

Circular Dichroism on the Edge of Quantum Hall Systems: From Many-Body Chern Number to Anisotropy Measurements

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Quantum Hall states are characterized by a topological invariant, the many-body Chern number, which determines their quantized Hall conductivity. This invariant also emerges in circular dichroic responses, namely, by applying a circular drive and comparing excitation rates for opposite orientations. This work explores the dichroic response of confined, isolated quantum Hall systems, where bulk and edge contributions cancel exactly: When the edge response is properly isolated, the circular dichroic signal becomes quantized, serving as a direct and elegant probe of the many-body Chern number encoded in the edge physics. We demonstrate that this quantized edge response is entirely captured by low-energy chiral edge modes, allowing for a universal description of this effect based on Wen's edge theory. Its low-energy nature implies that the quantized edge response can be distinguished from the bulk response in the frequency domain. The edge response is also shown to be a sensitive diagnostic of geometric features. This opens the possibility of characterizing the shape of quantum Hall droplets through edge spectroscopic measurements, without requiring knowledge of the system's boundary profile. We illustrate our findings using realistic models of integer and fractional Chern insulators, with different edge geometries, and propose detection schemes suitable for ultracold atoms.

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Introduction—Since the discovery of the quantum Hall (QH) effects [1], the robust quantization of macroscopic observables has been recognized as the manifestation of topology in physics [2–5], sparking tremendous attention in both solid state [6,7] and quantum engineered settings [8,9]. This vast exploration of topological states has revealed various quantized effects, accessible through a rich variety of experimental probes, such as transport, interferometry, spectroscopy, and tomography [6–9].

Coupling light to topological matter can lead to novel phenomena [10–12], and in particular, to quantized optical responses [13–23]. As a prominent example, circular dichroism (CD) has been proposed as a powerful probe

for extracting many-body topological invariants and quantum geometry in a broad class of topological systems [15,23–35]. This approach has been demonstrated in ultracold atoms, where the CD response was measured to evaluate the Chern number of Floquet-engineered Bloch bands [36]; it was also applied in solid-state qubit systems to extract the full quantum geometric tensor [37–40] and to reveal exotic monopoles [41,42]. The quantized CD response of strongly correlated fractional Chern insulators (FCIs), which reflects the underlying many-body Chern number [34], was also demonstrated through numerical studies [25].

Crucially, the relation between the many-body Chern number of a 2D Chern insulator (CI) and its quantized CD response implicitly assumes that the measured response entirely emanates from the bulk [15,34,43,44]. In fact, in a confined and isolated system (e.g., a quantum gas in a trap), the edge contribution to the CD response exactly cancels the quantized bulk response [Fig. 1(a)]; this effect can be attributed to the vanishing net conductivity of closed settings. Identifying schemes that isolate the bulk

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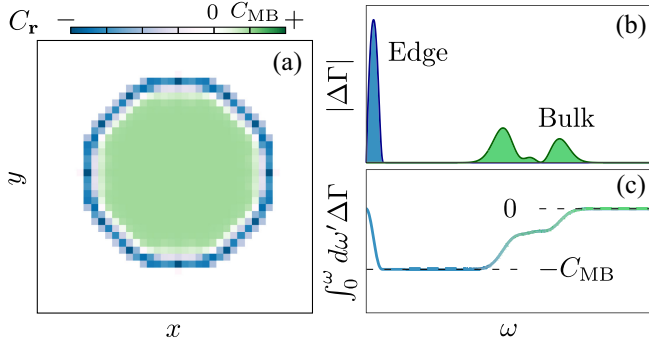


FIG. 1. (a) The dichroic response of a QH state is quantized in the bulk according to the many-body Chern number C_{MB} [Eq. (2)]. In a confined system, this bulk response is perfectly cancelled by the edge, as evidenced by the local Chern-marker description [Eq. (4)]. (b) The bulk and edge dichroic responses can be resolved in the frequency domain. (c) The response associated with low-energy edge modes is quantized, reflecting the many-body Chern number C_{MB} of the topological insulating state.

response [15] has thus been essential in this quantized CD framework.

