

Contamination-Free Resolution of Cantor Spectra and Magic-Angle Localization in Moiré Quasicrystals and Aperiodic Media

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Quasicrystalline Hamiltonians host Cantor-like spectra, gaps, and critical states, yet conventional finite-patch diagonalization is often compromised by spectral pollution: spurious eigenvalues absent from the infinite system. We introduce a certified spectral-computation framework for aperiodic media, computing infinite-volume spectra of general Hermitian Hamiltonians with rigorous *a posteriori* error bounds and without imposing artificial boundaries or artificial periodicity. The method resolves fine spectral features and fractal dimensions universally across aperiodic systems. For twisted Penrose bilayers, angle-resolved certified spectra identify a magic angle at $\theta \approx 27.228^\circ$, where ring-worm interference produces twist-controlled confinement. Across one- and two-dimensional quasicrystals, the results show how fractal spectral geometry organizes localization: lower box-counting dimension correlates with larger inverse participation ratio in substitution chains, while dodecagonal tilings exhibit compactly supported and algebraically confined critical modes. The framework also scales to three dimensions, delivering contamination-free, high-resolution spectra for a quasicrystal arising in acoustic metamaterials. Further examples involving quantum walks, continuum superlattices, longer-range hopping, and inhomogeneous magnetic fields demonstrate the broad reach of certified spectral computation as a route to predictive spectral design in aperiodic media.

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Quasicrystals are aperiodic solids with long-range order and rotational symmetries forbidden in periodic crystals. Since their discovery in the 1980s [1], they have become relevant to renewable energy, superconducting, high-strength, low-friction, and photonic materials [2–6]. Their quasiperiodic order produces distinctive electronic spectra and transport: dodecagonal structures can host nearly flat bands, sharply structured densities of states, and localized resonances [7], while quasiperiodic spectra often contain gaps and localized states [8–13]. When topologically non-trivial, these gaps can support quantum-Hall-like responses without periodicity [14]. Quasicrystals also exhibit rich correlated and wave-transport phenomena, including superconductivity [15–17], Kondo screening [18], magnetic

transitions [12], and fractal spectra associated with protected transport, localization, and nonlinear optics [6,19–21].

These phenomena are commonly studied by diagonalizing large finite patches of a quasicrystal [14,22], but boundary truncation often introduces spectral pollution: eigenvalues absent from the infinite system (Fig. 1). Periodic supercells provide an alternative [23,24], but their construction is tiling-dependent and can still produce artificial pseudogaps in the density of states [25]. Such uncontrolled errors make it difficult to identify genuine gaps or estimate box-counting dimensions reliably. In Supplemental Material (SM) [26], we show that supercell spectra can be useful approximants when equipped with an *a posteriori* error certificate, which our method provides. The same analysis exposes their intrinsic limitations: supercells require compatible cut-and-project data and magnetic-flux commensurability, and are cumbersome or unavailable for inflation-generated tilings such as Stampfli and Socolar, twisted-bilayer models, and nonuniform magnetic fields.

We introduce a certified method for computing spectra of general quasicrystal Hamiltonians without spectral pollution. The convergence theory for the spectral-enclosure construction is developed in the mathematical companion paper [42]; here we develop these enclosures as a physics tool for quasicrystalline models. They yield intrinsic gaps,

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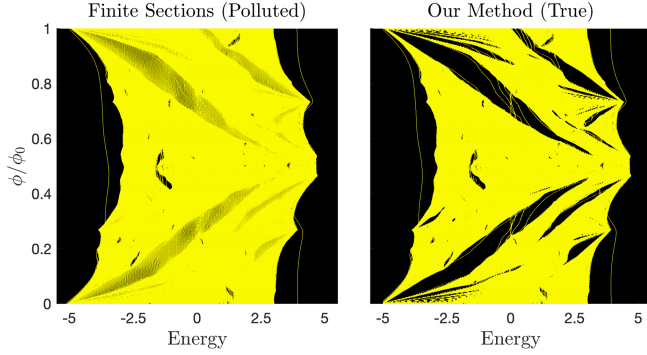


FIG. 1. Spectrum of a Stampfli-tiling model for twisted bilayer graphene in a uniform magnetic field, plotted versus flux ϕ/ϕ_0 . Left: finite-section approximation, showing spectral pollution. Right: certified spectral enclosure from our method, free of spurious states and resolving intrinsic gaps and fractal spectral structure.

box-counting dimensions, Lebesgue measures, and localization diagnostics for the infinite aperiodic operator itself, rather than for truncated patches or uncontrolled periodic approximants. The method gives convergent, pollution-free spectral computations beyond one dimension and for general quasicrystalline lattices, addressing a long-standing challenge in quasicrystals and aperiodic media [43]. It acts as a spectral microscope for aperiodic systems, resolving features that control localization, critical confinement, and twist-angle sensitivity when Bloch theory is unavailable.

