

Infrared Singularities of Multileg Amplitudes with a Massive Particle at Three Loops

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We determine the complete three-loop QCD soft anomalous dimension for multileg amplitudes involving a single massive colored particle and any number of massless ones. This is achieved by applying a novel strategy based on a light-cone expansion of correlators of semi-infinite Wilson lines using the method of regions. The resulting region integrals depend exclusively on rescaling-invariant ratios that remain finite in the limit. We evaluate these integrals using differential equation techniques. The result is written in terms of uniform weight five generalized polylogarithms of a 12-letter alphabet in three variables, and is compatible with the massless limit as well as with two- and three-particle collinear factorization.

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Infrared singularities are an important feature of on-shell gauge-theory amplitudes. These singularities admit a highly constrained, universal structure [1–41]. Their origin in soft and collinear loop momentum regions guarantees their all-order factorization from the finite hard amplitude. Using renormalization-group equations, factorization can be used to prove that the singularities of any amplitude exponentiate in terms of the so-called soft anomalous dimensions. This universal quantity is therefore of unique importance in the study of amplitudes [25].

In QCD, infrared singularities present one of the key challenges for collider physics. Their cancellation in observables involves a sum over states of different multiplicity, requiring sophisticated subtraction techniques [42–48]. The singularities also give rise to enhanced logarithmic corrections, which often need to be resummed to achieve precise predictions. In any of these applications, the soft anomalous dimension is essential.

The soft anomalous dimension for multileg massless scattering is currently known to three loops [36]. However, in the case of massive particles, the state of the art is still two loops [49–51]. Important collider physics applications, such as resummation of top production cross sections [54–62], require knowledge of the three-loop soft anomalous dimension for processes involving heavy quarks alongside light quarks and gluons. Recently, first results for such corrections were obtained [63], namely those

associated with the interaction between one massive particle and two massless ones. In this Letter we compute the remaining contributions, and provide the complete three-loop result for the soft anomalous dimension in amplitudes involving one massive particle and any number of massless ones.

The soft anomalous dimension is remarkably simple in comparison with finite corrections to multileg amplitudes. In particular, its multileg components are spin-independent and feature rescaling symmetry with respect to the external momenta, as expected from its relation with Wilson-line correlators. This symmetry significantly reduces the number of independent kinematic variables on which the soft anomalous dimension depends, especially so in massless scattering [19–22,64]. However, it has been difficult to make use of this simplification when computing the soft anomalous dimension because multiplicative renormalizability [1–4] of Wilson-line correlators only holds when the Wilson lines are nonlightlike. Our novel strategy for computing the soft anomalous dimension [65] directly addressed this issue. We use the method of regions (MoR) [66–77] to perform an asymptotic light-cone expansion of correlators of timelike Wilson lines. In this way, the integrals we compute involve only the kinematic variables that remain finite in the limit of interest.

One-mass soft anomalous dimension—Consider a renormalized amplitude $\mathcal{M}$ with one massive particle p_Q and any number of massless particles $\{p_i\}$ [78]. Infrared singularities can be factorized as $\mathcal{M} = Z\mathcal{H}$ where the singularities are given by

$$Z = \text{P exp} \left\{ - \int_0^\mu \frac{d\tau}{\tau} \Gamma(\tau, \alpha_s(\tau^2)) \right\} \quad (1)$$

and $\mathcal{H}$ is the finite hard amplitude. The one-mass soft anomalous dimension is