In this work, we follow a different approach and set the focus on the quantized CD response associated with the edge of quantum Hall systems. We demonstrate that this quantized CD response is entirely captured by low-energy edge modes, as described by Wen’s chiral Luttinger liquid (χ LL) theory [45–48]. This result has multiple consequences: (i) the edge CD response is universal, it depends only on the specific topological order, and can be extracted from the edge of any QH state; (ii) the bulk and edge CD responses, which are both quantized in insulating states, can be individually isolated by resolving the dichroic response in the frequency domain [Fig. 1(b)]; (iii) the edge response is also a sensitive diagnostic of geometric features, which can be used to probe the shape of QH droplets. We argue that these quantized responses can be individually measured in cold-atom experiments [36], in the presence of well-designed trapping potentials [49,50]. We illustrate our findings by studying realistic models of Chern insulator and fractional Chern insulator states, considering different edge geometries and edge-mode configurations.

Quantized CD: Bulk versus edge contributions—We consider a general class of Chern insulators characterized by the many-body Chern number, C_{MB} , which can be integer or fractional depending on the nature of the state [3]. Such gapped many-body states exhibit a quantized Hall response, $\sigma_{xy}/\sigma_0 = -C_{\text{MB}}$, where σ_0 denotes the conductivity quantum [3]. Interestingly, one can relate this transport coefficient to a dissipative optical response, using Kramers-Kronig relations [51]: $\sigma_{xy} = (2/\pi) \int_0^\infty \omega^{-1} \sigma_{xy}^I(\omega) d\omega$. Here, the optical Hall conductivity $\sigma_{xy}^I(\omega)$ can be expressed in terms of excitation rates upon circular driving [15]

$$\sigma_{xy}^I(\omega) = \hbar\omega(\Gamma_+ - \Gamma_-)/8AE^2, \quad (1)$$

where E is the strength of the perturbing electric field $\mathbf{E}_\pm(\mathbf{r}, t) = E(\hat{e}_x \pm i\hat{e}_y)e^{-i\omega t} + \text{c.c.}$, $\Gamma_\pm(\omega)$ are the excitation rates associated with polarization $\pm$ and frequency ω , and A is the system’s area. Combining these results indicates that the many-body Chern number can be directly related to integrated excitation rates [15,25,34]

$$\frac{\hbar^2}{2AE^2} (\Gamma_+^{\text{int}} - \Gamma_-^{\text{int}}) = C_{\text{MB}}, \quad (2)$$

where we introduced the notation $\Gamma_\pm^{\text{int}} = \int_0^\infty \Gamma_\pm(\omega) d\omega$. Quantized CD [Eq. (2)] was measured in quantum engineered systems, where the excitation rates upon circular driving can be finely monitored, e.g., using band-mapping techniques [36]. This quantization law was also verified numerically for FCIs [25].

Crucially, Eq. (2) pertains to a purely *bulk* response and implicitly disregards any contributions to the excitation rates originating from the edge region. In fact, for any isolated and confined system, the edge and bulk responses perfectly cancel each other [15,43,44]

$$(\Gamma_+^{\text{int}} - \Gamma_-^{\text{int}})_{\text{edge}} = -(\Gamma_+^{\text{int}} - \Gamma_-^{\text{int}})_{\text{bulk}}, \quad (3)$$

which can be traced back to the vanishing Hall conductivity of any isolated setting with boundaries. For a noninteracting system, this can be appreciated by noting that the CD response in Eq. (2) can be expressed in terms of the local (real-space) Chern marker [15,52]

$$\frac{\hbar^2}{2E^2} (\Gamma_+^{\text{int}} - \Gamma_-^{\text{int}}) = \sum_{\mathbf{r}} C_{\mathbf{r}}. \quad (4)$$

As illustrated in Fig. 1(a), the bulk contributes to the quantized response in Eq. (2), $\sum_{\text{bulk}} C_{\mathbf{r}} = AC_{\text{MB}}$, which is exactly compensated by the edge response, $\sum_{\mathbf{r}} C_{\mathbf{r}} = 0$. In what follows, we rigorously establish that the edge CD response can be selectively isolated, thereby providing direct access to the many-body Chern number, an insightful manifestation of the bulk-edge correspondence.

