

Thin-Film Drainage Becomes Singular at Saddles

Simeon Djambov^{1,2}, Alice Marcotte³, François Gallaire¹, and Pier Giuseppe Ledda^{4,*}

¹Laboratory of Fluid Mechanics and Instabilities, EPFL, Lausanne CH-1015, Switzerland

²LadHyX, CNRS, Ecole Polytechnique, Institut Polytechnique de Paris, Palaiseau, France

³Institut Jean le Rond d'Alembert, CNRS, Sorbonne Université, Paris, France

⁴Department of Civil, Environmental Engineering and Architecture, University of Cagliari, 09123 Cagliari, Italy



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From coating curved parts to creeping lava, thin-film gravitational drainage over complex surfaces can produce abrupt accumulation, usually associated with sharp edges or defects. Here, experiments, simulations, and asymptotics show that a smooth saddle alone generates a logarithmically singular ridge, regularized within a dynamic region where drainage balances hydrostatic pressure and capillarity. In a Newtonian lubrication picture, this mechanism can inform redistribution in manufacturing and geophysics, motivating extensions to complex rheology and interacting geometric features.

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Viscous drainage of thin liquid films over complex topographies arises in settings spanning coating and manufacturing processes [1,2], geophysical processes [3], and deposition and erosion in geological systems [4–6]. In these settings, substrate geometry can generate strong local variations in film thickness. Such variations affect coating in glove manufacturing and shell uniformity in investment casting [7] and control lava propagation and subsequent deposition at geophysical scales [8,9]. Identifying the geometric features that preferentially accumulate liquid is therefore relevant across a broad range of scales and applications.

In the viscous thin-film limit, the dynamics is governed by gravity, viscosity, substrate curvature, and capillarity [10–14]. Classical drainage is relatively well understood on flat substrates [15–18] and on simple curved geometries such as cylinders, cones, spheres, and funnels. In these cases, drainage is predominantly converging or diverging under the action of gravity along the surface, whose variation, set by the substrate curvature, determines the film thickness distribution [19,20]. Much less is known when these two tendencies coexist locally. Specifically, the signs of the principal curvatures determine the local topology of gravity-driven drainage. Opposite-sign curvatures, occurring in a saddle, produce a hyperbolic flow with drainage locally converging along one principal direction and diverging along the other (Fig. 1). Sharp features are known to organize thin-film and interfacial dynamics, including defects [21], contact

lines [22], and edges [18]. This raises the question of whether smooth interior features such as saddles, which occur naturally on surfaces with nonuniform curvature, can similarly organize thin-film drainage.

In this Letter, we combine experiments, numerical simulations, and theory to show that the hyperbolic drainage topology near a smooth saddle focuses the film into a localized ridge whose leading-order profile is logarithmically singular, without requiring edges, corners, contact lines, or imposed defects. The apparent singularity induced by the hyperbolic flow is regularized within a region where drainage, hydrostatic pressure, and capillarity balance. Its width is not set by a preexisting geometric length, but is selected by the flow and shrinks as drainage proceeds.

We test this mechanism in a controlled setting using a Newtonian fluid on a smooth substrate containing a single nondegenerate saddle. Lengths are scaled by a characteristic length L , and the substrate is prescribed as $\hat{\eta}(\hat{x}, \hat{y}) = L - L \cos(\hat{y}/L) - (\hat{x}^2/2L)$. Near the origin, this surface reduces to $\hat{\eta} \simeq (\hat{y}^2/2L) - (\hat{x}^2/2L)$, so that the saddle has opposite curvatures in the two orthogonal principal directions $\hat{x}$ and $\hat{y}$, and gravity acts downward along the $\hat{z}$ direction (Fig. 1). The periodic dependence in $\hat{y}$ introduces neighboring maxima that delimit the drainage basin feeding the central saddle, thereby uniquely fixing its liquid supply and ensuring reproducible late-time drainage, independent of the initial conditions [12]. The experiment focuses on a minimal feeding unit, a central saddle surrounded by two maxima. Near the saddle, the local quadratic form controls the singular behavior derived below. A silicone oil (Carl Roth) of density $\rho = 970 \pm 5 \text{ kg m}^{-3}$, kinematic viscosity $1000 \pm 50 \text{ mm}^2 \text{ s}^{-1}$ (from manufacturer's datasheet), and surface tension $\gamma = 21 \pm 0.2 \text{ mN m}^{-1}$ [23] was used in the experiments. Three geometrically similar substrates