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$$\begin{aligned}
 \Gamma = & \sum_{i < j} \mathbf{T}_i \cdot \mathbf{T}_j \gamma_{\text{cusp}}(\alpha_s) \log \left(-\frac{\tau^2}{2p_i \cdot p_j} \right) + \sum_i \gamma_i(\alpha_s) + \sum_i \mathbf{T}_i \cdot \mathbf{T}_Q \gamma_{\text{cusp}}(\alpha_s) \log \left(-\frac{\tau \sqrt{p_Q^2}}{2p_i \cdot p_Q} \right) + \Omega_Q(\alpha_s) \\
 & + \sum_{i < j < k < q} \sum_{(u,v,w) \in P} \mathbf{T}_{uv;wq} \mathcal{F}_{0,4}(\rho_{uvwq}, \rho_{vwuq}) + \mathcal{F}_{0,3} \sum_i \sum_{j < k, j, k \neq i} \mathbf{T}_{ij;ik} + \sum_{i < j < k} \sum_{(u,v,w) \in P} \mathbf{T}_{uv;wQ} \mathcal{F}_{1,3}(r_{uwQ}, r_{vwQ}, r_{uvQ}) \\
 & + \sum_{i < j} \mathbf{T}_{iQ;jQ} \mathcal{F}_{1,2}(r_{ijQ}) + \mathcal{O}(\alpha_s^4),
 \end{aligned} \tag{2}$$

where $\mathbf{T}_i$ and p_i are the color generator [79] and outgoing four-momentum associated with parton i . We assume here that all scalar products $p_i \cdot p_j$ are spacelike; physical kinematics can be reached by analytic continuation. The dipole terms entering the first two lines in Eq. (2) start at one loop, while the quadrupole ones in the subsequent lines start at three loops, $\mathcal{F}_{h,l} = \sum_{n=3}^{\infty} (\alpha_s/4\pi)^n \mathcal{F}_{h,l}^{(n)}$. Here, the subscripts h and l count the number of heavy and light particles, respectively. In Eq. (2) we suppress any contribution associated with two or more heavy particles, and any new structures appearing from four loops [39–41]. The lightlike γ_{cusp} is known to four loops [80–93], while the color-singlet terms, γ_i and Ω_Q , are known to four loops [92–97] and three loops [52,53,98], respectively [99].

In Eq. (2) $(u, v; w) \in P \equiv \{(i, k; j), (j, k; i), (i, j; k)\}$ defines the three quadrupole channels, having color structures

$$\mathbf{T}_{uv;wq} \equiv f^{abe} f^{cde} \{\mathbf{T}_u^a, \mathbf{T}_v^b, \mathbf{T}_w^c, \mathbf{T}_q^d\}_+, \tag{3}$$

where the curly brackets are defined, as in Refs. [39,40,104], as a symmetric sum over all permutations of the n generators it contains, divided by $n!$ [105]. Beyond Bose symmetry, the three channels in P are related by the Jacobi identity $f^{abe} f^{cde} + f^{ade} f^{bce} = f^{ace} f^{bde}$.

A salient feature of the quadrupole corrections is that they depend on the momenta only via rescaling-invariant ratios [19–22,36,64,106] of the four-momenta p_u , or equivalently the four-velocities β_u . In the massless case, these are

$$\rho_{ijkq} \equiv \frac{p_i \cdot p_j p_k \cdot p_q}{p_i \cdot p_k p_j \cdot p_q} = \frac{\beta_i \cdot \beta_j \beta_k \cdot \beta_q}{\beta_i \cdot \beta_k \beta_j \cdot \beta_q}, \tag{4}$$

where $\mathcal{F}_{0,4}$ involves just two independent cross ratios. In contrast, in the one-mass case, there are three independent rescaling-invariant ratios of the form [63]

$$r_{ijQ} \equiv \frac{p_Q^2 p_i \cdot p_j}{2p_i \cdot p_Q p_j \cdot p_Q} = \frac{\beta_Q^2 \beta_i \cdot \beta_j}{2\beta_i \cdot \beta_Q \beta_j \cdot \beta_Q}. \tag{5}$$