We apply the method to several quasicrystalline platforms. For dodecagonal models motivated by twisted bilayer graphene [44] and PI-PEO quasicrystals [45], certified magnetic spectra reveal intrinsic Hofstadter-like gap hierarchies and spectral dimensions, free of finite-section artifacts. We show that these systems, realized in soft-matter and metamaterial settings [46–51], exhibit compactly supported modes and algebraically confined critical states. For twisted Penrose bilayers, the angle-resolved spectrum identifies a new magic angle, $\theta \approx 27.228^\circ$, where ring-worm interference confines central modes. The End Matter extends the framework to unitary fiber-optics models, continuum semiconductor-superlattice models, lattice-shaken Haldane-type systems, inhomogeneous magnetic fields, and an acoustic-network model of a 3D quasicrystal, for which we obtain a contamination-free high-resolution spectrum.

The models—The standard 1D model is a hopping Hamiltonian with aperiodic on-site potential $\{\omega_j\}_{j \in \mathbb{Z}}$:

$$H_{\alpha,\omega} = \sum_{j \in \mathbb{Z}} (|j\rangle\langle j-1| + |j\rangle\langle j+1| + \alpha\omega_j|j\rangle\langle j|), \quad (1)$$

where $\alpha \geq 0$ is the coupling strength. For Fibonacci [52–54], Thue–Morse [55–57], and period-doubling [43,58,59] sequences $\{\omega_j\}$, $H_{\alpha,\omega}$ has a fractal spectrum of zero

Lebesgue measure for $\alpha > 0$. Far less is known for two-dimensional models.

In 2D, we study a single-particle tight-binding Hamiltonian on a quasicrystalline graph in a perpendicular magnetic field. Hopping between vertices carries a Peierls phase $\theta_{j,k}$ [60,61], giving

$$H_{\mathbf{B}} = - \sum_{\langle j,k \rangle} (e^{i\theta_{j,k}}|j\rangle\langle k| + e^{-i\theta_{j,k}}|k\rangle\langle j|),$$

where $\theta_{k,j} = -\theta_{j,k}$. With on-site terms absent, the spectrum is governed solely by quasicrystalline geometry and magnetic flux, in contrast to Eq. (1) [62].

For moiré platforms, we model a twisted bilayer Penrose quasicrystal by exponentially coupled tight-binding layers [63,64]. Let $|j, \ell\rangle$ denote a localized orbital at vertex j in layer $\ell \in \{\text{bot}, \text{top}\}$. We take

$$H = -(H_{\text{bot}} + H_{\text{top}}) + T + T^*, \quad H_{\ell} = \sum_{\langle j,k \rangle_{\ell}} |j, \ell\rangle\langle k, \ell|,$$

$$T = \sum_{j,m} e^{-|r_{jm}(\theta)|/0.1} |j, \text{bot}\rangle\langle m, \text{top}|,$$

where $\langle j, k \rangle_{\ell}$ denotes nearest-neighbor pairs within layer ℓ , $r_{jm}(\theta)$ is the three-dimensional displacement between opposite-layer sites after relative twist θ , and T^* denotes the adjoint of T .

Methods—Many of these spectra are fractal, and our algorithm is naturally compared with the construction of the Cantor set. At scale n , it produces a finite union of intervals Γ_n such that the spectrum $\text{Sp}(H) \subseteq \Gamma_n$; as n increases, these certified enclosures converge to the true spectrum.

The core test, illustrated in Fig. 2, is based on singular values. In finite dimensions, a real number z is an eigenvalue of a Hermitian matrix H if and only if $H - zI$ is singular; equivalently $\sigma_{\inf}(H - zI) = \inf_{\|u\|=1} \|(H - zI)u\| = 0$, where $\sigma_{\inf}$ is the smallest singular value. For a self-adjoint operator, $\sigma_{\inf}(H - zI) = \text{dist}(z, \text{Sp}(H))$. Thus singular values provide a gap detector: a positive lower bound certifies that z lies outside the spectrum, while a small value places z close to it.