Quantized CD as a low-energy edge response—The latter discussion brings us to further investigate the CD response associated with the edge region, the nature of which is *a priori* not obvious. Although, in principle, excitations localized at the edge can potentially occur at high energies (of the order of the bulk gap), we demonstrate that the edge CD response arises exclusively from low-energy, gapless modes localized at the system’s boundary; see Figs. 1(b) and 1(c). The quantized edge dichroism can indeed be understood from the viewpoint of the χ LL theory in the large particle-number ($N \gg 1$) limit. For simplicity, we first consider a QH droplet with a circular boundary and a single chiral edge mode, such as an integer QH state [53,54] or a Laughlin-type fractional QH state [55] at filling fraction $\nu = -C_{\text{MB}}$ in the lowest Landau level.

Generalizations to more sophisticated settings with multiple edge modes and anisotropic boundaries will be discussed in the following paragraphs.

Using Fermi's golden rule, we write the integrated rates entering Eq. (3) as $(\Gamma_{\pm}^{\text{int}})_{\text{edge}} = 2\pi E^2 S_{\pm} / \hbar^2$, where the structure factor $S_{\pm}$ is given by [15,44,56,57]

$$S_{\pm} = \sum_{n \in \text{edge}} \left| \int (x \pm iy) \langle n | \hat{\rho}(\mathbf{r}) | 0 \rangle d\mathbf{r} \right|^2 = \sum_{n \in \text{edge}} M_{\pm}^{n,0}. \quad (5)$$

Here, $\hat{\rho}(\mathbf{r})$ is the particle density operator, $|0\rangle$ denotes the ground state and the sum runs over the set of low-energy edge modes [47]. Since the bulk is incompressible, contributions to $S_{\pm}$ can only come from a thin layer at the edge of the system, at radius R , with a typical thickness set by the magnetic length. We can thus approximate $S_{\pm} \simeq R^2 \sum_n |\langle n | \hat{\rho}_{\pm 1} | 0 \rangle|^2$, where $\hat{\rho}_{\ell} = \int e^{i\ell\theta} \hat{\rho}_{\text{eff}}(\theta) d\theta$ is the angular Fourier transform of a one-dimensional edge density, $\hat{\rho}_{\text{eff}}(\theta) = \int \hat{\rho}(\mathbf{r}) r dr$. Following Wen's bosonization [46,47,63,64], we identify $\hat{\rho}_{+\ell} = \sqrt{|C_{\text{MB}}|} \ell \hat{b}_{\ell}^{\dagger}$, with the operator $\hat{b}_{\ell}^{\dagger}$ creating a bosonic excitation along the edge of the system, and $\hat{\rho}_{-\ell} = \hat{\rho}_{+\ell}^{\dagger}$ as its annihilation; here we explicitly used the relation between the filling fraction and the many-body Chern number. The ground state $|0\rangle$ is naturally interpreted as the vacuum of bosonic excitations and the excited states $|n\rangle$ as the states that can be obtained from it through the action of $\hat{b}_{\ell}^{\dagger}$. Since the edge mode carries a definite chirality, one obtains $S_{+} = |C_{\text{MB}}|A/\pi$, while the opposite orientation yields $S_{-} = 0$; here and from now on we assume $C_{\text{MB}} < 0$ [57]. As anticipated, we exactly recover Eqs. (2) and (3), where the sum rule is entirely exhausted by the gapless edge mode. As a consequence, edge and bulk CD responses can be distinguished via frequency selection, as we explicitly demonstrate below.