The massless quadrupole terms, $\mathcal{F}_{0,3}$ and $\mathcal{F}_{0,4}$, were computed at three loops [36,37],

$$\mathcal{F}_{0,3}^{(3)} = 32[2\zeta(2)\zeta(3) + \zeta(5)] \equiv 32C,$$

$$\begin{aligned}
 \mathcal{F}_{0,4}^{(3)}(\rho_{ijkq}, \rho_{kjiq}) &= 16[F_{0,4}(1 - \bar{z}, 1 - z) - F_{0,4}(z, \bar{z})], \\
 F_{0,4}(z, \bar{z}) &\equiv \mathcal{L}_{10101}(z) + 2\zeta(2)[\mathcal{L}_{100}(z) + \mathcal{L}_{001}(z)],
 \end{aligned} \tag{6}$$

where the variables z and $\bar{z}$ are defined by

$$\rho_{ijkq} \equiv z\bar{z}, \quad \rho_{kjiq} \equiv (1 - z)(1 - \bar{z}), \tag{7}$$

and $\mathcal{L}(z)$ are single-valued [107,108] harmonic polylogarithms. They are single-valued when $z = (\bar{z})^*$. The expansion of $F_{0,4}(z, \bar{z})$ in ordinary (multivalued) generalized polylogarithms (GPLs) of z and $\bar{z}$ has been provided in Ref. [36]. Note that $\mathcal{F}_{0,4}$ in Eq. (6) is antisymmetric under $z \leftrightarrow 1 - \bar{z}$, in line with Bose symmetry and the antisymmetry of the color structure $\mathbf{T}_{ik;jq}$ under $i \leftrightarrow k$ permutation.

The one-mass three-line quadrupole, which depends on a single ratio r_{ijQ} , was obtained in Ref. [63], $\mathcal{F}_{1,2}^{(3)}(r_{ijQ}) = 2F_{h2}(r_{ijQ})$. Thus the only unknown ingredient in Eq. (2) prior to this work is $\mathcal{F}_{1,3}^{(3)}$ of the penultimate line, which describes long-distance interaction between the heavy particle and three massless ones. In this Letter we compute $\mathcal{F}_{1,3}^{(3)}$ using a novel method based on the light-cone expansion of Wilson-line correlators; representative diagrams are shown in Fig. 1 [109]. We proceed by reviewing the result and then summarize the key features of the new computation strategy.

The result—To present the result for $\mathcal{F}_{1,3}^{(3)}$ we define, in analogy with $\mathcal{F}_{0,4}^{(3)}$ in Eq. (6),

$$\begin{aligned}
 \mathcal{F}_{1,3}^{(3)}(r_{ijQ}, r_{jkQ}, r_{ikQ}) \\
 = 16[F_{1,3}(x; 1 - \bar{z}, 1 - z) - F_{1,3}(x; z, \bar{z})],
 \end{aligned} \tag{8}$$

where the variables $\{x, z, \bar{z}\}$ are defined by

$$\begin{aligned}
 r_{ijQ} &\equiv \frac{\beta_Q^2 \beta_i \cdot \beta_j}{2\beta_i \cdot \beta_Q \beta_j \cdot \beta_Q} = \frac{x(x + z - \bar{z})}{(1 - z)(1 - \bar{z})}, \\
 r_{jkQ} &\equiv \frac{\beta_Q^2 \beta_j \cdot \beta_k}{2\beta_j \cdot \beta_Q \beta_k \cdot \beta_Q} = \frac{x(x + z - \bar{z})}{z\bar{z}}, \\
 r_{ikQ} &\equiv \frac{\beta_Q^2 \beta_i \cdot \beta_k}{2\beta_i \cdot \beta_Q \beta_k \cdot \beta_Q} = \frac{x(x + z - \bar{z})}{(1 - z)(1 - \bar{z})z\bar{z}},
 \end{aligned} \tag{9}$$

