

Haagerup Symmetry in $(E_8)_1$?

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We suggest that the chiral $(e_8)_1$ theory—in many senses the simplest vertex operator algebra—may have Haagerup symmetry $\mathcal{H}_i$ for $i = 1, 2, 3$. Likewise, we suggest that the nonchiral $(E_8)_1$ Wess-Zumino-Witten model may have $\mathcal{H}_i \times \mathcal{H}_i^{\text{op}}$ symmetry, and that gauging the diagonal symmetry gives a $c = 8$ theory with $\mathcal{Z}(\mathcal{H}_3)$ symmetry, which is the theory predicted in Evans and Gannon [Commun. Math. Phys. **307**, 463 (2011)]. Along the way, we show that $(E_8)_1$ also has a $\text{Fib} \times \text{Fib}^{\text{op}}$ symmetry, and that gauging the diagonal symmetry gives the $(G_2)_1 \times (F_4)_1$ Wess-Zumino-Witten model, explaining the well-known conformal embedding $(G_2)_1 \times (F_4)_1 \subset (E_8)_1$. Finally, we suggest a relation to theories with $\mathcal{H}_3$ symmetry at $c = 2, 6$, complimenting the discussion with new modular bootstrap results.

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Symmetry has long been a central organizing principle in theoretical physics, underlying phenomena from crystalline order to particle interactions. Traditional notions of symmetry are rooted in groups of transformations that act on physical systems, but recent advances in quantum field theory and condensed matter physics have revealed a richer landscape of “noninvertible” symmetries, which cannot be described by ordinary groups, instead generalizing to algebraic structures known as fusion categories. Among these, the Haagerup categories [1,2] stand out for being the simplest examples resisting realization in familiar quantum theories. In this Letter, we propose that the conformal field theory associated with the exceptional Lie algebra E_8 at level one—one of the simplest two-dimensional theories—naturally hosts Haagerup-type symmetry. These results provide a concrete new setting in which exotic symmetry structures may emerge in otherwise well-known quantum systems.

The Haagerup fusion categories $\mathcal{H}_i$ for $i = 1, 2, 3$ were originally identified in [1,2], with the latter two having six simple objects and the following fusion ring:

$$\begin{aligned}\alpha^3 &= 1, & \alpha\rho &= \rho\alpha^2, \\ \rho^2 &= 1 + \rho + \alpha\rho + \alpha^2\rho,\end{aligned}\quad (1)$$

as is explained in more detail in Supplemental Material (SM) [3]. An explicit construction of a gapless theory with Haagerup symmetry was given in [31], though this theory has multiple vacua [32]. On the other hand, a variety of recent results point to the existence of a gapless theory with a unique vacuum with Haagerup symmetry—for example, a series of numerical results have hinted at the existence of such a theory around central charge $c \approx 2$ [35–39].

In significantly older work, Evans and Gannon argued for the existence of $c \in 8\mathbb{Z}$ theories with $\mathcal{Z}(\mathcal{H}_3)$ symmetry, with $\mathcal{Z}(\mathcal{H}_3)$ being the Drinfeld center of $\mathcal{H}_3$ [40]. Though they were unable to give a complete construction of such theories, they were able to use the modular data of $\mathcal{Z}(\mathcal{H}_3)$ to constrain the modular characters, and in the case of $c = 8$ they noticed an intriguing connection to the chiral $(e_8)_1$ theory, as well as its sub vertex operator algebra $(e_6)_1 \times (\mathfrak{a}_2)_1$.

In this Letter, we offer an explanation for this connection at the level of the nonchiral $(E_8)_1$ Wess-Zumino-Witten (WZW) model; to avoid confusion, we will denote the chiral theory by $(e_8)_1$, and the nonchiral theory by $(E_8)_1$. In particular, we will suggest that $(E_8)_1$ has an $\mathcal{H}_3 \times \mathcal{H}_3^{\text{op}}$ symmetry [41], and that gauging the diagonal $\mathcal{H}_3$ symmetry (using recent techniques for noninvertible gauging developed in, e.g., [42–45]) gives the putative $\mathcal{Z}(\mathcal{H}_3)$ theory predicted by Evans and Gannon. The role of the invertible $\mathbb{Z}_3$ element in the diagonal $\mathcal{H}_3$ is played by the $\mathbb{Z}_{3B}$ conjugacy class in E_8 , whose centralizer is $E_6 \times A_2$.

