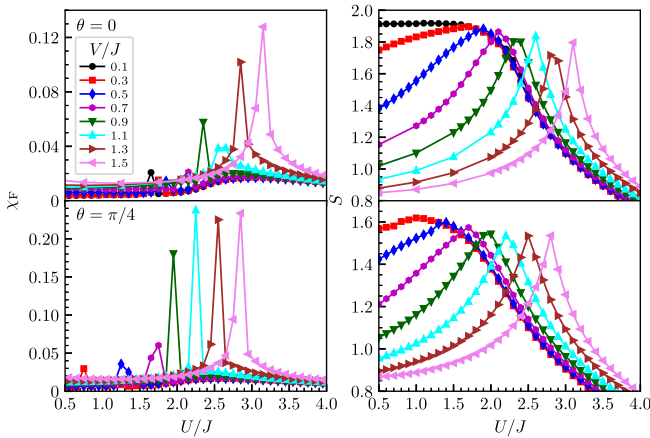


FIG. 7. Fidelity susceptibility χ_F (left) and entanglement entropy S (right) from iDMRG plotted as a function of U at the Mott to HI transition for $\theta = 0$ (upper) and $\theta = \pi/4$ (lower) and different V (legend applies to all subfigures).

of S and ξ are broader, while χ_F is overall smaller but with a relatively sharp feature (for an enlarged scale see Fig. 7). In fact, we find that χ_F remains always sharp even as other maxima broaden for smaller V , as illustrated in Fig. 7. Hence, we obtain all data points for the phase diagrams from the maxima of χ_F only, but the maxima are always consistent with the other quantities.

Finally, we examine the effect of the number of states kept M , which plays the role of a cutoff, so the quantitative behavior directly at phase transitions may crucially depend on M . Hence, for the scaling of Eq. (A3) shown in the insets of Fig. 5 a large value $M = 800$ was used. However, for the location of the maxima, the value of M is much less important as illustrated in Fig. 8 where the $M = 800$ data points are on top of points with lower M except for $M = 50$. We find that using $M = 100$ allows for a good sampling of transition points in parameter space and gives identical values as $M = 800$ as checked at selected points.

Appendix B: Superfluid phases—In the bosonic limit ($\theta = 0$), the phase diagram exhibits a SF-PSF phase transition that has attracted interest in dimension $d = 1$ [77–79] and higher [79–83] due to its unusual Ising quantum critical point characterized by a $U(1) \times Z_2$ conformal field theory with fractional central charge $c = 3/2$. For corresponding gapless phases in the spin-1 chain see Ref. [39]. In the 1D model there is no breaking of the continuous $U(1)$ symmetry, but quasi long-range order exists, which is characterized by algebraically decaying pair correlations $\langle (\hat{b}_i^\dagger)^2 (\hat{b}_{i+j})^2 \rangle$ in both phases. On the other hand the single particle correlation $\langle \hat{b}_i^\dagger \hat{b}_{i+j} \rangle$ decays exponentially in the PSF phase but remains long range in the SF phase, which we confirmed numerically [40]. Both superfluid phases require $K_+ > 2$, while the $(-)$ sector of the continuum Hamiltonian in Eq. (7) is always gapped. Consequently, all phase transitions between the superfluid and insulating phases are driven by the g_1 interaction and are of BKT type.

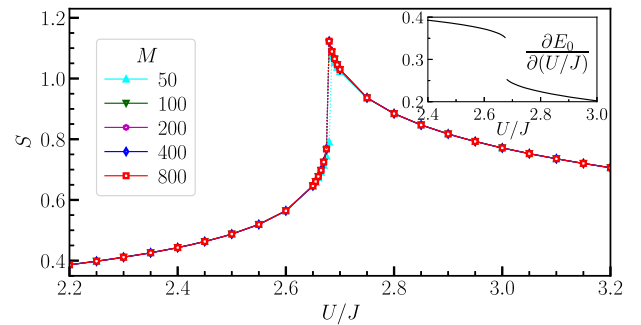


FIG. 8. Entanglement entropy S and derivative of ground-state energy E_0 (inset) over U for $V = 3J$, $\theta = \pi$. The jump at $U = 2.68J$ indicates a first order CDW-Mott transition.

