

Master Stability for Traveling Waves on Networks

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We present a framework for determining effectively the spectrum and stability of traveling waves on networks with symmetries, such as rings and lattices, by computing master stability curves (MSCs). Unlike traditional methods, MSCs are independent of system size and can be readily used to assess wave destabilization and multistability in small and large networks.

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Since the pioneering works of Fermi, Pasta, Ulam, and Tsingou [1,2], discrete traveling waves (DTWs) are studied in areas such as neuroscience [3–5], nonlinear optics [6–8], chemical reactions [9], crystal dislocations [10–12], Ising-like phase transitions [13,14], conservation laws [15], coupled oscillators [16,17], traffic jams [18], etc. DTWs have received significant attention both analytically [19,20] and numerically [21]. DTWs have also been observed in discretizations of continuous spatially extended systems, such as discrete-nonlinear-Schrödinger or Frenkel-Kantorova-type models [22–24]. Waves with time-varying profile (e.g., discrete breathers) have also been considered [25,26].

Considering the importance of DTWs, there is a fundamental need for a unifying theory describing their spectral properties. Characterizing the stability of DTWs has remained challenging often tackled with specific approximations [8]. Rigorous results can be obtained for special cases; for instance, for zero wave speed [27] and in weakly coupled rings [28]. Similarly, the stability of DTWs has been studied previously on feed-forward ring networks [29,30], and feed-forward discrete tori [31].

In this Letter, we outline a master stability theory for DTWs on networks and coupled systems of ordinary differential equations that are invariant under cyclic index shift ($\mathbb{Z}_N$ -equivariant) [32]. Notable examples include systems of N identical nodes coupled in a (possibly nonlocal) ring, spatially discretized PDEs with periodic boundary conditions, and systems with higher-order coupling topologies [33] invariant under cyclic index shift,

as well as—by little extension—multichromatic networks, discrete tori, and systems with higher spatial embedding order (see Supplemental Material [34] for details).

The proposed framework is a generalization of the master stability function approach for synchronization in networks (when all nodes behave dynamically the same) developed by Pecora and Carroll [35], and shares similarities with classic methods for assessing the stability of traveling waves in continuous systems [36]. Specifically, we recast the higher dimensional Floquet spectral problem as a delay differential equation, possibly of mixed type, the master stability equation. Using the derived equation, we introduce a numerical scheme based on the continuation of a suitable two-point boundary value problem (2PBVP) to assess the stability of such waves independent of network size by computing master stability curves (MSCs).

Setup—To set the stage, we consider the N -dimensional network with d -dimensional internal dynamics given by

$$x'_n = F(x_n) + \sum_{1 \leq |m| \leq r} H_m(x_n, x_{n+m}), \quad x_n(t) \in \mathbb{R}^d. \quad (1)$$

Here, x_n , $n = 1, 2, \dots, N$, represents a node in the network, and x'_n denotes its first derivative with respect to time t . The function F defines the uncoupled dynamics at each node x_n . Each node x_n is coupled to nodes x_{n+m} (using the convention $x_j = x_{j+N}$ for $j \in \mathbb{Z}$) through the functions H_m , $1 \leq |m| \leq r$, where r is the coupling radius of the network. A comprehensive treatment of systems featuring higher-order interactions (see, e.g., [33]) can be found in Supplemental Material [34]. Note that system (1) is equivariant under cyclic permutation $x_n \mapsto x_{n+1} \pmod{N}$ and hence supports DTWs [32].

Our results pertain to the stability of DTWs. More precisely, let $\gamma = (x_1, \dots, x_N)$ be a DTW of system (1), where each component of γ is of the form

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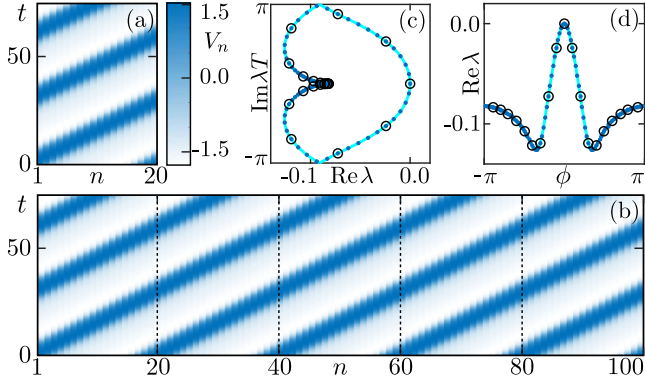