We will also see that $(e_8)_1$ is self-dual under certain gaugings, which would imply that $(e_8)_1$ has $\mathcal{H}_i$ symmetry for $i = 1, 2, 3$, as well as a number of additional duality defects. Finally, we will discuss connections to the $c \approx 2$

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theory predicted in [36–39], suggest the existence of a $c \approx 6$ counterpart, and compliment the discussion with new modular bootstrap results, summarized in Figs. 2 and 3. Details will be given in upcoming work [46].

Let us also note that the SM [3], which gives a simple prescription for computing the change in fusion categories upon discrete gauging, may be of independent interest to the reader.

Warm-up: Automorphisms of E_8 —By far the most well-understood class of symmetries in two-dimensional rational conformal field theories (RCFTs) are those generated by Verlinde lines [47,48]. Verlinde lines commute not only with the Virasoro generators, but also with the full chiral algebra of the RCFT. It is the latter property that enables their straightforward classification, and for diagonal RCFTs they can be shown to be in one-to-one correspondence with RCFT primaries, further sharing the same fusion rules.

However, Verlinde lines are not the only symmetries of RCFTs. Indeed, there can be a variety of line operators that are topological, but which do not commute with the chiral algebra. This Letter will focus on a certain class of such symmetries in the $(E_8)_1$ WZW model [49].

The basic idea can be illustrated via the following simple example. Consider the well-known conformal embedding $(E_7)_1 \times (A_1)_1 \subset (E_8)_1$. Unlike $(E_8)_1$, which has no non-trivial primary operators, and hence whose torus partition function can be written solely in terms of the vacuum character,

$$Z_{E_8}(\tau) := |\chi_0^{E_8}|^2, \quad (2)$$

with

$$\chi_0^{E_8} = q^{-1/3}(1 + 248q + 4124q^2 + 34752q^3 + \dots), \quad (3)$$

each of $(E_7)_1$ and $(A_1)_1$ has a single nontrivial primary, with respective conformal weights $h = \frac{3}{4}$ and $\frac{1}{4}$. As such, the diagonal torus partition function is written as a sum of four terms,

$$Z_{E_7 \times A_1}(\tau) = |\chi_0^{E_7} \chi_0^{A_1}|^2 + |\chi_{3/4}^{E_7} \chi_0^{A_1}|^2 + |\chi_0^{E_7} \chi_{1/4}^{A_1}|^2 + |\chi_{3/4}^{E_7} \chi_{1/4}^{A_1}|^2. \quad (4)$$

In this diagonal theory, there is a Verlinde line corresponding to each nontrivial primary, which in total generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. In particular, we have topological lines η_i for $i = 1, 2$ such that

$$\begin{aligned} \eta_1 |1\rangle^{E_7} &= |1\rangle^{E_7}, & \eta_1 |3/4\rangle^{E_7} &= -|3/4\rangle^{E_7}, \\ \eta_2 |1\rangle^{A_1} &= |1\rangle^{A_1}, & \eta_2 |1/4\rangle^{A_1} &= -|1/4\rangle^{A_1}, \end{aligned} \quad (5)$$

and the remaining actions are trivial.

It is straightforward to verify that gauging the diagonal symmetry generated by $\eta := \eta_1 \eta_2$ takes us from the diagonal $(E_7)_1 \times (A_1)_1$ WZW model to the $(E_8)_1$ WZW model.

Indeed, computing the twisted-twined partition functions,

$$\begin{aligned} Z_{E_7 \times A_1} |^\eta &= |\chi_0^{E_7} \chi_0^{A_1}|^2 - |\chi_{3/4}^{E_7} \chi_0^{A_1}|^2 - |\chi_0^{E_7} \chi_{1/4}^{A_1}|^2 + |\chi_{3/4}^{E_7} \chi_{1/4}^{A_1}|^2, \\ Z_{E_7 \times A_1} |_\eta &= \chi_0^{E_7} \chi_0^{A_1} \bar{\chi}_{3/4}^{E_7} \bar{\chi}_{1/4}^{A_1} + \chi_0^{E_7} \chi_{1/4}^{A_1} \bar{\chi}_{3/4}^{E_7} \bar{\chi}_0^{A_1} + \text{c.c.}, \\ Z_{E_7 \times A_1} |_\eta^\eta &= \chi_0^{E_7} \chi_0^{A_1} \bar{\chi}_{3/4}^{E_7} \bar{\chi}_{1/4}^{A_1} - \chi_0^{E_7} \chi_{1/4}^{A_1} \bar{\chi}_{3/4}^{E_7} \bar{\chi}_0^{A_1} + \text{c.c.}, \end{aligned} \quad (6)$$