For an infinite quasicrystal, finite-patch diagonalization imposes artificial boundaries and can create spurious gap eigenvalues. We instead use *rectangular* residual tests. Fixing an enumeration of vertices, trial states are supported on the first n sites, while the residual $(H - zI)u$ is evaluated on the enlarged set of $f(n)$ sites containing all couplings from that support: $\Phi_n(z, H) = \sigma_{\inf}(P_{f(n)}(H - zI)P_n^*)$. Here P_n^* embeds the n -site trial space into the full Hilbert space, and $P_{f(n)}$ projects the residual onto the enlarged patch. For finite-range hopping, choosing $f(n) > n$ to include the interaction halo gives $\Phi_n(z, H) \downarrow \text{dist}(z, \text{Sp}(H))$. Thus, with the certified error bound supplied by the method, energies whose error remains bounded away from zero are excluded

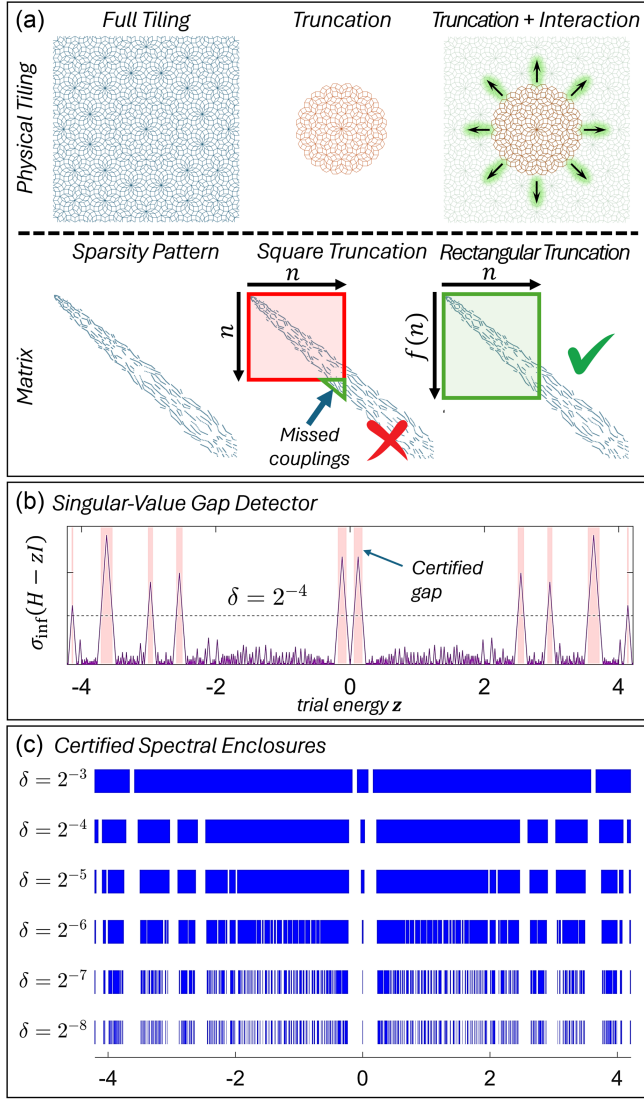


FIG. 2. Schematic of the certified spectral algorithm. (a) Square finite sections impose artificial boundaries and can generate spectral pollution; rectangular residual tests instead evaluate $(H - zI)u$ on all sites coupled to the trial support. (b) Singular-value gap test: for Hermitian finite matrices, $z \in \text{Sp}(H)$ if and only if $\sigma_{\text{inf}}(H - zI) = 0$, while for self-adjoint infinite operators $\sigma_{\text{inf}}(H - zI) = \text{dist}(z, \text{Sp}(H))$. A positive lower bound therefore certifies separation from the spectrum. (c) Scanning a δ mesh removes energies certified to lie outside the spectrum, leaving an enclosure with *a posteriori* error $E_{\delta,n}$. Refining n and δ yields pollution-free approximations of the aperiodic spectrum.

as gaps; the remaining energies form a spectral enclosure at the chosen resolution.

For a mesh $G_{\delta,n}$ of size δ spanning $[-n, n]$, we scan trial energies z and remove intervals certified to lie outside the spectrum, leaving an enclosure $\Gamma_{\delta,n}$. On the retained grid points $\Gamma_{\delta,n} \cap G_{\delta,n}$, we report the *a posteriori* error $E_{\delta,n} = \max_{z \in \Gamma_{\delta,n} \cap G_{\delta,n}} \Phi_n(z, H)$. Thus every retained energy lies within $E_{\delta,n}$ of the infinite-system spectrum. Increasing

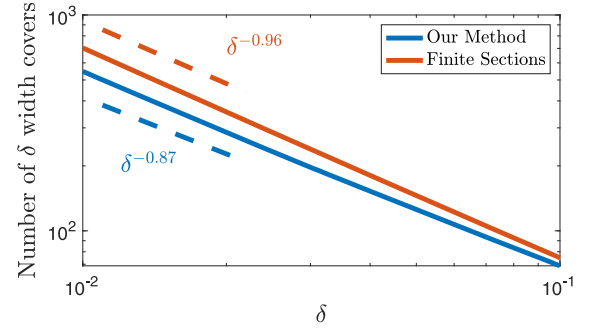


FIG. 3. Box-counting dimension of the Stampfli-tiling spectrum: finite sections versus certified enclosures. Spectral pollution prevents finite sections from reaching a stable scaling regime, whereas our method yields controlled estimates from pollution-free spectra.

n and decreasing δ yields pollution-free spectral approximations with controlled resolution. Details of the convergence hierarchy, and of the box-counting-dimension and Lebesgue-measure computations, are given in SM [26]. We use floating point arithmetic in this Letter and the algorithms can be implemented with interval arithmetic.