*Contact author: piergiuseppe.ledda@unica.it

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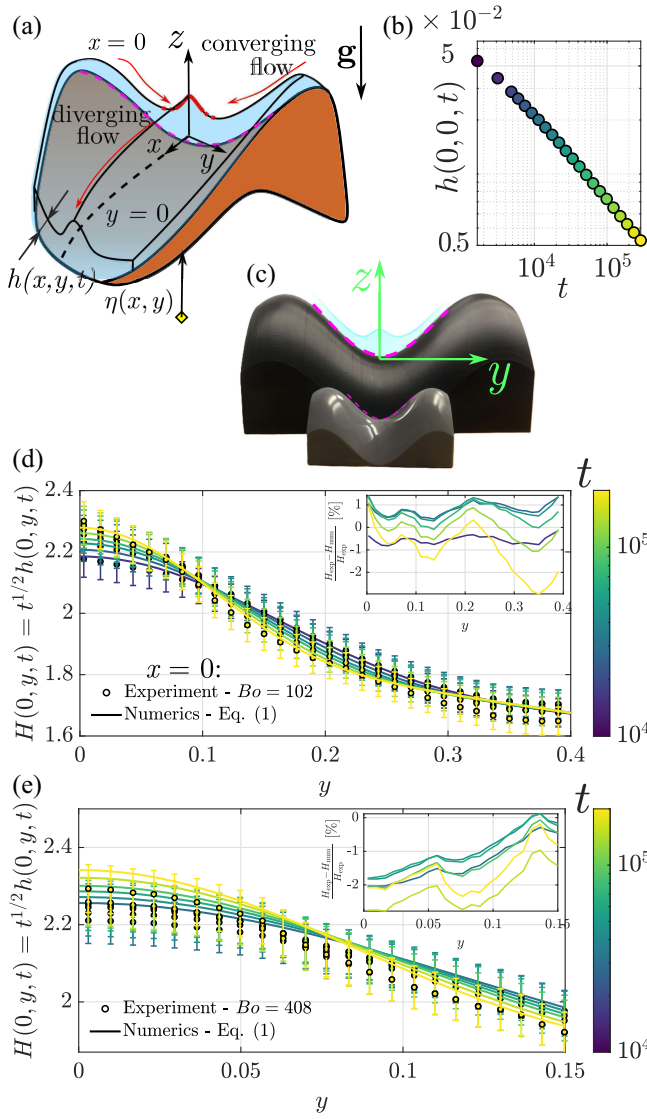


FIG. 1. (a) Sketch of the drainage of a thin film over a saddle-shaped substrate. Liquid drains from elevated regions toward lower regions, producing converging inflow toward the saddle and diverging outflow away from it. (b) Experimental saddle thickness as a function of time, for the $\nu = 10^{-3} \text{ m}^2 \text{ s}^{-1}$ silicone oil, and $Bo = 102$. (c) Experimental photo of two of the three employed substrates with a sketch of the imaged thickness along $x = 0$. (d),(e) Rescaled thickness profiles $H = t^{1/2}h$ along $x = 0$ for (d) $Bo = 102$ and (e) $Bo = 408$. Circles denote experiments and solid lines numerical solutions of Eq. (1) for initial non-dimensional thickness $h_0 = 0.1$; colors indicate increasing time. Insets show the relative experiment-simulation difference $(H_{\text{exp}} - H_{\text{num}})/H_{\text{exp}}$.

reproducing the prescribed saddle geometry were fabricated by 3D printing, as detailed in Supplemental Material [24]. Their length scales were $L = 15 \pm 0.025$, 30 ± 0.1 , and 40 ± 0.1 mm. The corresponding Bond numbers, measuring the effect of gravity against surface tension, are $Bo = \rho g L^2 / \gamma = 102 \pm 1$, 408 ± 5 , and 725 ± 9 , respectively.

The draining film was imaged with a uniform LED panel placed behind the substrate, while a camera equipped with a macro lens recorded the transmitted light from the opposite side, along the plane $\hat{x} = 0$ (Fig. 1). The thickness profile $\hat{h}(\hat{x} = 0, \hat{y}, \hat{t})$ was extracted by subtracting the dry-substrate reference image before coating [24]. The uncertainty on the measured film thickness $\hat{h}$ is primarily associated with the calibration factor converting pixel distances into physical lengths. This calibration was obtained by fitting the dry-substrate profile with the prescribed parabolic geometry. The resulting uncertainty on the calibration factor is within $\pm 2.5\%$ [24].