taking their sum, and dividing by the order of the group (two), we immediately land upon

$$Z_{E_7 \times A_1 / \eta}(\tau) = |\chi_0^{E_7} \chi_0^{A_1} + \chi_{3/4}^{E_7} \chi_{1/4}^{A_1}|^2, \quad (7)$$

which is equivalent to the $(E_8)_1$ partition function in (2), upon recalling the well-known fact that

$$\chi_0^{E_8} = \chi_0^{E_7} \chi_0^{A_1} + \chi_{3/4}^{E_7} \chi_{1/4}^{A_1}, \quad (8)$$

which follows from the decomposition $248 \rightarrow (133, 1) \oplus (1, 3) \oplus (56, 2)$.

Because it was possible to go from $(E_7)_1 \times (A_1)_1$ to $(E_8)_1$ by gauging a discrete symmetry, there must exist a symmetry that can be gauged to go back. This symmetry will clearly not commute with the chiral algebra of E_8 , and instead should be such that its centralizer in E_8 is $E_7 \times A_1$. Of course, the symmetry in this case is well-known—it is the subgroup $\mathbb{Z}_{2A} \subset \text{Aut}(E_8)$ (see, e.g., Ref. [52]).

However, let us assume that we did not know this. A simpler way to arrive at this is to “lift” the Verlinde lines of $(E_7)_1 \times (A_1)_1$ in the following way (this lifting procedure is discussed in more detail in SM). Since η is trivial in $(E_8)_1$, let us focus on η_2 . We now use the action of η_2 on primaries in $(A_1)_1$ to define two operations in $(E_8)_1$, acting on the chiral and antichiral characters as

$$\begin{aligned} \hat{\eta}_L &: (\chi_0^{E_7} \chi_0^{A_1}, \chi_{3/4}^{E_7} \chi_{1/4}^{A_1}) \rightarrow (\chi_0^{E_7} \chi_0^{A_1}, -\chi_{3/4}^{E_7} \chi_{1/4}^{A_1}), \\ \hat{\eta}_R &: (\bar{\chi}_0^{E_7} \bar{\chi}_0^{A_1}, \bar{\chi}_{3/4}^{E_7} \bar{\chi}_{1/4}^{A_1}) \rightarrow (\bar{\chi}_0^{E_7} \bar{\chi}_0^{A_1}, -\bar{\chi}_{3/4}^{E_7} \bar{\chi}_{1/4}^{A_1}). \end{aligned} \quad (9)$$

Each of these can be thought of as defining a symmetry in one of the two chiral halves $(e_8)_1$ making up the $(E_8)_1$ theory.

While both of these symmetries are anomalous (see, e.g., Table 2 in [51]), the diagonal symmetry $\hat{\eta} := \hat{\eta}_L \hat{\eta}_R$ is nonanomalous, and gauging it takes us back to $(E_7)_1 \times (A_1)_1$ —in other words, $\hat{\eta}$ generates $\mathbb{Z}_{2A}$. This can be seen by computing the twisted-twined partition functions via (9) and the modular S matrix, giving

$$\begin{aligned} Z_{E_8}(\tau) |^{\hat{\eta}} &= |\chi_0^{E_7} \chi_0^{A_1} - \chi_{3/4}^{E_7} \chi_{1/4}^{A_1}|^2, \\ Z_{E_8}(\tau) |_{\hat{\eta}} &= |\chi_{3/4}^{E_7} \chi_0^{A_1} + \chi_0^{E_7} \chi_{1/4}^{A_1}|^2, \\ Z_{E_8}(\tau) |_{\hat{\eta}}^{\hat{\eta}} &= |\chi_{3/4}^{E_7} \chi_0^{A_1} - \chi_0^{E_7} \chi_{1/4}^{A_1}|^2, \end{aligned} \quad (10)$$

and taking the relevant sum to reproduce (4).