The two distinct superfluid phases (SF and PSF) correspond to different locked values of the fields Φ_- and Θ_- depending on the value of K_- [39]. The correlation functions characterize the different gapless phases for $K_+ > 2$. For the slowly varying part of the single particle function we find [40]

$$\begin{aligned} \langle \hat{b}_0^\dagger \hat{b}_j \rangle_{\text{uni}} &\simeq \frac{1}{2} \left(\langle \hat{S}_{1,0}^- \hat{S}_{1,j}^+ \rangle_{\text{uni}} + \langle \hat{S}_{2,0}^- \hat{S}_{2,j}^+ \rangle_{\text{uni}} \right) \\ &\propto |x|^{-\frac{1}{2K_+}} \left(\langle e^{i\sqrt{\pi}\Theta_-(0)} e^{-i\sqrt{\pi}\Theta_-(x)} \rangle + \text{H.c.} \right), \quad (\text{B1}) \end{aligned}$$

where only the leading asymptotic behavior is retained. Generally, in the $(-)$ sector there is a competition between the g_2 and g_3 interactions which have scaling dimensions K_- and $1/K_-$, respectively. For $K_- > 1$, the Θ_- -field is locked at $\Theta_- = 0$. In this case, the expectation values involving the vertex operators of Θ_- contribute a constant to the prefactor of $\langle \hat{b}_0^\dagger \hat{b}_j \rangle_{\text{uni}}$, so the single-particle function decays as a power law, which is the hallmark of the SF phase. However, for $K_- < 1$, we have $\Phi_- = 0$, while Θ_- fluctuates freely, resulting in an exponential decay of the single particle function. Hence, the change of single particle decay and the SF-PSF transition is determined by $K_- = 1$. The pair-superfluid phase in Fig. 3 is pushed to smaller values of V by the CDW phase, which we attribute to the g_4 interaction in Eq. (7) in combination with the pinned field $\Phi_- = 0$. For larger V and $K_+ < 2$ the HI-CDW Ising transition is also characterized by $K_- = 1$ as discussed above and therefore is in line with the SF-PSF transition, resulting again in a multicritical point where two gapless and two gapped phases meet, consistent with the numerical data in the inset of Fig. 3 for $\theta = 0$. Note that the slowly varying part of the pair correlation does not

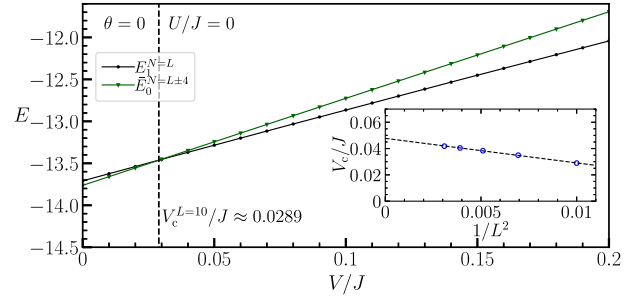


FIG. 9. Crossing of energy levels that represent the SF phase (green symbols) and the HI phase (black symbols). E_n^N denotes the energy of the n th excitation of the N -particle sector. Inset: extrapolation to $L \rightarrow \infty$.

depend on Θ_- :

$$\begin{aligned} \langle (\hat{b}_0^\dagger)^2 (\hat{b}_j)^2 \rangle_{\text{uni}} &\simeq \langle \hat{S}_{1,0}^- \hat{S}_{2,0}^- \hat{S}_{1,j}^+ \hat{S}_{2,j}^+ \rangle_{\text{uni}} \\ &\propto \langle e^{i\sqrt{4\pi}\Theta_+(0)} e^{-i\sqrt{4\pi}\Theta_+(x)} \rangle \propto |x|^{-\frac{2}{K_+}}. \quad (\text{B2}) \end{aligned}$$

It remains algebraically decaying in either gapless phase as confirmed numerically [40].

For numerically locating the gapless to gapped BKT phase transitions, we use the level spectroscopy approach [64–70] where finite-size transition points are identified using excited energy level degeneracies, which are then extrapolated to the thermodynamic limit using exact diagonalization on periodic finite size systems. Figure 9 shows the relevant energy level crossing as a function of V . In the inset the corresponding crossings for different finite L are extrapolated to an infinite system size, which determines one SF/HI transition point to an accuracy of $0.01J$. This procedure is repeated for a large number of different values of U to obtain the insets of Fig. 3.