FIG. 1. Master stability curve (MSC) of DTWs in system (7) of N dissipatively coupled FitzHugh-Nagumo oscillators $x_n = (V_n, W_n)$, $V_n(t)$ and $W_n(t) \in \mathbb{R}$. (a),(b) Space-time plots of DTWs with wave number $k = 1/20$, (a) $N = 20$ and (b) $N = 100$. (c) Floquet exponents with $\text{Re}(\lambda) \geq -0.3$ of DTW (black circles) in (a), of DTW (blue dots) in (b), and MSC (cyan). (d) FESs and dispersion relation, i.e., MSC in (c) shown in $(\phi, \text{Re}(\lambda))$ -projection.

$$x_n(t) = y(t - n\tau), \quad y(t) \in \mathbb{R}^d, \quad (2)$$

where y is the T -periodic profile of the wave propagating with constant wave speed $c = 1/\tau$ (one unit of space per unit of time). In finite networks, the consistency relation $k = \tau/T$, $k \in \mathbb{Q}$, must hold, where k is the wave number of the DTW ($1/k$ being the wavelength).

Figures 1(a) and 1(b) show two examples of such DTWs with wave number $k = 1/20$ and constant wave speed $c \approx 0.6764$ in system (7) with (a) $N = 20$ and (b) $N = 100$ nodes. In both cases, the wave profile is the same since, for $k = M_0/N_0$ and $\text{gcd}(M_0, N_0) = 1$, the DTWs corresponding to the profile y can be supported in a network of size $N = KN_0$, where $K \in \mathbb{N}$, by concatenating K copies of the profile. This does not change the wave number or period. As such, Fig. 1(a) shows the minimal network size realization $N_0 = 20$ of profile y .

Stability of the DTW γ can be assessed by computing its Floquet exponent spectrum (FES). Specifically, a Floquet exponent (FE) $\lambda \in \mathbb{C}$ and its nonzero bundle $z = (z_1, \dots, z_N)$ can be computed as solutions of the Nd -dimensional Floquet 2PBVP

$$\begin{aligned} z'_n(t) &= A(t - n\tau)z_n(t) + \sum_{1 \leq |m| \leq r} B_m(t - n\tau)z_{n+m}(t), \\ z_n(t) &= e^{-\lambda T} z_n(t + T), \end{aligned} \quad (3)$$

where $z_n(t) \in \mathbb{C}^d$, with T -periodic coefficient functions

$$\begin{aligned} A(t) &= DF(y(t)) + \sum_{1 \leq |m| \leq r} D_1 H_m[y(t), y(t - m\tau)], \\ B_m(t) &= D_2 H_m[y(t), y(t - m\tau)]. \end{aligned}$$

Here, $D_1 H_m$ and $D_2 H_m$ denote the Jacobians of H_m with respect to its first and second arguments, respectively. The FES has size Nd (counting multiplicity) and consists of all FEs with imaginary parts inside the interval $[-\pi i/T, \pi i/T]$. Particularly, if all $\text{Re}(\lambda) < 0$ except for the trivial exponent $\lambda = 0$ (Goldstone mode), then the DTW γ is (orbitally) stable. There are multiple ways to numerically obtain the FES, ranging from explicitly solving the Floquet 2PBVP to finding the eigenvalues of the monodromy matrix [37]. In any case, the computations become infeasible or inaccurate for large system size N , or period T .