Having established that a quantized CD edge response is expected at low frequencies, we now illustrate this result by considering a bosonic Laughlin-type FCI state at filling factor $\nu = 1/2$, in the Harper-Hofstadter model with hard-core interactions [65–77]. Here, we add a weak harmonic potential $V_{m,n} = U_h(m^2 + n^2)$, which helps isolating a well-defined edge-mode branch [50,68,78], and stabilizes a Laughlin-like state with a circular edge. We study the CD response of this FCI state by calculating the coupling matrix elements $M_{\pm}^{n,0}$ in Eq. (5) for all (bulk and edge) eigenstates $|n\rangle$; we show our results in Fig. 2 as a function of the magnetic flux ϕ . In agreement with the χ LL prediction, we find $M_{+}^{n,0} \neq 0$ and $M_{-}^{n,0} = 0$ at low energy for a broad range of flux [Figs. 2(a) and 2(c)]. Since a single edge mode couples to the CD probe, the corresponding matrix element is directly related to the many-body Chern number $|C_{\text{MB}}| = 1/2$, as we demonstrate in Fig. 2(b). We note that this result can be used to identify the flux window within which the FCI state is stabilized. From Fig. 2(a), we also find that the energy of the relevant low-energy

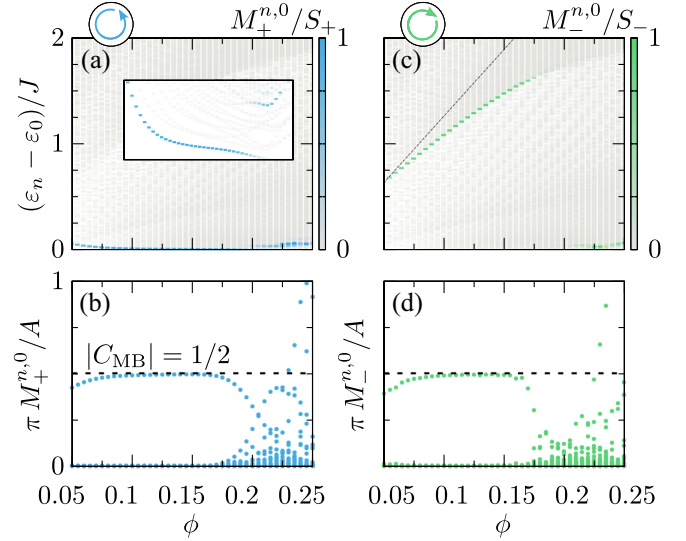


FIG. 2. (a) Energy spectrum for $N = 2$ hard-core bosons on a 12×12 square lattice, in the presence of a weak harmonic confinement of strength $U_h = 3.5 \times 10^{-3}J$, as a function of the flux ϕ . The points have been colored according to the magnitude of the matrix elements $M_{+}^{n,0}$. The inset shows a close-up on the low-energy features. (b) The same matrix elements, without energy resolution, and compared to the theory prediction (black dashed line). (c),(d) Same but for the matrix elements $M_{-}^{n,0}$, associated with the bulk response. Black dashes in (c) show the continuum-limit cyclotron gap $\hbar\omega_c/J = 4\pi\phi$.

mode decreases with increasing magnetic flux, as expected from a semiclassical description of the edge. In contrast, the probe with negative orientation couples to a single high-energy (bulk) mode, located approximately at the continuum cyclotron energy $\hbar\omega_c/J = 4\pi\phi$ [Fig. 2(c)]. In accordance with Eq. (3), the corresponding matrix element is also related to the many-body Chern number $|C_{\text{MB}}|$, hence exactly compensating the low-energy edge response. We note that it is the cyclotron gap (and not the many-body gap, which is much smaller) that sets the energy scale at which absorption of opposite angular-momentum quanta occurs [57].

This example illustrates how the bulk gap naturally induces a frequency separation, allowing isolation of the quantized CD edge response from the bulk; we performed similar calculations on other models [57], further highlighting the broad application of the result.

Multiple edge modes—The perfect edge dichroism ($S_{+} \neq 0$ and $S_{-} = 0$) discussed earlier is a distinctive result of a single edge mode circulating along a circular boundary. While the result $S_{-} = 0$ typically fails when multiple edge channels are present, our χ LL calculations predict that the difference $S_{+} - S_{-} = |C_{\text{MB}}|A/\pi$ is robustly quantized [57], consistent with Eqs. (2) and (3). This multi-edge-mode scenario is relevant for hierarchy [47,79] or bilayer [80] QH states, and more generally, in the presence of edge reconstruction [81–83]. Here, we illustrate this key result