Of course, the exact same techniques can be used more generally. For example, gauging the diagonal $\mathbb{Z}_3$ center

symmetry of $(E_6)_1 \times (A_2)_1$ gives rise again to $(E_8)_1$, and one can go back by gauging $\mathbb{Z}_{3B} \subset \text{Aut}(E_8)$, which does not commute with the chiral algebra. The action of $\mathbb{Z}_{3B}$ can be defined as above, by first decomposing the characters of $(E_8)_1$ into those of $(E_6)_1 \times (A_2)_1$,

$$\chi_0^{E_8} = \chi_0^{E_6} \chi_0^{A_2} + \chi_{2/3}^{E_6} \chi_{1/3}^{A_2} + \chi_{2/3}^{E_6} \chi_{1/3}^{A_2}, \quad (11)$$

and then “lifting” the action of the center symmetry of one of the two factors to $(e_8)_1$.

In both of the cases mentioned above, the symmetry in question was an element of $\text{Aut}(E_8)$, and hence its existence was no surprise. A slightly more interesting example is to consider the conformal embedding $(G_2)_1 \times (F_4)_1 \subset (E_8)_1$. As described in SM, each of $(G_2)_1$ and $(F_4)_1$ has a Fibonacci symmetry, and we may gauge the diagonal symmetry to obtain $(E_8)_1$. Conversely, the Verlinde lines of $(G_2)_1 \times (F_4)_1$ may be lifted to a $\text{Fib} \times \text{Fib}^{\text{op}}$ symmetry of $(E_8)_1$, and gauging the diagonal symmetry takes us back to $(G_2)_1 \times (F_4)_1$.

Haagerup symmetry—Having introduced the basic idea, we may now move on to the case of actual interest in this Letter. The starting point is the result of [40,53], which claims that any vector-valued modular form whose modular data is that of the Drinfeld center $\mathcal{Z}(\mathcal{H}_3)$ and has central charge $c = 8$ must have characters of the following form:

$$\begin{aligned} \chi_0^{\mathcal{Z}(\mathcal{H}_3)} &= q^{2/3}(q^{-1} + (6 + 13\gamma) + (120 + 78\gamma)q \\ &\quad + (956 + 351\gamma)q^2 + (6010 + 1235\gamma)q^3 + \dots), \\ \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)} &= q^{2/3}\left((80 - 13\gamma) + (1250 - 78\gamma)q \right. \\ &\quad \left. + (10630 - 351\gamma)q^2 + (65042 - 1235\gamma)q^3 + \dots\right), \\ \chi_{\pi_2}^{\mathcal{Z}(\mathcal{H}_3)} &= \chi_{\sigma_1}^{\mathcal{Z}(\mathcal{H}_3)} \\ &= q^{2/3}(81 + 1377q + 11583q^2 + 71037q^3 + \dots), \end{aligned} \quad (12)$$

where $\gamma = 0$ or 1 and the characters are labeled by elements of $\mathcal{Z}(\mathcal{H}_3)$. Note that we have written only four of the 12 total characters; the others can be found in [40]. We refer to the putative diagonal, nonchiral conformal field theory with these characters as the “ $\mathcal{Z}(\mathcal{H}_3)$ theory” [54].

The key observation is that we have the following relation between characters:

$$\begin{aligned} \chi_0^{E_8} &= \chi_0^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)} + 2\chi_{\pi_2}^{\mathcal{Z}(\mathcal{H}_3)}, \\ &= \chi_0^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)} + 2\chi_{\sigma_1}^{\mathcal{Z}(\mathcal{H}_3)}, \\ &= \chi_0^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\pi_2}^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\sigma_1}^{\mathcal{Z}(\mathcal{H}_3)}. \end{aligned} \quad (13)$$

Focusing for the moment on the first of these, this relationship suggests that the $(E_8)_1$ WZW model can be

obtained by gauging the algebra object $\mathcal{A} = 1 \oplus \pi_1 \oplus 2\pi_2$ in the putative $\mathcal{Z}(\mathcal{H}_3)$ theory. Upon gauging this algebra object, a standard module category analysis shows that the resulting theory has symmetry $\mathcal{H}_3 \times \mathcal{H}_3^{\text{op}}$, which leads us to the claim that $(E_8)_1$ has $\mathcal{H}_3 \times \mathcal{H}_3^{\text{op}}$ symmetry [55]. This can also be shown by lifting the Verlinde lines of $\mathcal{Z}(\mathcal{H}_3)$ to $(E_8)_1$ via the same procedure as for $(E_7)_1 \times (A_1)_1 \subset (E_8)_1$, as is discussed in SM. Here in the main text, we will content ourselves with briefly describing the action of this symmetry, and confirming that gauging of the diagonal Haagerup symmetry gives the $\mathcal{Z}(\mathcal{H}_3)$ theory [56].