Master stability equation—Our approach overcomes these issues by leveraging Theorem 1 (Supplemental Material [34]), which states that every bundle z of DTW γ satisfying (3) is one-to-one with a nonzero periodic solution of the problem

$$\begin{aligned} \zeta'(t) &= (A(t) - \lambda I)\zeta(t) + \sum_{1 \leq |m| \leq r} e^{im\phi} B_m(t)\zeta(t - m\tau), \\ \zeta(t) &= \zeta(t + T), \end{aligned} \quad (4)$$

the *master stability equation* (MSE), where $\zeta(t) \in \mathbb{R}^d$ and $\phi = 2\pi l/N$, for some $l = 1, \dots, N$. Equation (4) serves as an extension of the master stability function by Pecora and Carroll to traveling waves on networks. Moreover, for a simple FE λ , its associated bundle z is

$$z_n(t) = \zeta(t - n\tau)e^{\lambda t + in\phi} \quad (5)$$

for a periodic profile ζ satisfying the MSE (4) for the corresponding value of λ and ϕ (see Theorem 1 in Supplemental Material [34]). Equation (5) presents a projection on the network modes and a discrete analog of the plane wave ansatz in continuous systems [36].

The FE problem (3) is effectively solved by considering the MSE (4) and treating ϕ as a continuous parameter. This is exemplified in Fig. 1(c), where we show the N largest FEs (in real part) of the waves shown in (a) for $N = 20$ (black circles) and (b) for $N = 100$ (blue dots). Notice that both waves are stable and that FEs appear to delineate a curve suggesting an accumulation of FEs of DTWs as $N \rightarrow \infty$, a phenomenon previously observed in feed-forward rings of Duffing oscillators [17]. Indeed, by continuously varying ϕ , the solutions $(\phi, \lambda(\phi))$ of (4) trace a set of curves, the *master stability curves* (MSCs), which contain the full FES for all network sizes $N = KN_0$, $K = 1, 2, \dots$; see the MSC (cyan) in Fig. 1(c). This is an advantage of the MSC framework: instead of computing the FESs of the DTWs with wave profile y and wave number k for networks with size KN_0 , the MSCs contain all FESs for all sizes. Indeed, whenever $e^{i\phi}$ is a KN_0 th root of unity, $K = 1, 2, \dots$, then the associated $\lambda(\phi)$ is a FE of the DTW in the network of size KN_0 , and a solution of (3). Moreover, the spectrum in the limit $K \rightarrow \infty$ (approximating the lattice case $K = \infty$) densely fills the MSCs.

Figure 1(d) shows the $(\phi, \text{Re}(\lambda(\phi)))$ -projection of the MSC in panel (c), where the equidistant spacing between FEs is easily observed. In analogy to continuous systems, we refer to this projection as (a branch of) the dispersion relation, relating a perturbation with wave number $\phi/2\pi$ to the corresponding growth rate.

Computing MCSs by means of numerical continuation of a 2PBVP—We use numerical methods for the continuation of periodic solutions to delay differential equations [38]. Given the wave profile y of a DTW γ , its period T , and wavenumber k , we require a solution triple (λ, ζ, ϕ) of the MSE (4) to initialize the 2PBVP. One such triple is $(\lambda, \zeta, \phi) = (0, y', 0)$ where y' can be computed using numerical differentiation. Fixing the norm and angle of $\zeta(0)$, pseudoarclength continuation in (ϕ, λ) [39] defines a one-parameter family of solutions for (4), of which the projection onto the $(\text{Re}(\lambda), \text{Im}(\lambda))$ -plane is an MSC. For more details on the numerical implementation see Supplemental Material [34].

We remark that the FES of a DTW might be comprised of several, disconnected MSCs. In general, we find the number of MSCs to be of the order of d . Additional MSCs can be initialized by computing triples (λ, ζ, ϕ) for a given perturbation wave number ϕ (see Proposition 1, Supplemental Material [34]).

Applications of the master stability framework—In the following, we illustrate how the master stability framework allows studying the stability of wave profiles independently of a specific embedding into a given network of size N . It is then immediately interesting to see how the MSC changes for different wave profiles. To that aim, we introduce the profile equation corresponding to Eq. (1) in a coordinate frame comoving with wave profile y , i.e.,

$$y'(t) = F(y(t)) + \sum_{1 \leq |m| \leq r} H_m[y(t), y(t - m\tau)], \quad (6)$$

which can be readily obtained by imposing (2) on system (1). Notice that τ can be considered a continuous parameter in Eq. (6). This allows applying continuation techniques to Eq. (6) to study how profile and period change as a one-dimensional solution family when τ (or a system parameter) is varied. The consistency relation implies a variation of k according to $k = \tau/T$. When $k = p/q \in \mathbb{Q}$, the corresponding profile can be embedded in a suitable network of size $N = Kq$, for any $K \in \mathbb{N}$, by concatenating Kp wave profiles and sampling with time window $p\tau$. For simplicity, we present examples for $p = 1$. Below, we show how simultaneous numerical continuation of (4) and (6) in τ (and therefore the wave number) can be used to explore (A) wave destabilization and (B) the presence of stable bound multipulses.