Figure 3 illustrates the need for certified spectra when estimating box-counting dimensions. Spectral pollution in finite-patch diagonalization distorts the scaling window and can produce misleading slopes in log-log plots. Our method instead provides spectral enclosures certified at resolution δ , excluding artificial components and stabilizing the scaling regime needed for reliable fractal-dimension estimates.

1D quasicrystals—Figure 4 shows the certified, pollution-free spectra of three one-dimensional quasicrystals versus coupling strength. In each case the spectrum forms a Cantor-like set with a hierarchy of gaps for $\alpha > 0$. For fixed coupling, Fig. 5 compares the box-counting dimension with the maximal inverse participation ratio,

$$\text{IPR}_{\text{max}} = \max_m \sum_i |\psi_i^{(m)}|^4 / (\sum_i |\psi_i^{(m)}|^2)^2,$$

where $\psi^{(m)}$ are eigenvectors of finite truncations of $H_{\alpha,\omega}$.

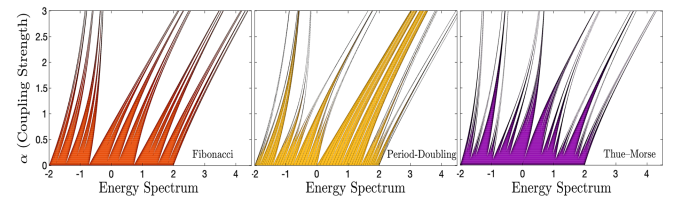


FIG. 4. Certified spectra of one-dimensional quasicrystal Hamiltonians versus coupling strength α for Fibonacci (left), period-doubling (middle), and Thue-Morse (right) potentials. Increasing α opens a hierarchy of gaps and produces increasingly fragmented, Cantor-like spectra. For similar illustrations, see Refs. [43,65].

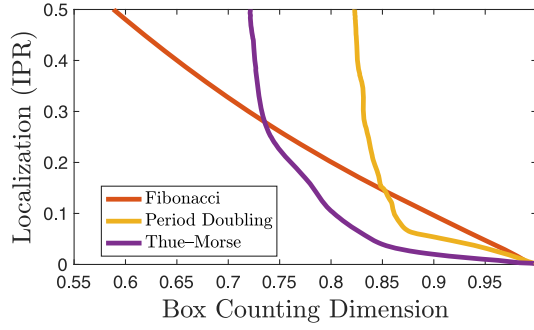


FIG. 5. Spectral geometry versus localization in one-dimensional quasicrystals. The maximal inverse participation ratio, IPR_{max} , computed from finite-truncation eigenvectors, is plotted against the box-counting dimension of the certified spectrum for Fibonacci, period-doubling, and Thue-Morse chains. Lower spectral dimension correlates with stronger real-space confinement.

This figure reveals an inverse correlation between spectral fractal dimension and wave function localization: more fragmented spectra have lower box-counting dimension and larger IPR, while spectra closer to interval-like sets have smaller IPR. Since IPR measures real-space confinement, this provides a transport-relevant spectral diagnostic, consistent with experimental links between fractal dimension and dynamical spreading [66].

2D quasicrystals—Figure 6 identifies critical states in the Socolar and Stampfli tilings, whose amplitudes decay algebraically with radius, $|\varphi_i| \sim R_i^{-\alpha}$, with $\alpha \approx 1.02$ and $\alpha \approx 2.33$, respectively. The construction and validation of compactly supported modes and algebraically decaying approximate eigenstates are described in SM [26]; in both cases the key diagnostic is the normalized residual: zero for

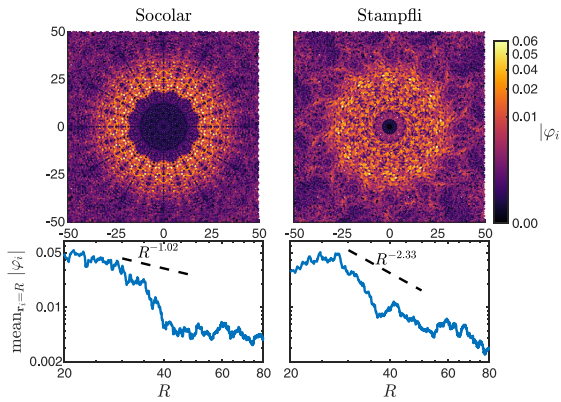


FIG. 6. Algebraically confined modes in the Socolar (left) and Stampfli (right) tilings. Top: spatial amplitude $|\varphi_i|$, showing long-range power-law tails. Bottom: angularly averaged radial decay, $\langle |\varphi_i| \rangle_{|r_i|=R} \sim R^{-\alpha}$, with exponents $\alpha \approx 1.02$ and 2.33 , respectively. Normalized residual tests under increasing rectangular truncations, described in SM [26] certify spectral proximity of the approximate eigenvalues.