Lengths are nondimensionalized by L , so that the non-dimensional film thickness is $h = \hat{h}/L$, and time by the drainage scale $\tau_d = \nu/(gL)$ such that $t = \hat{t}/\tau_d$, where g is the gravitational acceleration. The experimental observations span the nondimensional time range $10^4 \lesssim t \lesssim 5 \times 10^5$. The measured saddle thickness decays with time [Fig. 1(b)]. Motivated by the standard late-time drainage law, we rescale the thickness as $H := t^{1/2}h = (g\hat{t}/\nu L)^{1/2}\hat{h}$, independent of the thickness initial condition [12]. The uncertainty on the rescaled thickness H thus includes also the contribution from viscosity. Although the thickness appears to follow the $t^{-1/2}$ scaling, as shown in Fig. 1(b) for $Bo = 102$, the rescaled profiles retain a time dependence in the immediate vicinity of the saddle, exhibiting a maximum that slightly grows over time [Figs. 1(d) and 1(e)].

We model viscous drainage in the lubrication limit by a conservation law written in the natural coordinate system of the curved substrate [14],

$$\mathcal{V} \partial_t h + \nabla \cdot \left(\mathbf{q}_d + \mathbf{q}_r + \frac{1}{Bo} \mathbf{q}_\gamma \right) = 0. \quad (1)$$

Here, $\mathbf{q}_d = \frac{1}{3} h^3 \mathbf{g}_t$ is the leading gravity-driven drainage flux, where $\mathbf{g}_t$ is tangential gravity; $\mathbf{q}_r$ collects the next-order hydrostatic-pressure flux corrections, and $\mathbf{q}_\gamma$ is the capillary flux. The factor $\mathcal{V}$ is a thickness-dependent geometric correction and reduces to unity as the film drains, $\mathcal{V} = 1 + \mathcal{O}(t^{-1/2})$. Explicit expressions for $\mathcal{V}$, $\mathbf{q}_r$, and $\mathbf{q}_\gamma$ are reported in Supplemental Material [24]. Equation (1) is solved via a custom formulation in COMSOL, previously validated against experiments [14], with further detail reported in Supplemental Material [24]. The numerical profiles reproduce experimental measurements, as quantified also by the relative difference in the insets [Figs. 1(d) and 1(e)]. The deviations are a few percent, in line with the experimental uncertainties.

We first consider the late-time drainage problem for $Bo \rightarrow \infty$. Writing $h(x, y, t) = t^{-1/2} H_d(x, y) + o(t^{-1/2})$, $t^{-1/2} \ll 1$, the $O(1)$ balance in Eq. (1) retains only the leading drainage flux and yields [12,14]

$$-\frac{H_d}{2} + \frac{1}{3} \nabla \cdot (H_d^3 \mathbf{g}_t) = 0, \quad (2)$$

where $\mathbf{g}_t = -(\nabla_{xy} \eta / w^2)$, $w = (1 + \eta_x^2 + \eta_y^2)^{1/2}$, with $\nabla_{xy} = [\partial_x, \partial_y]$, from which it follows $\nabla \cdot \mathbf{g}_t = -\mathcal{K}/w$, where $\mathcal{K}$ is twice the classical mean substrate curvature [14]. Equation (2) becomes

$$\nabla_{xy} \eta \cdot \nabla_{xy} H_d^2 + w \left(\frac{2}{3} \mathcal{K} H_d^2 + w \right) = 0, \quad (3)$$

solved by the method of characteristics [24] with initial conditions imposed at the substrate maxima [14]. Figure 2 compares the drainage solution H_d with numerical solutions of Eq. (1) without capillarity. The geometry produces