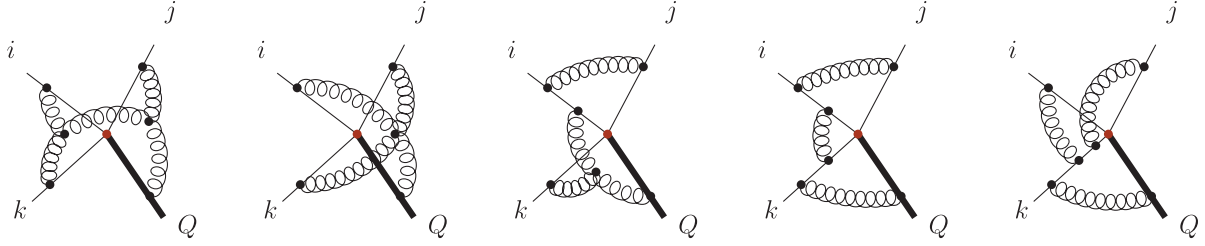


FIG. 1. Representative three-loop diagrams contributing to the quadrupole structure. The thick line corresponds to the timelike Wilson line with velocity β_Q , while the thin lines to Wilson lines with nearly lightlike velocities. The two leftmost diagrams depict connected webs, W_{1111} , the middle one is a representative diagram of the W_{1112} web, and the two rightmost ones belong to multiple gluon-exchange webs, W_{1122} and W_{1113} , respectively.

allowing us to express $F_{1,3}$ using GPLs without square roots. In [110] we provide the explicit expression of $F_{1,3}$, written in terms of uniform weight-five GPLs of $\{x, z, \bar{z}\}$ and Riemann zeta values with rational numerical coefficients. Permutation symmetry in $i \leftrightarrow k$ is realized in Eq. (8) via $z \leftrightarrow 1 - \bar{z}$ antisymmetry, just as in Eq. (6). Note that $F_{1,3}$ is invariant under the following Galois symmetries:

$$F_{1,3}(x; z, \bar{z}) = F_{1,3}(-x; \bar{z}, z) = F_{1,3}(-x - z + \bar{z}; z, \bar{z}), \quad (10)$$

related to flipping the sign of the square roots that appear upon solving (9) for $\{x, z, \bar{z}\}$. The full one-mass quadrupole in Eq. (2) is obtained by performing the three permutations on $F_{1,3}$ of Eq. (8), as prescribed by P [111].

In the new variables, the alphabet of $\mathcal{F}_{1,3}^{(3)}$ is

$$\begin{aligned} \{\omega_i\}^{\text{phy}} = & \{z, \bar{z}, 1 - z, 1 - \bar{z}\} \cup \{x, \bar{z} - z - x\} \\ & \cup \{x + z, \bar{z} - x, 1 - z - x, 1 - \bar{z} + x, z + x \\ & - z\bar{z}, \bar{z} - x - z\bar{z}\}. \end{aligned} \quad (11)$$

These 12 letters represent physical collinear singularities, $\beta_u \cdot \beta_v \rightarrow 0$. The first group of four letters corresponds to $r_{uvQ} \rightarrow \infty$ and coincides with the full alphabet of $\mathcal{F}_{0,4}^{(3)}$ in Eq. (6). The next group of two letters corresponds to $r_{uvQ} \rightarrow 0$, while the final group of six letters to $r_{uvQ} \rightarrow 1$ [112]. The first symbol entry of $\mathcal{F}_{1,3}^{(3)}$, associated with its channel discontinuities, is more restricted and only consists of the three ratios r_{uvQ} , with branch cuts for $r_{uvQ} > 0$.