Wave destabilization—Starting with the DTWs shown in Fig. 1, we increase k from $k_1 = 1/20$ to $k_3 = 1/15$, see Fig. 2(a), and notice a qualitative change of the MSC of the underlying wave profile as shown in Fig. 2(b).

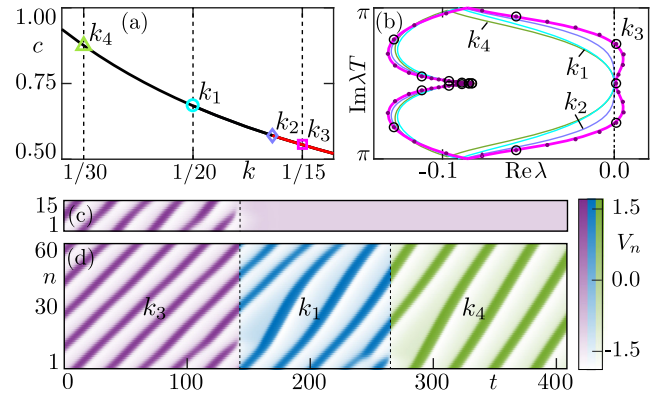


FIG. 2. Wave destabilization in system (7) of N dissipatively coupled FitzHugh-Nagumo oscillators $x_n = (V_n, W_n)$. (a) Numerical continuation of wave profile shown in Fig. 1 parameterized by wave number k . The curve's color indicates the curvature of the MSC at $\lambda = 0$; red (black) represents positive (negative) curvature. (b) MSCs of waves at values k_1 (cyan circle), k_2 (mauve diamond), k_3 (magenta square), and k_4 (green triangle) shown in (a). Also shown are the FEs of the wave profile with wave number $k = k_3$ for embedding dimension $N = 15$ (black circles) and $N = 60$ (purple dots). (c), (d) Long-term dynamics of system (7) starting at unstable wave profile at k_3 for $N = 15$ (c) and $N = 60$ (d).

Approximately at $k_2 \approx 0.0617$, the MSC displays a curvature change (from negative to positive) around $\lambda = 0$, causing a small curve segment to extend into the positive half plane. The wave profiles for which the MSC has positive curvature at $\lambda = 0$ (to the right of k_2) are colored red in (a). The change in curvature is well pronounced for $k = k_3$, see Fig. 2(b). Also plotted is the FES of the corresponding DTW in a ring of $N = 15$ (black circles) and $N = 60$ (purple dots), which are both unstable. In the network with fewer nodes, this causes the wave to die out after some transient which is largely determined by the distance of the initial condition used for simulation from the exact unstable wave profile, see Fig. 2(c). In the larger network, we observe another phenomenon: multistability of DTWs. Indeed, (d) shows the two-step transition of a DTW starting near an unstable profile with $k = k_3$, to $k = k_1$, and settling on $k = k_4$. Both wave profiles with wave numbers k_1 and k_4 are stable, but the simulation in Fig. 2(d) missed the basin of stability of the DTW at $k = k_1$.

Wave localization and bound two-pulses—Continuing the wave profile shown in Fig. 1 to smaller wave numbers k , we observe an oscillatory limiting behavior toward the asymptotic wave speed $c \approx 1.0049$ as k approaches 0; see Fig. 3(a). Simultaneously, the minimal embedding dimension of the profile grows beyond bound, the wave profile localizes in a small number of nodes (a pulse) and its period tends to infinity. This oscillatory behavior can be attributed to the presence of a homoclinic bifurcation, presumably of Shilnikov type, in the profile equation at $k = 0$, possibly leading to multipulses near $k = 0$ [40]. Another mechanism