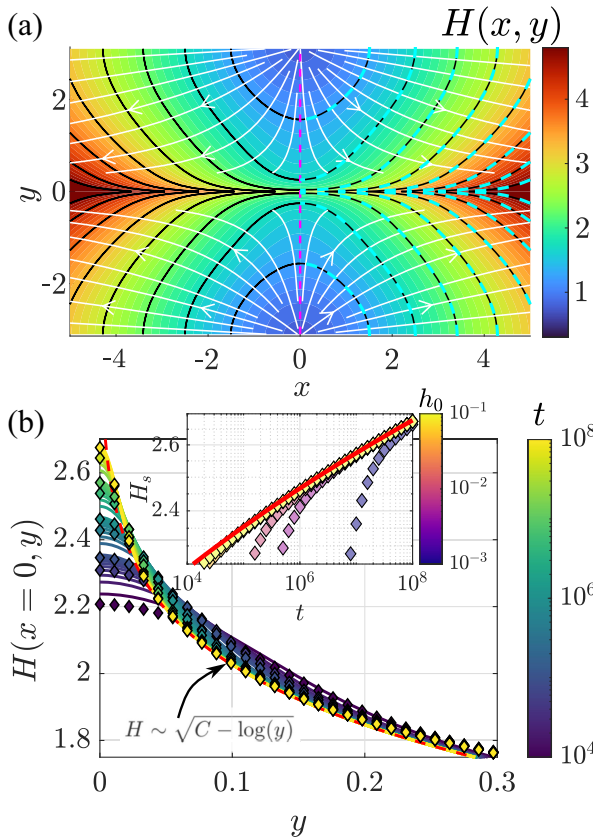


FIG. 2. Capillary-free problem. (a) Late-time drainage field for $h(x, y, 0) = 0.1$ at $t = 10^8$. Color map and isocontours (black) of the numerical rescaled thickness $H(x, y)$. Cyan dashed lines show the drainage solution H_d [Eq. (3)], and white curves denote the numerical drainage flux $\frac{1}{3} h^3 \mathbf{g}_t$ streamlines. (b) Rescaled thickness profiles for increasing times. Diamonds are numerical solutions $H(x = 0, y)$, colored solid lines are theoretical solutions $\mathcal{H}(x = 0, y)$ obtained by solving the saddle layer problem Eq. (6), and the red dashed line is the near-saddle drainage solution H_d^* with $C \approx 1.8$. Inset: numerical rescaled saddle thickness $H_s := H(0, 0)$ versus time, for different initial uniform thicknesses h_0 . The theory (red solid line) captures both the ridge profile and the slow growth of H_s , independently of the initial condition.

inflow along y and outflow along x . Away from the saddle, $H = t^{1/2} h$ approaches H_d , whereas near $y = 0$ a time-dependent ridge forms. Its height and lateral extent remain time dependent [diamonds in Fig. 2(b)], even though the rescaled thickness has already collapsed onto the drainage profile [Eq. (3)], away from $y = 0$. Conversely, the drainage solution H_d grows without bound as one approaches $y = 0$.

This behavior follows from the local saddle geometry. Near $(x, y) = (0, 0)$, the substrate is approximated by $\eta(x, y) \approx (y^2/2) - (x^2/2)$, so that $\eta_x \approx -x$, $\eta_y \approx y$, $w \approx 1$, while $\mathcal{K} \approx 0$. Equation (3) thus reduces to $-x\partial_x(H_d^2) + y\partial_y(H_d^2) + 1 = 0$. Along the symmetry line $x = 0$, where $\partial_x H_d = 0$, this gives

$$\frac{dH_d^2}{dy} = -\frac{1}{y} \Rightarrow H_d(0, y) = (C - \log|y|)^{1/2} := H_d^*(y). \quad (4)$$

The constant C is set by Eq. (3) by solving the drainage problem along the line $x = 0$, as detailed in Supplemental Material [24]. For the global (far from the saddle) substrate topology under consideration, $C \approx 1.8$. Thus, unlike previously studied geometries, the saddle forces a logarithmic divergence of the leading-order late-time profile, which follows from the local hyperbolic geometry $\eta \sim y^2 - x^2$. The singularity is not caused by a sharp edge, contact line, or imposed defect, but by the structure of the gravitational drainage field near a smooth saddle.