The collinear limits of $\mathcal{F}_{1,3}^{(3)}$ are consistent with the collinear factorization constraints [63,104], yielding

$$\mathcal{F}_{1,3}^{(3)}(0, r, r) = -\mathcal{F}_{1,3}^{(3)}(r, 0, r) = 2\mathcal{F}_{0,3}^{(3)}, \quad (12)$$

or equivalently, for $F_{1,3}(x; z, \bar{z})$ [113],

$$F_{1,3}(0; 1, 1) = 4C, \quad F_{1,3}(0; 0, 0) = 0. \quad (13)$$

The light-cone limit $\beta_Q^2 \rightarrow 0$ is obtained by taking $x \rightarrow 0$,

$$F_{1,3}(0; z, \bar{z}) = F_{0,4}(z, \bar{z}), \quad (14)$$

recovering the known result [36] of Eq. (6). Equation (14) also represents the triple collinear limit, $\beta_i || \beta_j || \beta_k$ [65,104]. Figure 2 shows $F_{1,3}(x; z, \bar{z})$ in the Euclidean domain, where it is real-valued. The figure illustrates that the computed result satisfies both the two-particle collinear limit (13) and the massless limit (14).

Computation strategy—We start from the correlator [114] of four timelike Wilson lines ($\beta_U^2 > 0$),

$$\left\langle \Phi_{\beta_i}^{(m)} \Phi_{\beta_j}^{(m)} \Phi_{\beta_k}^{(m)} \Phi_{\beta_Q}^{(m)} \right\rangle = \exp \left[\sum_n \left(\frac{\alpha_s}{4\pi} \right)^n w^{(n)} \right], \quad (15)$$

which is multiplicatively renormalizable. Specifically, the renormalization of the multi-Wilson-line vertex involves the very same quadrupole terms appearing in Γ of Eq. (2), barring the fact that in the latter three or four velocities are expanded near the light cone. As indicated by the superscript (m) , the correlator is regularized in the infrared. While the precise regularization prescription does not affect the ultraviolet pole we aim to extract, we follow previous work [32,36], and introduce exponential suppression of the

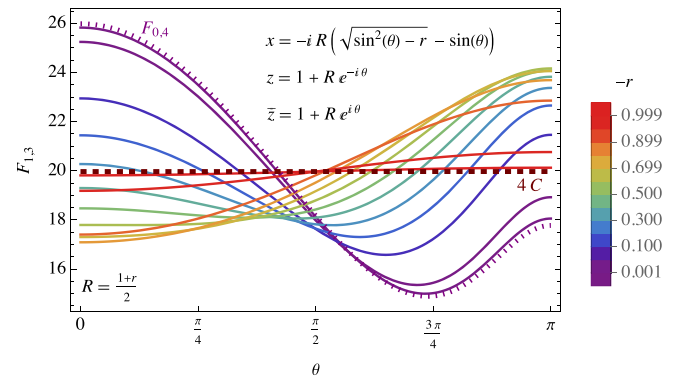


FIG. 2. $F_{1,3}(x; z, \bar{z})$ in Euclidean kinematics with $(z, \bar{z})$ parametrized in polar coordinates. It is sampled along the line $R = (1 + r)/2$ at fixed values of $r = r_{ijQ} < 0$, interpolating between the collinear limit $\beta_j || \beta_k$ ($R = 0$) of Eq. (13) and the lightlike limit of Eq. (14) at $r = 0$.

gluon coupling to the Wilson line at large distances, inversely proportional to $m\sqrt{\beta_U^2}$. Equivalently, in momentum space, the regulator induces a $m\sqrt{\beta_U^2}$ shift in the denominator of the eikonal propagator. The dependence of the regulator on the Wilson-line squared velocity ensures that the correlator remains invariant under the rescaling of each velocity.