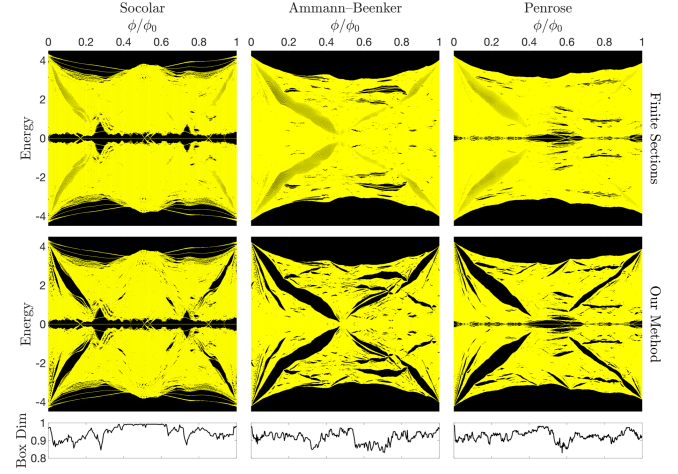


FIG. 7. Magnetic spectra of Socolar (left), Ammann-Beenker (middle), and rhombic Penrose (right) tilings versus flux ϕ/ϕ_0 . Top: finite-section approximations, showing boundary-induced spectral pollution. Middle: certified spectral enclosures from our method, resolving intrinsic gap structure. Bottom: box-counting dimension of the certified spectra versus flux, quantifying spectral fractality. A periodic-supercell comparison for a modified Ammann-Beenker magnetic Hamiltonian, together with singular-value certification of the supercell eigenvalues, is given in SM [26].

exact compact modes and controlled for approximate modes. Small residuals certify spectral proximity, but alone do not establish bulk localization or convergence to a fixed eigenfunction. These states are neither extended nor exponentially localized, but power-law confined, consistent with the underlying fractal spectral structure. Figure 7 shows Hofstadter-like spectra for the rhombic Penrose and Ammann-Beenker tilings in a uniform magnetic field, agreeing with previous computations [14,24] and validating the proposed method. Figures 1 and 7 compare the Stampfli

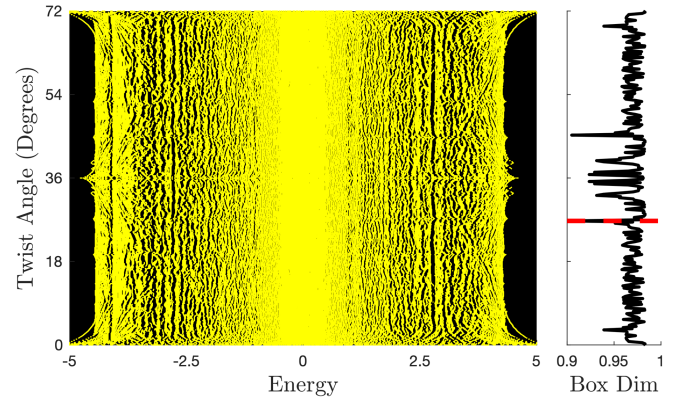


FIG. 8. Angle-resolved certified spectrum and box-counting dimension of the twisted rhombic Penrose bilayer. Left: energy spectrum versus twist angle $\theta \in [0, 72^\circ]$, corresponding to one fivefold angular period. Right: box-counting dimension versus θ , with a dip near the straight-worm angle $\theta = 36^\circ$ and a pronounced minimum at the ring-worm angle $\theta \approx 27.228^\circ$ (red line).

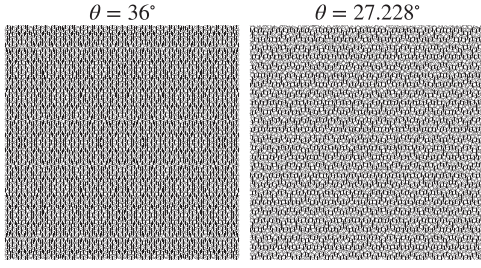


FIG. 9. Moiré geometry of the twisted Penrose bilayer at special angles. Left: at the straight-worm angle $\theta = 36^\circ$ identified in [64], interference lines form extended quasi-one-dimensional channels along fivefold directions. Right: at $\theta \approx 27.228^\circ$, the lines close into concentric ring-worms surrounding a central star-worm domain.

and Socolar spectra with finite-section calculations: finite patches generate spurious eigenvalues that obscure gaps, whereas our certified enclosures remain free of artificial states.