The singularity can be regularized without reference to the global shape of the substrate, by introducing a minimal local model that includes hydrostatic effects and, for completeness, capillary terms on flat surfaces [10,18]. The metric is thus taken as Cartesian, and the leading-order effect of the saddle geometry is retained through the tangential gravity, $\mathbf{g}_t \approx x\mathbf{e}_x - y\mathbf{e}_y$. Introducing the local rescaled thickness $\mathcal{H}$, $h(x, y, t) = \delta \mathcal{H}(x, y) + o(\delta)$, $\delta := t^{-1/2} \ll 1$. The local thin-film equation reads as [10]

$$-\frac{\mathcal{H}}{2} + \nabla_{xy} \cdot \left[\frac{\mathcal{H}^3}{3} \left(x\mathbf{e}_x - y\mathbf{e}_y - \delta \nabla_{xy} \mathcal{H} + \frac{\delta}{\text{Bo}} \nabla_{xy} \nabla_{xy}^2 \mathcal{H} \right) \right] = 0. \quad (5)$$

We begin by considering the case without capillarity ($\text{Bo} \rightarrow \infty$). Compared with the drainage problem, this local balance retains the same saddle-induced drainage flux but also includes the hydrostatic-pressure correction, $-\delta \mathcal{H}^3 \nabla_{xy} \mathcal{H}$, which becomes important as the drainage solution develops large gradients. We thus seek a narrow region localized in the transverse direction by introducing the stretched coordinate ζ such that $y = \delta^\alpha \zeta$. In Eq. (5), the drainage contribution along y scales as δ^α , because it is proportional to y , whereas the transverse hydrostatic contribution scales as $\delta(\partial_y \mathcal{H}) \sim \delta^{1-\alpha}$. Balancing these two terms gives $\delta^\alpha \sim \delta^{1-\alpha}$, hence $\alpha = \frac{1}{2}$ and $\zeta = y t^{1/4}$.

The regularizing region width thus scales as $y \sim t^{-1/4}$. This distinguishes the saddle layer solution from both classical boundary layers [32] and self-similar solutions [18]: its width is dynamically selected rather than imposed by a fixed geometrical length, while its amplitude remains tied to the logarithmic drainage profile. We now restrict the analysis to the symmetry plane $x = 0$, where $\partial_x \mathcal{H} = 0$, to compare with the experimental ridge profile. The resulting ordinary differential equation reads

$$\mathcal{H}''\mathcal{H}^2 + 3\mathcal{H}\mathcal{H}'(\mathcal{H}' + \zeta) + \frac{3}{2} = 0, \quad (6)$$

where primes denote derivatives with respect to $\zeta = yt^{1/4}$. When solved for $\zeta \geq 0$, the solution must satisfy the symmetry condition $\mathcal{H}'(0) = 0$, and match the logarithmic behavior, $\mathcal{H}(\zeta \gg 1) \sim H_d^* = (C - \log(\zeta t^{-1/4}))^{1/2}$. Thus, the leading-order time dependence enters through the matching condition. The large- ζ asymptotics of Eq. (6) recovers the logarithmic form (4), allowing numerical matching at a single point in the overlap region. The matched solution reproduces the numerical profiles close to the saddle for the model without capillarity [Fig. 2(b)], thereby confirming the local balance mechanism. In particular, the numerical rescaled saddle thickness $H_s := H(0,0)$ collapses for different initial conditions and increases slowly with time [inset of Fig. 2(b)], consistent with the persistent sharpening of the ridge observed in the simulations. Although the physical thickness $h = t^{-1/2}\mathcal{H}$ decreases, the growth of $\mathcal{H}$ shows that thinning is asymptotically slower there than far from the saddle, signaling a local relative accumulation of fluid.

At finite Bond number, numerical solutions deviate systematically from the capillary-free prediction (Fig. 3). The deviation becomes stronger as the saddle layer narrows with time, indicating that capillary smoothing becomes comparable to the local hydrostatic correction in Eq. (5). Near the saddle, the leading transverse balance is between drainage, hydrostatic pressure, and capillarity, $\partial_y(\mathcal{H}^3 y) \sim \delta \partial_y(\mathcal{H}^3 \partial_y \mathcal{H}) \sim \text{Bo}^{-1} \delta \partial_y(\mathcal{H}^3 \partial_y \mathcal{H})$. The saddle layer generates its own transverse scale $\ell_b = \hat{\ell}_b/L \sim \delta^{1/2} = t^{-1/4}$ and, therefore, its own modified Bond number $\text{Bo}^* = (\rho g \ell_b^2 / \gamma)$. Capillarity enters in the dominant balance with drainage and hydrostatic pressure when $\text{Bo}^* = \mathcal{O}(1)$, i.e., under the distinguished scaling $\text{Bo} = \text{Bo}^* \delta^{-1} = \text{Bo}^* t^{1/2}$. The dominant balances, using $\partial_y \sim \delta^{-\alpha}$, are $\delta^\alpha \sim \delta^{1-\alpha} \sim \delta^{2-3\alpha}$, which give $\alpha = \frac{1}{2}$. Capillarity therefore enters the same $t^{-1/4}$ ridge width scaling without altering it, leading to the local equation