Having defined the regularized correlator (15), we follow the strategy described in Ref. [65], and use the MoR [66–77] to perform an asymptotic light-cone expansion in

$$\beta_I^2 \sim \lambda_i, \quad \beta_J^2 \sim \lambda_j, \quad \beta_K^2 \sim \lambda_k. \quad (16)$$

We stress that the naive limit—dubbed the “hard region”—corresponding to taking the squared velocities in (16) to zero in the integrand, amounts to losing infrared regularization along the lightlike lines, therefore generating collinear singularities in addition to the ultraviolet ones. This leads to higher-order poles in ϵ , which cannot be linked with the renormalization of the multi-Wilson-line vertex. However, the full asymptotic expansion obtained by the MoR restores the multiplicatively renormalizable correlator [65], replacing the higher-order ϵ poles in each web with logarithms of the expansion parameters in (16). These logarithms ultimately cancel in the correlator in the sum of all webs, where one can read off the soft anomalous dimension. This strategy maximizes the simplification afforded by the lightlike limit, as each region integral is manifestly free of any dependence on the small squared velocities.

Upon computing the correlator in the Euclidean regime, where $\beta_U \cdot \beta_V < 0$ for any pair of velocities $U, V \in \{I, J, K, Q\}$, we guarantee that all the monomials in the Symanzik graph polynomials in any integral are non-negative, and hence there are no hidden regions [115,116]. Thus, the complete set of regions can be determined using the facets of the Newton polytope of the graph polynomials [68,70–75,77]. In this Letter we used the packages `pySecDec` [117] and `AmpRed` [77] to determine the region vectors.

Performing the perturbative expansion of the correlator (15), one finds that there are four types of four-line webs (see Fig. 1) [26–33,36,37] that contribute to the color quadrupole structure $\mathbf{T}_{uv;wQ}$ at three loops [36],

$$\left(\frac{\alpha_s}{4\pi}\right)^3 w^{(3)} = W_{1111} + W_{1112} + W_{1122} + W_{1113} + \dots, \quad (17)$$

where the subscripts on W count the number of attachments to each of the four Wilson lines. Each W term represents the sum of all webs of a given type in the Feynman gauge. The ellipsis represents webs that span fewer lines.

Connected webs— W_{1111} represents the sum of all connected diagrams spanning four lines. These feature either a

single four-gluon vertex or two three-gluon vertices (Fig. 1). These diagrams are special: they are the only ones that give rise to integrals depending on all three rescaling-invariant ratios of Eq. (9). We therefore focus our description of the calculation on these webs [118]. The structure of W_{1111} is

$$\begin{aligned} W_{1111} = & \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_K^c \mathbf{T}_Q^d [f^{abe} f^{cde} \mathcal{Y}_{(IJ;KQ)}(\{\alpha_{VU}\}) \\ & + f^{ace} f^{bde} \mathcal{Y}_{(IK;JQ)}(\{\alpha_{VU}\}) \\ & + f^{ade} f^{bce} \mathcal{Y}_{(JK;IQ)}(\{\alpha_{VU}\})]. \end{aligned} \quad (18)$$

The three channels are related by permutations, so let us now consider $\mathcal{Y}_{(IJ;KQ)}$. Relating the correlator (15) to the anomalous dimension (2) requires computing $\mathcal{Y}$ with four timelike Wilson lines, so prior to the expansion, the kinematic dependence is via the set $\{\alpha_{VU}\}$ of six rescaling-invariant variables,

$$\alpha_{VU} + \frac{1}{\alpha_{VU}} \equiv -\frac{2\beta_V \cdot \beta_U}{\sqrt{\beta_V^2 \beta_U^2}}. \quad (19)$$

Being connected, these diagrams feature a single ultraviolet pole in ϵ [4],

$$\mathcal{Y}_{(IJ;KQ)}(\{\alpha_{VU}\}) = \sum_{l=-1}^{\infty} \mathcal{Y}_{(IJ;KQ)}^{(l)}(\{\alpha_{VU}\}) \epsilon^l. \quad (20)$$

As a consequence it is sufficient to regularize the infrared on a single Wilson line [120]. We therefore drop the regulator m on the lines I, J and K , while keeping it on line Q , without affecting the ultraviolet $1/\epsilon$ pole we are interested in. In this setup the expansions in the three parameters of Eq. (16) commute.