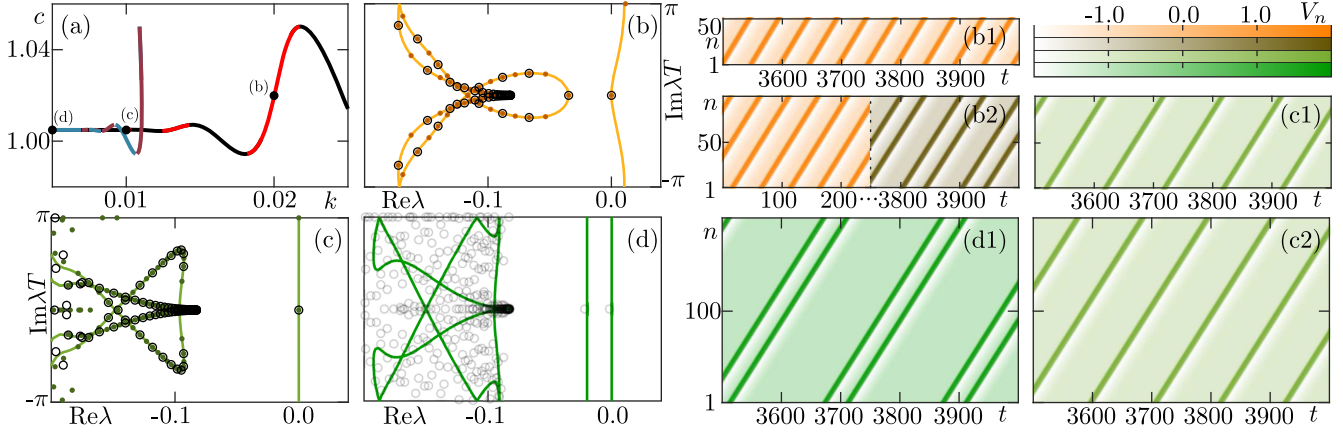


FIG. 3. Multistability and bound multipulses in system (7) of N dissipatively coupled FitzHugh-Nagumo oscillators $x_n = (V_n, W_n)$. (a) Numerical continuation of the wave profile shown in Fig. 1 (main profile), and of the bound two-pulse wave profile (secondary profile). Colors represent the curvature of the MSCs attached to $\lambda = 0$; namely, red (black) represents positive (negative) curvature for the main profile, while dark red (blue) represents positive (negative) curvature for the secondary profile. Positive curvature means that the curve extends to the right half plane close to $\lambda = 0$ indicating instability of the corresponding DTW for sufficiently large embedding dimension N . (b) FES when $N = 50$ (black circles) and $N = 100$ (brown dots), together with MSCs (orange curves) of the main profile at $k = 1/50$. (c) FES when $N = 100$ (black circles) and $N = 200$ (green dots), together with MSCs (light green curves) of the main profile at $k = 1/100$. (d) FES [computed with same numerical tolerances as in (b) and (c)] when $N = 200$ (gray circles) together with MSCs (dark green curve) of the secondary profile at $k = 1/200$. (b1), (c1) Long-term dynamics of the main profile at points indicated in (a) for different embedding dimensions N . (d1) Long-term dynamics of the secondary profile as indicated in (a) in a network with $N = 200$ nodes.

for the bifurcation of bound multipulses from a single pulse in delay differential equations has been studied in [41]. In what follows, we briefly illustrate how the presence of this object manifests in system (7) for $k = 0.02$, 0.01 and $k = 0.05$, and minimal embedding dimension $N = 50$, 100 and 200 by showing the existence of stable bound two-pulses for sufficiently small k . The results are contained in Fig. 3.