$$\begin{aligned} \mathcal{H}''\mathcal{H}^2 - \frac{1}{\text{Bo}^*} (3\mathcal{H}\mathcal{H}'\mathcal{H}''' + \mathcal{H}^2\mathcal{H}^{IV}) \\ + 3\mathcal{H}\mathcal{H}'(\mathcal{H}' + \zeta) + \frac{3}{2} = 0, \end{aligned} \quad (7)$$

subject to the symmetry conditions $\mathcal{H}'(0) = \mathcal{H}'''(0) = 0$, together with matching to the logarithmic profile for both

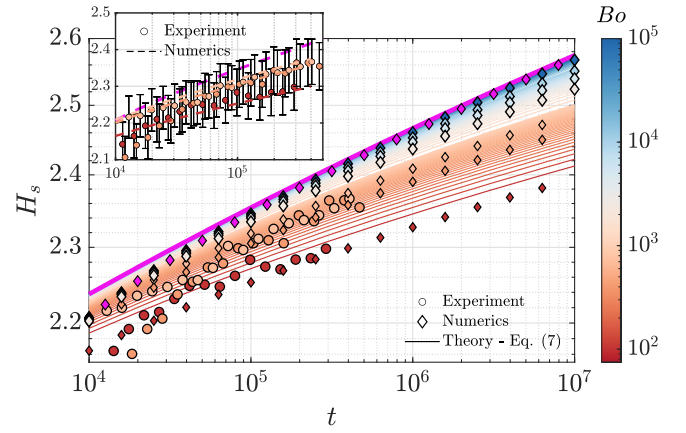


FIG. 3. Regularization of the saddle singularity along $x = 0$. Finite-capillarity case: rescaled saddle thickness $H_s = H(0,0)$ versus time. Diamonds are numerical solutions of H (with $h_0 = 0.1$) for $\text{Bo} \in [102, 10^5]$, from the bottom to the top: $\text{Bo} = 102, 408, 725$ (same values as in the experiments), and $\text{Bo} = 2500, 5000, 10^4, 10^5$. Circles are experimental measurements with $\text{Bo} = 102, 408, 725$, colored lines are finite-capillarity saddle-layer predictions, and the magenta line shows the capillary-free limit. In the inset, experiments are reported with their error bars, while the corresponding numerical simulations are shown as dashed lines, and the magenta dashed line is the numerical case without capillarity.

value and slope. The resulting solutions agree well with both numerical simulations and experiments (Fig. 3), without any fitting parameter, confirming that capillarity modifies the ridge profile but not the underlying ridge width scaling. Residual small discrepancies in Fig. 3 can be attributed, at early times, to the film not yet reaching the asymptotic $t^{-1/2}$ regime far from the saddle and the matching occurring in a region where the local metric is no longer well approximated as Cartesian, and finite Bond number effects in the drainage solution [11]. Furthermore, the comparatively larger experimental-numerical discrepancy observed for $\text{Bo} = 408$ likely results from the 3D printing resolution, whose magnitude relative to the geometric length scale L is largest in this case [24]. Eventually, a systematic expansion of the full model Eq. (1) leads to the same dominant balances and problems studied here, as shown in Supplemental Material [24]. Additional measurements performed with a lower-viscosity silicone oil ($\nu = 350 \pm 17 \text{ mm}^2 \text{ s}^{-1}$) show the same redistribution mechanism [24].