The asymptotic expansion $\mathcal{T}$ of the Laurent coefficients in Eq. (20) takes the form a log-power expansion. Specifically, at the leading order in ϵ , $\mathcal{O}(\epsilon^{-1})$, we have

$$\begin{aligned} \mathcal{T}_{\lambda_i, \lambda_j, \lambda_k} [\mathcal{Y}_{(IJ;KQ)}^{(-1)}(\{\alpha_{VU}\})] \\ = \mathcal{Y}_{(ij;kQ)}^{(-1)}(r_{ikQ}, r_{jkQ}, r_{ijQ}; \lambda_i, \lambda_j, \lambda_k) + \mathcal{O}(\lambda), \end{aligned} \quad (21)$$

where $\mathcal{Y}_{(ij;kQ)}^{(-1)}$ on the right-hand side depends on the kinematics via the ratios r_{uvQ} as well as the three logarithms of the expansion parameters λ_u of Eq. (16). We compute the asymptotic light-cone expansion using the MoR, which yields a sum over region integrals,

$$\begin{aligned} \mathcal{T}_{\lambda_i, \lambda_j, \lambda_k} [\mathcal{Y}_{(IJ;KQ)}(\{\alpha_{VU}\})] \\ = \sum_{[n_i, n_j, n_k]} \lambda_i^{n_i} \lambda_j^{n_j} \lambda_k^{n_k} \mathcal{Y}_{(ij;kQ)}^{[n_i, n_j, n_k]}(r_{ikQ}, r_{jkQ}, r_{ijQ}), \end{aligned} \quad (22)$$

where we neglected all subleading powers in the expansion parameters λ_i , λ_j , and λ_k . Here, the regions are designated by $[n_i, n_j, n_k]$, identifying their scaling law $\lambda_u^{n_u \epsilon}$ for $u \in \{i, j, k\}$, where n_u are integers. In each region integral we have scaled out the nonanalytic dependence on the expansion parameters. Crucially, any given region integral $\mathcal{Y}_{(ij;kQ)}^{[n_i, n_j, n_k]}$ depends *only* on the three r_{uvQ} ratios, and not on the expansion parameters. Individual region integrals feature higher-order poles in ϵ . These cancel in the sum of

regions, leaving a trace in $\mathcal{Y}_{(ij;kQ)}^{(-1)}$ as logarithms of the three expansion parameters, in line with Eq. (21).

Applying the MoR to W_{1111} , we get 56 regions for each of the three channels. Most of the region integrals are simple to evaluate; the most complicated eight region integrals depend on all three r_{uvQ} ratios via polylogarithms. These are the hard region, where the three Wilson lines are taken to be strictly lightlike, and seven other regions which feature one, two, or three collinear modes,

$$\begin{aligned} \mathcal{T}_{\lambda_i, \lambda_j, \lambda_k} [W_{1111}] = & \text{Diagram 1} = \text{Diagram 2} + \lambda_i^{-\epsilon} \text{Diagram 3} + \lambda_j^{-\epsilon} \text{Diagram 4} + \lambda_k^{-\epsilon} \text{Diagram 5} \\ & + \lambda_i^{-\epsilon} \lambda_j^{-\epsilon} \text{Diagram 6} + \lambda_i^{-\epsilon} \lambda_k^{-\epsilon} \text{Diagram 7} + \lambda_j^{-\epsilon} \lambda_k^{-\epsilon} \text{Diagram 8} + \lambda_i^{-\epsilon} \lambda_j^{-\epsilon} \lambda_k^{-\epsilon} \text{Diagram 9} + \dots \end{aligned} \quad (23)$$

Here, the gluons are color-coded according to the loop-momentum modes [65]: the blue gluons are hard, while the green, orange, and brown gluons are collinear to the i , j , and k directions, respectively. The blob represents all the possible ways in which the four gluons connect through a single four-gluon vertex or two three-gluon vertices.