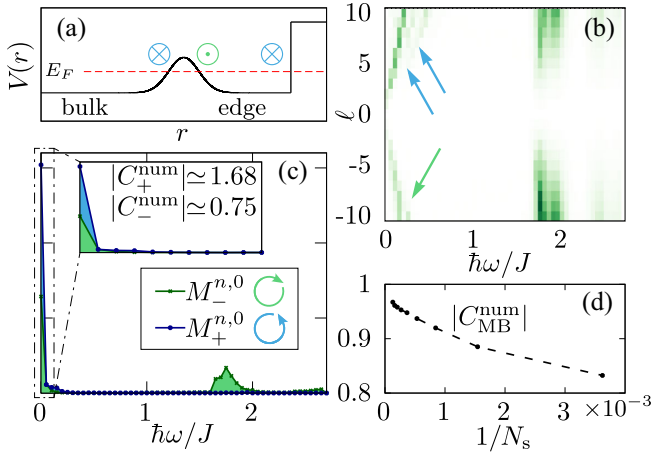


FIG. 3. (a) Multiple edge modes are introduced in a noninteracting integer quantum Hall state in Harper-Hofstadter bands by including, in addition to a circular hard wall at $r_{\text{wall}} = 14.5$, a ring-shaped potential bump of strength $0.5J$, centered two sites away from the walls. (b) Angular-momentum-resolved spectroscopy: Matrix elements $M_e^{n,0} = |\int d^2\mathbf{r} e^{i\ell\theta} \langle n | \hat{\rho}(\mathbf{r}) | 0 \rangle|^2$, as a function of frequency, $\omega_{n,0} = (\epsilon_n - \epsilon_0)/\hbar$, and transferred (optical) angular momentum ℓ ; ϵ_n is the energy of the n th eigenstate. The presence of multiple edge modes is highlighted by the arrows. (c) CD matrix elements $M_{\pm}^{n,0}$ [Eq. (5)], as a function of $\omega_{n,0}$, with the inset zooming in on the low-energy features. The numerically extracted Chern number contributions, $C_{\text{MB}}^{\text{num}} = C_+^{\text{num}} - C_-^{\text{num}}$, are calculated by isolating the low-energy response ($\hbar\omega \leq 1J$). The matrix elements are averaged over a small frequency window (of width $0.05J/\hbar$) for visualization. (d) $C_{\text{MB}}^{\text{num}}$ approaches the quantized value for increasing system size, denoted by the total number of occupied sites, N_s . We used $\phi = 2/7$ and $\epsilon_F = -1.5J$.

by considering: (i) a noninteracting CI state with reconstructed edges displaying three counter-propagating edge modes; and (ii) a strongly interacting two-component bosonic Halperin state displaying two co-propagating edge modes [47,84].

We first consider a CI of noninteracting fermions on the Harper-Hofstadter model, at flux $\phi = 2/7$ per plaquette, confined in a circular box. By setting the Fermi energy in the lowest gap, the CI state exhibits a single edge mode ($|C_{\text{MB}}| = 1$). We then add a small Gaussian potential bump, in the vicinity of the edge, in view of generating additional (counter-propagating) edge modes; see the sketch in Fig. 3(a). We confirm the appearance of two additional edge modes by performing angular-momentum resolved spectroscopy on the edge of the QH system [50,85], as we show in Fig. 3(b). We then represent the CD matrix elements $M_{\pm}^{n,0}$, as a function of frequency $\omega_{n,0} = (\epsilon_n - \epsilon_0)/\hbar$, in Fig. 3(c). Given the circular shape of the boundary, the two strong signals observed at low energy, $M_+^{n,0} \sim M_-^{n,0}$, indicate the presence of counter-propagating edge modes. We verify that their difference approaches the quantized result, $S_+ - S_- = |C_{\text{MB}}|A/\pi$, as one increases the system size; see Fig. 3(d).