However, real saddles do not always balance gravity exactly along the principal directions, and fabrication tolerances can introduce a small nonzero curvature. Its effect is quantifiable and leaves the mechanism intact, as follows. A generic saddle with nonzero mean curvature may be written locally as $\eta \simeq \frac{1}{2}(y^2 - (1 - \mathcal{K}_0)x^2)$ with $\mathcal{K}_0$ being twice the saddle mean curvature, here assumed to be small. Along $x = 0$, Eq. (3) becomes $y(dH_d^2/dy) + \frac{2}{3}\mathcal{K}_0 H_d^2 + 1 = 0$, whose solution reads ($y > 0$)

$$H_d = \left\{ \frac{3}{2\mathcal{K}_0} \left[\left(\frac{y}{\exp[C]} \right)^{-2\mathcal{K}_0/3} - 1 \right] \right\}^{1/2} := \tilde{H}_d, \quad (8)$$

where C is set by global drainage, as before. For $\mathcal{K}_0 > 0$, H_d diverges algebraically, whereas for $\mathcal{K}_0 < 0$ it approaches the value $(-3/(2\mathcal{K}_0))^{1/2}$, with an infinite slope for $-3/2 < \mathcal{K}_0 < 0$. However, drainage holds only up to the saddle layer, whose $t^{-1/4}$ width scaling follows from Eq. (5), unaffected by $\mathcal{K}_0$, and imposes small slopes and $H_d = \mathcal{O}(1)$. Consequently, the power law can be converted into an exponential, namely $(y/\exp[C])^{-2\mathcal{K}_0/3} = \exp\{[2\mathcal{K}_0/3](C - \log y)\} = \exp[\frac{2}{3}\mathcal{K}_0 H_d^{*2}]$, and $\tilde{H}_d$ expanded for small $\mathcal{K}_0 H_d^{*2}$:

$$\tilde{H}_d = H_d^* \left[1 + \frac{\mathcal{K}_0}{6} (H_d^*)^2 + \mathcal{O}(\mathcal{K}_0^2 (H_d^*)^4) \right]. \quad (9)$$

Hence, for saddles with comparable principal curvatures, a small nonzero mean curvature produces only a perturbative correction to the logarithmic profile and leaves the regularization-layer width scaling unchanged. The ridge-formation mechanism therefore persists for a broad class of saddles with comparable principal curvatures. Because the drainage competition depends only on the local hyperbolic structure, it should arise generically on complex topographies, as numerically confirmed for another global substrate shape [24].

In summary, a smooth saddle generates a logarithmic ridge within a through-flow drainage state regularized by a dynamically selected region. Therefore, smooth saddles are not neutral elements on complex surfaces and act as organizing centers for coating flows alongside contact lines, substrate edges and defects. Restoring dimensions, the following scaling laws hold:

$$\frac{\hat{h}_s}{\hat{\ell}_b} \sim \frac{\hat{\ell}_b}{L} \left[C + \log \left(\frac{L}{\hat{\ell}_b} \right) \right]^{1/2}, \quad \hat{\ell}_b \sim L \left(\frac{\nu}{gL\hat{t}} \right)^{1/4}, \quad (10)$$

obtained by assuming $\zeta \sim 1$ at matching. Eliminating time yields a geometric relation between the instantaneous saddle thickness $\hat{h}_s$ and ridge width $\hat{\ell}_b$ that contains no explicit dependence on viscosity, gravity, and drainage time. This relation may guide local geometric compensation before shell fabrication. Within the Newtonian case considered here, the representative coating parameters for investment casting $\nu = 10^{-4} \text{ m}^2 \text{ s}^{-1}$, $L = 10 \text{ mm}$ and $\hat{t} = 60 \text{ s}$ give a ridge approximately 0.6 mm wide and a saddle thickness of order $10^2 \text{ }\mu\text{m}$, corresponding to an increase in maximum thickness of $\approx 100\%$ compared to the one developing on a cylindrical surface of radius equal to L [19], but in a ridge ten times narrower. At geophysical scales, the geometric scaling may help infer topography and lava properties from preserved ridges in mountainlike passes [33]. These examples motivate extensions to

interactions among multiple geometrical features, evolving substrates [34], complex and time-dependent rheology, as well as continuous injection and downstream evolution of propagating currents.

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P. G. L. and F. G. conceived the project. P. G. L. supervised the research and coordinated the research group. P. G. L., S. D., and F. G. developed the theory. P. G. L. and S. D. prepared and performed the numerical simulations. A. M. and P. G. L. designed and prepared the experiments. A. M. performed the experiments and postprocessed the data. P. G. L. prepared the draft. All authors contributed to the revision of the manuscript.

Data availability—The data that support the findings of this article are openly available [35].

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