The idea is similar to before—namely, each copy of $\mathcal{H}_3$ acts on a single chiral half, with the action given by the usual action of $\mathcal{H}_3$ on elements of $\mathcal{Z}(\mathcal{H}_3)$. Focusing on the left movers, we have [4,31] (see also SM),

$$\begin{aligned} \chi_0^{\mathcal{Z}(\mathcal{H}_3)}: \hat{\alpha}_L &= 1, & \hat{\rho}_L &= d_\rho, \\ \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)}: \hat{\alpha}_L &= 1, & \hat{\rho}_L &= -d_\rho^{-1}, \\ \chi_{\pi_2}^{\mathcal{Z}(\mathcal{H}_3)}: \hat{\alpha}_L &= \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, & \hat{\rho}_L &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \quad (14)$$

where $d_\rho := \frac{1}{2}(3 + \sqrt{13})$ and $\omega := e^{2\pi i/3}$, and likewise for the right movers. Using these results and (13), we then arrive at the following partition functions with insertions of the diagonal operators $\hat{\alpha} := \hat{\alpha}_L \hat{\alpha}_R^{-1}$ and $\hat{\rho} := \hat{\rho}_L \hat{\rho}_R$:

$$\begin{aligned} Z_{E_8} \Big|_{\hat{\alpha}} &= \left| \chi_0^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)} + (\omega + \omega^2) \chi_{\pi_2}^{\mathcal{Z}(\mathcal{H}_3)} \right|^2, \\ Z_{E_8} \Big|_{\hat{\rho}} &= \left| d_\rho \chi_0^{\mathcal{Z}(\mathcal{H}_3)} - d_\rho^{-1} \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)} \right|^2. \end{aligned} \quad (15)$$

Making use of the S matrix for $\mathcal{Z}(\mathcal{H}_3)$ reviewed in SM, we further obtain the twisted partition functions,

$$\begin{aligned} Z_{E_8} \Big|_{\hat{\alpha}} &= \left| \chi_{\sigma_1}^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\sigma_2}^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\sigma_3}^{\mathcal{Z}(\mathcal{H}_3)} \right|^2, \\ Z_{E_8} \Big|_{\hat{\rho}} &= \left| \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\pi_2}^{\mathcal{Z}(\mathcal{H}_3)} + \sum_{i=1}^3 \chi_{\sigma_i}^{\mathcal{Z}(\mathcal{H}_3)} + \sum_{i=1}^6 \chi_{\mu_i}^{\mathcal{Z}(\mathcal{H}_3)} \right|^2. \end{aligned} \quad (16)$$

Of course, similar results hold for $\hat{\alpha}^2, \hat{\alpha}\hat{\rho}$, and $\hat{\alpha}^2\hat{\rho}$.

We now claim that gauging the diagonal algebra $\mathcal{A}^{\text{diag}} := 1 \oplus \hat{\alpha} \oplus \hat{\alpha}^2 \oplus \hat{\rho} \oplus \hat{\alpha}\hat{\rho} \oplus \hat{\alpha}^2\hat{\rho}$ gives the $\mathcal{Z}(\mathcal{H}_3)$ theory, with the diagonal modular invariant

$$\begin{aligned} Z_{\mathcal{Z}(\mathcal{H}_3)} &= \left| \chi_0^{\mathcal{Z}(\mathcal{H}_3)} \right|^2 + \left| \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)} \right|^2 + \left| \chi_{\pi_2}^{\mathcal{Z}(\mathcal{H}_3)} \right|^2 \\ &\quad + \sum_{i=1}^3 \left| \chi_{\sigma_i}^{\mathcal{Z}(\mathcal{H}_3)} \right|^2 + \sum_{i=1}^6 \left| \chi_{\mu_i}^{\mathcal{Z}(\mathcal{H}_3)} \right|^2. \end{aligned} \quad (17)$$