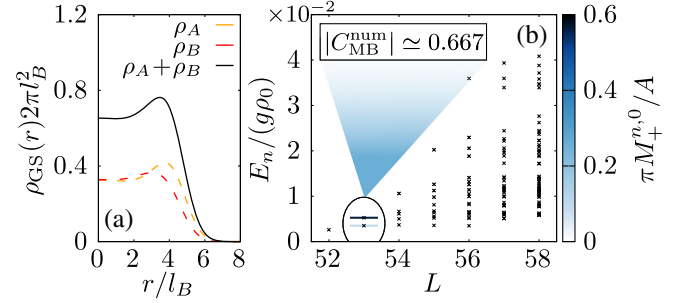


FIG. 4. (a) Radial density profile $\rho_A(\rho_B)$ of the $A(B)$ component of a Halperin (2,2,1) state [yellow (or red) dashes], and the combined density $\rho_A + \rho_B$ (black solid line). The bulk's density approaches $\rho_0 = \nu/(2\pi l_B^2)$, with $\nu = |C_{\text{MB}}| = 2/3$. (b) Low-lying portion of the system's spectrum E_n (black crosses) as a function of its angular momentum L ; horizontal colored lines display the $M_+^{n,0}$ matrix elements; both edge modes at $L = L_{(2,2,1)} + 1$ can be seen to carry a nonzero spectral weight, which together add up to the many body Chern number C_{MB} (inset).

We then consider the incompressible bosonic Halperin (2,2,1) state [84], which appears in the continuum as the densest ground state for a strongly interacting two-component system under a magnetic field [57], at angular momentum $L_{(2,2,1)} = N_A(N_A - 1) + N_B(N_B - 1) + N_A N_B$ with $N_{A/B}$ the number of bosons. This incompressible state has a bulk filling fraction $\nu = |C_{\text{MB}}| = 2/3$, and supports two co-propagating edge modes along its boundary [47]. We numerically study a system of $N_A = 5$ and $N_B = 6$ particles using exact diagonalization, and consider contact two-body interactions, $V_{12} = g\delta^{(2)}(r_1 - r_2)$. Figure 4(a) displays the radial density profile, which exhibits a clear plateau at $\rho_0 = (|C_{\text{MB}}|/2\pi l_B^2)$ ($l_B = \sqrt{\hbar/qB}$ denotes the magnetic length), indicating the formation of an incompressible bulk state. Figure 4(b) illustrates the many-body energy spectrum as a function of the total angular momentum, highlighting a nondegenerate ground state at $L = L_{(2,2,1)}$ close to zero energy, and the emergence of low-energy excitations for $L > L_{(2,2,1)}$. Note that we included an anharmonic ring-shaped box potential to make the edge modes dispersive [86,87]. A detailed examination of the matrix elements $M_{\pm}^{n,0}$ reveals the edge-mode contributions to the dichroic response. Below the bulk energy gap, the matrix elements $M_{\pm}^{n,0} \approx 0$ remain negligible [57]. On the other hand, as we show in Fig. 4(b), $M_+^{n,0} \neq 0$ exclusively for the two co-propagating low-energy edge modes located at $L - L_{(2,2,1)} = 1$, consistent with the angular-momentum transferred by the circular probe. One verifies that the sum of these two matrix elements yields the many-body Chern number of the underlying Halperin state, $|C_{\text{MB}}| = 2/3$, hence validating our theory in this strongly interacting multimode setting.

Anisotropic droplets—The quantized nature of the edge CD response is not restricted to systems with circular

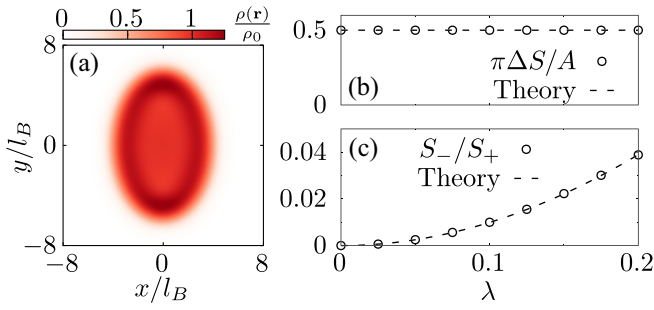


FIG. 5. (a) Density $\rho(x, y)$ of a Laughlin liquid in an anisotropic (elliptic) anharmonic trap, for a fixed ellipticity parameter $\lambda = 0.2$. (b) The two normalized rates $S_{\pm}$ (black circles) saturate the edge CD sum rule for all the ellipticities λ considered, with the expected value $|C_{MB}| = 1/2$ (black dashes). (c) The ratio S_-/S_+ reveals λ via Eq. (6).

symmetry. Indeed, a generalization of our χ LL calculation demonstrates that the prediction $S_+ - S_- = |C_{MB}|A/\pi$ holds for arbitrary edge geometries [57].