A comparison with periodic supercells, presented in SM [26] for a modified Ammann-Beenker approximant, shows that the periodic eigenvalues move closer to the infinite quasiperiodic spectrum as the supercell order increases. Our singular-value test supplies the missing *a posteriori* error scale, which is needed to distinguish resolved gaps and box-counting slopes from finite-period artifacts.

Moiré materials—Figure 8 shows the certified spectrum and its box-counting dimension for the twisted Penrose bilayer as functions of $\theta \in [0, 72^\circ]$. At $\theta = 0$, the aligned layers form strongly coupled vertical dimers, giving the largest bandwidth. Near $\theta = 36^\circ$, worm interference lines appear [64]; the associated strongly coupled Fibonacci chains transfer spectral weight to larger $|E|$, producing an outward flare of the spectral edges. Across the angle range, the spectrum remains fractal, with dimension close to 0.98 and fluctuations at special twists. The most pronounced dip occurs at $\theta \approx \arccos[\tau^4/(6\tau - 2)] \approx 27.228^\circ$,

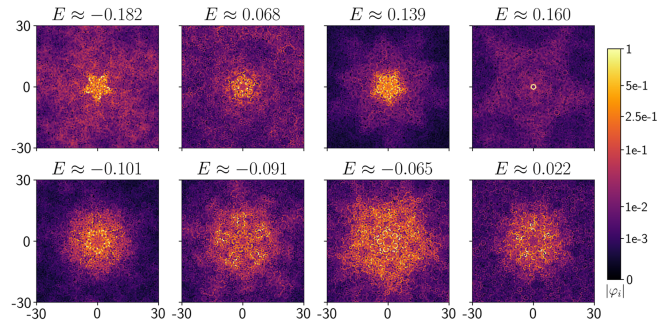


FIG. 10. Mode confinement in the twisted Penrose bilayer at $\theta \approx 27.228^\circ$. Top: modes resonant with the surrounding ring-worms, producing strong confinement in the central star-worm region. Bottom: modes that weakly leak through the ring-worm barrier while remaining concentrated near the center.

where $\tau = (1 + \sqrt{5})/2$; there, local interlayer hybridization is enhanced while the spectrum becomes sparser.

Figure 9 compares the lattice geometry at $\theta = 36^\circ$, where worm lines appear, and at $\theta \approx 27.228^\circ$, where ring-worm structures form. The corresponding modes in Fig. 10 show both strong confinement, with amplitudes concentrated in the central star-worm region, and weak leakage through the surrounding ring-worm barriers. These states indicate that ring-worms partition the quasicrystal into weakly connected domains. The dip in box-counting dimension at $\theta \approx 27.228^\circ$ therefore coincides with a twist-controlled confinement mechanism. This suggests a spectral criterion for magic angles in aperiodic moiré systems: special twists are identified by certified changes in fractal dimension and localization, rather than by Bloch-band flattening.

Discussion—The central result is a certified spectral microscope for analyzing aperiodic Hamiltonians. Instead of diagonalizing finite patches and relying on their eigenvalues to approximate the thermodynamic limit, the method computes spectral enclosures for the infinite operator with *a posteriori* error control. This tool enables reliable extraction of gaps, box-counting dimensions, Lebesgue measures, and localization diagnostics across tilings, magnetic fields, twist angles, and dimensions, without boundary-induced contamination.

Across these examples, certified spectra reveal quantitative links between fractal spectral geometry and confinement. In one dimension, lower box-counting dimension correlates with larger $\text{IPR}_{\max}$. In two-dimensional dodecagonal tilings, the spectra reveal compactly supported modes and algebraically confined critical states. In twisted Penrose bilayers, a sharp box-counting-dimension feature at $\theta \approx 27.228^\circ$ revealed ring-worm barriers and central confinement. This provides a model-agnostic route from twist angle to localization-relevant spectral structure in moiré systems, including aperiodic settings where Bloch bands are unavailable. Together, these results show that spectral geometry provides a quantitative design principle for localization and critical confinement in aperiodic media.

More generally, the lattice-independent formulation enables systematic studies across broad classes of quasicrystalline tilings and Hamiltonians, including on-site potentials and magnetic perturbations. Magnetic fields, which are technically cumbersome in aperiodic geometries despite their central role in lattice and fractal systems [67–72], can be incorporated directly while retaining rigorous spectral certification. The End Matter (Fig. 11) illustrates this flexibility through single-photon quantum walks, semiconductor superlattices, an acoustic-network 3D quasicrystalline analog of a third-order topological insulator, lattice-shaking-induced next-nearest-neighbor hopping, and spatially varying magnetic fields.