The (co)multiplication for the algebra object may be taken to be the most obvious one, where all coefficients in the expansion in terms of simple lines are fixed to the same value, namely the inverse square root of the order of the

algebra, i.e., $(3 + 3d_\rho^2)^{-1/2}$. Thanks to the transparent nature of the Haagerup fusion category [58], the gauging may be simplified significantly, and we are ultimately left to compute the following:

$$Z_{E_8/\mathcal{A}^{\text{diag}}} = \frac{1}{3 + 3d_\rho^2} \left[Z_{E_8} + 2(Z_{E_8}|_{\hat{\alpha}} + Z_{E_8}|_{\hat{\alpha}} + Z_{E_8}|_{\hat{\alpha}}) + 3Z_{E_8}|_{\hat{\rho},1} + 3 \sum_{i=0}^2 \left(Z_{E_8}|_{\hat{\rho},\hat{\alpha}^i\hat{\rho}} + Z_{E_8}|_{\hat{\rho},\hat{\alpha}^i\hat{\rho}} + Z_{E_8}|_{\hat{\rho},\hat{\alpha}^i\hat{\rho}} \right) \right], \quad (18)$$

where

$$Z|_{y,z}^x := \text{diagram} \quad (19)$$

The terms in the first line, together with the first term on the second line, are all straightforward to compute. The only nontrivial contributions are from the terms of the form $Z_{E_8}|_{\hat{\rho},\hat{\alpha}^i\hat{\rho}}$. These may be computed by explicitly solving the tube algebra, which in particular requires making use of the F symbols given in [33,58]; details can be found in SM. Doing so tells us, for example, that we have

$$\begin{aligned} Z_{E_8}|_{\hat{\rho},\hat{\rho}} &\approx \left| -0.128\chi_{\pi_1} + 0.845\chi_{\pi_2} + 0.422\chi_{\sigma_1} \right. \\ &\quad + 1.39(e^{-2\pi i \frac{1}{13}}\chi_{\sigma_2} + e^{2\pi i \frac{1}{13}}\chi_{\sigma_3}) \\ &\quad + 0.509(e^{2\pi i \frac{1}{13}}\chi_{\mu_1} + e^{-2\pi i \frac{1}{13}}\chi_{\mu_2}) \\ &\quad + 0.296(e^{2\pi i \frac{9}{13}}\chi_{\mu_3} + e^{-2\pi i \frac{9}{13}}\chi_{\mu_4}) \\ &\quad \left. - 1.65(e^{2\pi i \frac{3}{13}}\chi_{\mu_5} + e^{-2\pi i \frac{3}{13}}\chi_{\mu_6}) \right|^2, \end{aligned} \quad (20)$$

where we have written only numerical approximations of the coefficients, since their closed form expressions are complicated and unenlightening. The other partition functions are obtained similarly.

Computing all of the relevant twisted partition functions and inserting them into (18), one straightforwardly confirms that $Z_{E_8/\mathcal{A}^{\text{diag}}} = Z_{\mathcal{Z}(\mathcal{H}_3)}$, as claimed. Note that the $\hat{\alpha}$ appearing in $\mathcal{A}^{\text{diag}}$ is none other than the generator of $\mathbb{Z}_{3B} \subset \text{Aut}(E_8)$, as can be seen from the fact that

$$\begin{aligned} \chi_0^{E_6/A_2} \chi_0^{A_2} &= \chi_0^{\mathcal{Z}(\mathcal{H}_3)} + \chi_{\pi_1}^{\mathcal{Z}(\mathcal{H}_3)}, \\ \chi_{2/3}^{E_6/A_2} \chi_{1/3}^{A_2} + \chi_{2/3}^{E_6/A_2} \chi_{1/3}^{A_2} &= 2\chi_{\pi_2}^{\mathcal{Z}(\mathcal{H}_3)}, \end{aligned} \quad (21)$$

together with the action in (15).

Returning now to the other equalities in (13), it is easy to verify that one may go between the first and second lines by gauging the chiral $\hat{\alpha}_L$, which tells us that $(e_8)_1$ theory should also have $\mathcal{H}_2$ symmetry, since $\mathcal{H}_2$ and $\mathcal{H}_3$ are related by $\mathbb{Z}_3$ gauging [33]. Likewise, the third equality is obtained by gauging $\mathcal{A}_L = 1 \oplus \hat{\rho}_L \oplus \hat{\alpha}_L \hat{\rho}_L$, from which we learn that $(e_8)_1$ should also have $\mathcal{H}_1$ symmetry. Additional details

can be found in SM. Similar statements hold for the antichiral halves.

Let us close by noting that by a similar reasoning, the Monster CFT is also expected to have $\mathcal{H}_i \times \mathcal{H}_i^{\text{op}}$ symmetry for $i = 1, 2, 3$, though we do not give details here.