Interestingly, and quite generally, the ratio S_-/S_+ provides a direct measure of the anisotropy of the droplet. For instance, in the experimentally relevant case of an elliptic trap [88], $V = V(e^{2\lambda}x^2 + e^{-2\lambda}y^2)$, introducing an anisotropic edge, this ratio is given by [57]

$$S_-/S_+ = \tanh^2(\lambda). \quad (6)$$

We have verified this relation numerically, both for lattice and continuum models, for integer and fractional QH states, considering different types of confinement [57]. Figure 5 illustrates such a result for a Laughlin liquid trapped by an anisotropic (elliptic) anharmonic trap: in panel (a) we plot the ground state density; in (b) we verify the edge CD sum rule, while in (c) we demonstrate that the ratio S_-/S_+ gives access to the boundary's shape through Eq. (6).

A natural question arises: is it possible to distinguish the “intrinsic” contribution to $S_{\pm}$, stemming from counter-propagating modes, from the “extrinsic” contribution induced by boundary anisotropies? We argue that, at least in principle, the latter can be mitigated by modifying the geometry of the probe, e.g., replacing a circular probe with an elliptic one. In contrast, the intrinsic component should be robust against such geometric tuning. We leave a deeper investigation of this intriguing distinction for future work.

Summarizing, our results demonstrate that the edge CD response is not only a robust probe of topological invariants but also a sensitive diagnostic of geometric features. In experimental settings, this opens the possibility of characterizing the shape and symmetry of QH droplets through edge spectroscopic measurements, without requiring full knowledge of the boundary profile.

Local edge spectroscopy—We finally comment on the possibility of exciting the system with a generalized CD probe described by a time-dependent potential of the form

$U_{\pm}(\mathbf{r}, t) = Ef(r)e^{i(\pm\theta - \omega t)} + \text{c.c.}$, where $f(r)$ denotes a general profile function. Since the bulk is incompressible at low energy, the spatial profile of the probe is irrelevant deep in the bulk ($r \approx 0$). A straightforward generalization of our calculations above [57] yields $\Delta S = \sum_{n \in \text{edge}} (M_+^{n,0} - M_-^{n,0}) = |f(R)|^2 |C_{MB}|$: while the edge response encodes the many-body Chern number, it is also sensitive to the probe intensity at the edge ($r = R$). In particular, one recovers $\Delta S = |C_{MB}|A/\pi$ in the case of a uniform circular drive ($f(r) = r$). This analysis indicates that it is possible to extract the many-body Chern number of correlated insulators *locally* from the edge, e.g., employing a Laguerre-Gaussian-type probe [50,85] with $f(r) \approx 1$ at the system's edge and $f(r) \approx 0$ otherwise, which solely excites the boundary of the system [57].

Concluding remarks—Cold atoms in optical lattices [8] appear as a natural platform to measure the quantized edge response explored in this work: the edge-mode configuration could be finely controlled by adjusting the confining potential [49], and angular-momentum-sensitive spectroscopy could be applied locally on the edge of the cloud, using Laguerre-Gaussian beams [50,85,89]. These methods could be directly applied to few-atom FCI states recently realized in experiments [89,90]. In these settings, the edge spectrum is highly discretized, such that the relevant matrix elements $M_{\pm}$ could be directly extracted from Rabi oscillations [44,50,91]. It would be interesting to investigate extensions to non-Abelian topological matter, displaying exotic edge structures [92]. Finally, understanding how the intrinsic geometric degrees of freedom of fractional QH states [93,94] influence dichroic rates—both at the edge and in the bulk—constitutes an intriguing avenue for further investigation.

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Data availability—The data is available from the authors upon reasonable request.

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