In sum, certified spectral computation turns quasicrystal numerics from qualitative finite-approximant pictures into quantitative physical statements about the infinite aperiodic

operator. By eliminating spectral pollution and providing error control for gaps, spectral dimensions, and Lebesgue measures, the framework establishes certified spectral geometry as a design principle for electronic, photonic, acoustic, and cold-atom quasicrystals with twist, magnetic flux, longer-range hopping, or spatially varying fields.

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Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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Figure 11 stress tests the proposed spectral-enclosure framework. These examples show that unitary models, continuum operators, longer-range hopping, nonuniform magnetic fields, and three-dimensional systems, rather than being separate case studies, can be treated within a single certified spectral approach.

The first row of Fig. 11 shows the unitary Fibonacci operator, motivated by quantum walks, metal-insulator transitions, and topological transport in aperiodic media [73–77]. The application illustration is inspired by [75]. For each coin angle ϑ , spectral points $e^{i\varphi} \in \text{Sp}(U_\vartheta)$ are plotted at radius $\rho = \vartheta$, producing a spectral fan. The spectrum forms Cantor subsets of the unit circle, with spectral bundles separated by angular gaps. As ϑ increases, these gaps split and widen, indicating stronger fragmentation under Fibonacci modulation of the coin. The box-counting dimension is close to 1 for weak modulation; after a small initial rise, it decreases monotonically, showing a crossover from broad spectral arcs to a thinner Cantor set.

The second row shows the continuum Kronig-Penney quasicrystal [78,79]. At $\lambda = 0$ the model reduces to a free particle on the half line; as λ increases, $\text{Sp}(H_{\omega,\lambda}) \cap [0, 100]$ fragments into narrow components separated by gaps. A gap closing near $\lambda \approx 40.5$ (dashed line) merges two large components and reorganizes the spectrum. The local box-counting dimension of $\text{Sp}(H_{\omega,\lambda}) \cap [0, E_{\max}]$, computed for $E_{\max} = 60, 90, 100$, approaches 1 as higher energies are

included and shows a jump at the same λ , indicating that macroscopic gap rearrangements are encoded in the spectrum's fractal geometry.

The third row shows a nearest-neighbor tight-binding Hamiltonian on the isometric 3D Rauzy tiling [80],

$$H = \sum_{(j,k)} t_{jk}(|j\rangle\langle k| + |k\rangle\langle j|), \quad t_{jk} = \begin{cases} t_2, & \mathbf{r}_j - \mathbf{r}_k \parallel \mathbf{v}; \\ t_1, & \text{otherwise} \end{cases}$$

for a fixed direction $\mathbf{v}$, with $t_1 = 1$. Bonds parallel to $\mathbf{v}$ are strong when $t_2 > t_1$, while all others are weak. This gives a quasicrystalline analog of the anisotropic diamond-lattice model used to realize third-order topological insulators in acoustic metamaterials, where strong bonds form dimers along a high-symmetry direction. This dimerization competes with quasiperiodic order: the strong-bond network is long-range ordered but nonperiodic. Varying t_2/t_1 produces a fractal, Hofstadter-like spectrum in which a wedge opens and fills with dense bands that split and recombine into a hierarchy of gaps. Thus, rather than forming a single bulk gap, the Rauzy spectrum remains highly fragmented across the range shown. Across the full spectrum, the box-counting dimension remains high, $\approx 0.92\text{--}0.96$, indicating a thick Cantor spectrum. The Rauzy analog of the acoustic higher-order-insulator model therefore retains dense bulk resonances and a fractal gap hierarchy even deep in the