Relation to $c = 2, 6$ —As we have described above, we may gauge the $\mathbb{Z}_{3B}$ symmetry of $(E_8)_1$ theory to go to the $(E_6)_1 \times (A_2)_1$ WZW model. As such, the latter should contain a symmetry $(\mathcal{H}_3 \times \mathcal{H}_3^{\text{op}})/\mathbb{Z}_{3B}$, where $\mathbb{Z}_{3B}$ is the diagonal symmetry of the two copies of $\mathcal{H}_3$. As we discuss in SM, this category is equivalent to a double near-group extension of $\mathbb{Z}_3^2$. Choosing one of the dimension d_ρ^2 lines in this category, which we denote by $[\lambda]$, we may then gauge the binary algebra $\mathcal{A}_{[\lambda]} := 1 \oplus [\lambda]$, which takes us to the $\mathcal{Z}(\mathcal{H}_3)$ theory. Conversely, gauging the object $1 \oplus \pi_1$ in the $\mathcal{Z}(\mathcal{H}_3)$ takes us back to $(E_6)_1 \times (A_2)_1$. These relations are summarized in Fig. 1.

Based on this, one may hope that upon performing appropriate (asymmetric) quotients, the above will give rise

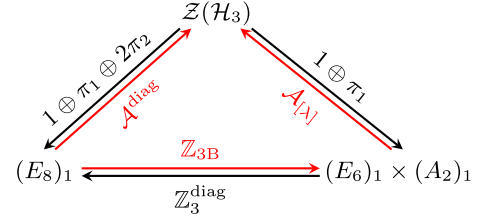


FIG. 1. Gaugings relating the $\mathcal{Z}(\mathcal{H}_3)$, $(E_8)_1$, and $(E_6)_1 \times (A_2)_1$ theories. Lines in red denote symmetries that do not commute with the full chiral algebra.

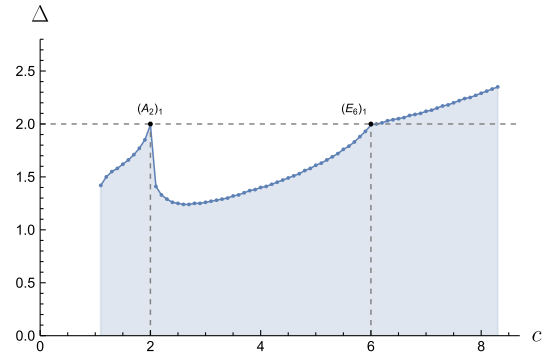


FIG. 2. Bounds on the dimension Δ of the lightest scalar in $\mathcal{V}_1$, as a function of c . The allowed region is shaded in blue. Note that both the $(A_2)_1$ and $(E_6)_1$ theories live at kinks on the boundary.

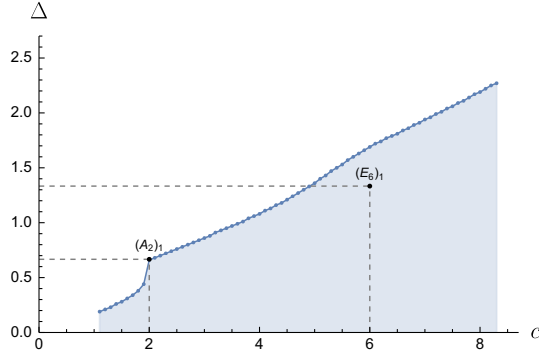


FIG. 3. Bounds on the dimension Δ of the lightest $\mathcal{V}_{\pi_2} \cup \mathcal{V}_{\sigma_1}$ (i.e., $\mathbb{Z}_3$ -charged) scalar, as a function of c . The $(A_2)_1$ theory again lives at a kink.

to theories with Haagerup symmetry at $c = 2, 6$. Details will be given in upcoming work [46]. Here, we offer just one tantalizing piece of evidence that this hope is correct—namely, we have performed numerical modular bootstrap for theories with $\mathcal{H}_i$ symmetry following the procedure in [5,59]. The technique is described in SM, with the main results given in Figs. 2 and 3. The first of these gives bounds on the dimension of the lightest scalar operator in the identity sector, showing prominent kinks at $c \approx 2, 6$ and $\Delta \approx 2$. The second gives bounds on the dimension of the lightest scalar operator in the $\pi_2 \cup \sigma_1$ sector, showing a kink at $c = 2$ and $\Delta \approx \frac{2}{3}$.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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