Figure 3(b) shows part of the FES (with largest real parts) of wave profiles at $k = 1/50$ alongside their corresponding MSCs. Notably, the MSC has split into two curves as compared to Fig. 1(c) (counting both segments on the left as one curve continuously parameterized by ϕ). As part of this transition (details not shown), the curvature of the rightmost MSC has changed from negative to positive locally at $\lambda = 0$, indicating that the corresponding DTWs are unstable for sufficiently large embedding dimension N . Indeed, the right-most curve satisfies the following property: $\lambda = 0$ whenever $e^{i\phi}$ is a 50th-root of unity. This implies that the DTW is stable in the network with $N = 50$ nodes. However, embedding the same wave profile in a network with $N = 100$, we observe that the corresponding DTW is unstable with a FE $\lambda = \lambda_0$ of the form $\text{Re}(\lambda_0) > 0$, $\text{Im}(\lambda_0 T) = i\pi$ whenever ϕ is an odd 100th root of unity. This is evidenced by the FES (black circles and orange dots) in Fig. 3(b) and direct simulations starting near the wave profile Figs. 3(b1) and 3(b2).

In Fig. 3(b2), we observe a DTW with nonequidistant pulses (a bound two-pulse). Interestingly, this DTW cannot

correspond to the wave profile at $k = 1/100$ in Fig. 3(a), which is orbitally stable when $N = 100$ as evidenced by the FES [Fig. 3(c)] and direct simulation [Fig. 3(c1)]; rather, it can be attributed to a period-doubling bifurcation. The two-pulse solution obtained in Fig. 3(b2) can be used to reconstruct a wave profile corresponding to the secondary curve of bound two-pulse wave profiles in Fig. 3(a) (dark red and blue). Furthermore, we observe that the bound two-pulse wave profile approaches a limit as k tends to zero; however, both profiles look different as they approach it. To illustrate this, we compare direct simulations in a $N = 200$ network when simulating near the original wave profile with $k = 1/100$ [Fig. 3(c2)] and the bound two-pulse wave profile with $k = 1/200$ [Fig. 3(d1)]. For the bound two-pulse wave profile, the DTW is stable in its minimal network embedding, evidenced by the FES shown in Fig. 3(d) and simulation. On the other hand, the original wave profile is unstable in the $N = 200$ network, where its largest unstable FE is close to zero in the real part. Indeed, the MSC in Fig. 3(c) attached to $\lambda = 0$ has become flat; this suggests a scaling of the real parts of FEs λ , namely $\text{Re}(\lambda) = \mathcal{O}(k)$ on the curve, as $k = 1/(cT) \rightarrow 0$ for finite values of c . This can result in long transients where the solution stays close to the profile, as shown in Fig. 3(c2). Figures 3(c) and 3(d) again emphasize the importance of our approach where classical methods for computing the FES begin to fail, yet all FEs must, in truth, lie on an MSC computed with our framework as a consequence of Theorem 1 (see Supplemental Material [34]).

Concluding remarks—The spectrum of traveling waves on finite networks (DTWs) with index-shift invariance is determined by the master stability equation and master stability curves (MSCs). These curves fully encapsulate the spectrum of DTWs and can be computed effectively using numerical path-following techniques for delay differential equations. Specifically, the original high-dimensional Floquet exponent problem is replaced by a low-dimensional boundary value problem with periodic boundary conditions and additional free parameters, in particular ϕ . This significantly reduces the computation time of analyzing the stability of DTWs and can also be applied in other contexts; see Supplemental Material [34]. While DTWs on a ring serve as primary examples, the method is readily applicable to systems with higher embedding dimension, such as discrete tori and beyond, as well as higher-order (beyond pairwise) interactions. Several extensions of this work (e.g., scaling of FEs, general equivariant networks, and time-delayed couplings) warrant future research, in particular, the relation of the MSC with a spectrum of DTWs on infinite lattices.

Example—Figures in this Letter are computed for N diffusively coupled FitzHugh-Nagumo (FHN) oscillators $x_n = (V_n, W_n)$, $1 \leq n \leq N \pmod{N}$ in the excitable regime, governed by

$$\begin{aligned} V'_n &= V_n - \frac{V_n^3}{3} - W_n + 0.2 + V_{n+1} - 2V_n + V_{n-1} \\ W'_n &= 0.08(V_n + 0.7 - 0.8W_n), \end{aligned} \quad (7)$$

where $V_n(t), W_n(t) \in \mathbb{R}$. The FHN system has been used to describe phenomenologically the propagation of action potentials along nerve axons [42–45].

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S. R. and A. G. contributed equally to the paper as main authors.

Data availability—The code and the data that support the findings of this study are openly available at the following [46].

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