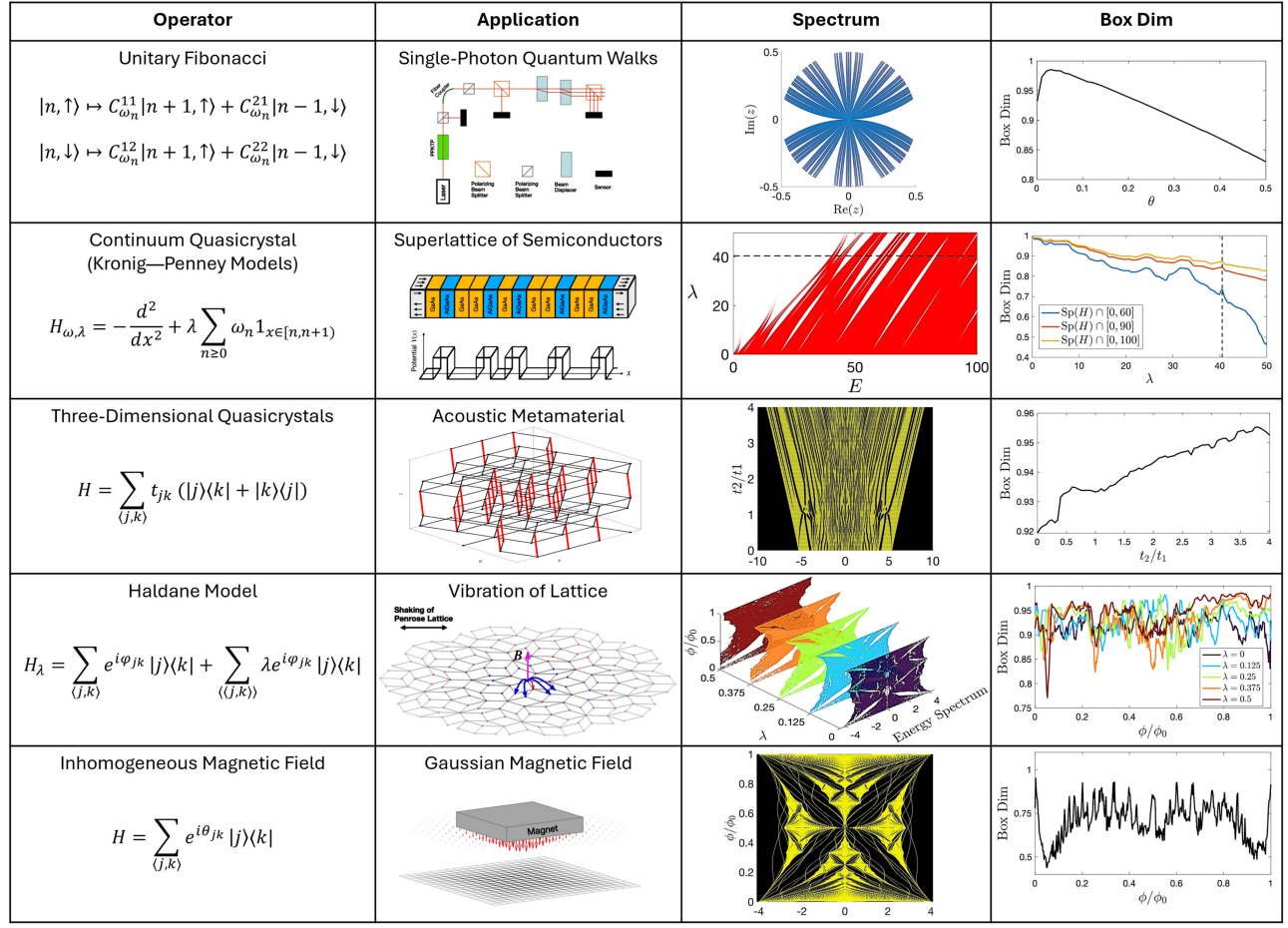


FIG. 11. Stress tests of the certified spectral framework beyond nearest-neighbor quasicrystal Hamiltonians: unitary quantum walks, continuum superlattices, a three-dimensional Rauzy acoustic network, Haldane-type longer-range hopping, and nonuniform magnetic fields. In each case the method resolves spectral structure and box-counting dimension without finite-section pollution.

dimerized regime. Putative third-order corner modes in finite samples must be interpreted against this critical non-Bloch bulk background, rather than within a clean bulk gap, making the model a natural test bed for higher-order topology in quasicrystals.

The fourth row of Fig. 11 shows a Haldane-type extension with next-nearest-neighbor (NNN) hopping [81]. Increasing λ mimics lattice shaking [82], enhancing indirect hopping in driven-lattice and moiré settings. The third column shows the Penrose spectrum in a perpendicular magnetic field as λ increases: additional hopping channels reshape quasiperiodic interference, closing some gaps, opening others, and broadening the spectrum. The combined tuning of magnetic flux and NNN hopping therefore drives spectral transitions and provides a route to engineering bandwidth, gap hierarchies, and localization-relevant spectral structure in quasicrystals.

The final row treats spinless particles on a square lattice in a perpendicular nonuniform field $B_z(r) = C_1 + (4C_2/\pi)e^{-(x^2+y^2)}$. In a Landau-gauge choice, only vertical hops $(x_1, y_1) \rightarrow (x_1, y_2)$ acquire the Peierls phase

$$\theta_{1,2} = C_1 x_1 (y_2 - y_1) + C_2 [\text{erf}(x_1) + 1] \times [\text{erf}(y_2) - \text{erf}(y_1)],$$

where erf is the error function. The model describes a Gaussian magnetic texture superposed on a uniform field, as in micromagnet-patterned two-dimensional materials, and is related to nonuniform-field settings in electron gases, graphene, and cold-atom optical lattices [83–90]. We fix $C_2 = 25$ and vary C_1 . The bulk spectrum remains close to the uniform-field Hofstadter spectrum, while the Gaussian perturbation creates levels that split off into spectral gaps.