We use the method of differential equations [121] to compute the region integrals. Using the packages AmpRed [77], Kira [122–124], and LiteRed [125,126], we derive the differential equation for a subset of the regions of one channel, obtaining the rest by permutations. In addition to the singularities at $r_{uvQ} \rightarrow \{0, 1, \infty\}$, represented by the alphabet (11), the differential equations display singularities also at $\Delta_1 \rightarrow 0$ and $\Delta_2 \rightarrow 0$, where

$$\Delta_1 = \Delta(r_{ijQ}, r_{ikQ}, r_{jkQ}), \quad \Delta_2 = \Delta_1 + 4r_{ijQ}r_{jkQ}r_{ikQ},$$

where Δ is the Källén function. Δ_1 and Δ_2 are quadratic and cubic polynomials in the r_{ijQ} variables, respectively. In the parametrization (9) this corresponds to the following extended alphabet:

$$\{\omega_i\} = \{\omega_i\}^{\text{phy}} \cup \{z - \bar{z}, 2x + z - \bar{z}\}, \quad (24)$$

where the last two letters originate exclusively in Δ_1 and Δ_2 [127]. These additional letters only appear in W_{1111} , not in any other contribution to the correlator.

In order to solve the differential equation, we bring it to an ϵ -factorized form [121]. To this end we first identify one uniformly transcendental integral. Based on the experience

from calculating the fully lightlike limit [36,37], one expects that this would be a property of the specific integral entering the anomalous dimension, i.e., $\mathcal{T}_{\lambda_i, \lambda_j, \lambda_k}[\mathcal{Y}_{(IJ;KQ)}]$. This is indeed so. Moreover, each region (characterized by distinct scaling $\lambda_i^{n_i \epsilon} \lambda_j^{n_j \epsilon} \lambda_k^{n_k \epsilon}$) is individually uniformly transcendental. Based on this, using the package Initial [129], we obtained a canonical form of the differential equations of each region. The general solution is then obtained order by order in ϵ in terms of generalized polylogarithms using PolyLogTools [130].

Important simplifications occur in W_{1111} upon summing all region integrals. As discussed in Supplemental Material [100], the result is compatible with a range of physical constraints, including Bose symmetry, the first-entry condition, a single ultraviolet $1/\epsilon$ pole, the known $\beta_Q^2 \rightarrow 0$ limit, obtained in the computation of the connected webs in Refs. [36,37], and the known collinear limit for the soft anomalous dimension, determined by the strict collinear factorization requirement [63,104]. These constraints, put together, fully fix the boundary data, yielding a unique result for W_{1111} . We stress that in contrast to individual region integrals, this result is free of higher-order poles in ϵ , and the leading order $\mathcal{Y}_{(ij;kQ)}^{(-1)}$ is free of singularities at Δ_1 and Δ_2 . It is a uniform weight-five function with GPLs of the physical alphabet (11) and logarithms of the expansion parameters.

Other webs—The remaining webs [131] in Eq. (17) do not involve functions of all three kinematic variables and are therefore simpler. The most complicated of these is

W_{1112} (see Fig. 1), which has one three-gluon vertex. The calculation strategy is similar to that of the connected webs. In contrast to W_{1111} , the W_{1112} integrals feature double poles in ϵ , and hence we computed them regularizing all four Wilson lines. Further details on the computation will be provided elsewhere [119]. The multi-gluon-exchange webs in Eq. (17), W_{1122} and W_{1113} (see Fig. 1), have been computed with all four velocities timelike in terms of GPLs [132,133]. It is therefore straightforward to expand these results according to Eq. (16).

In addition to the four-line webs in Eq. (17), three- and two-line webs also contribute to $F_{1,3}$ upon applying color conservation [36,37,134]. These contributions can either be obtained from known results [37] or deduced using Eq. (12). In the sum of the four-line webs and those with fewer lines, we find that the λ dependence entirely cancels out, so $F_{1,3}$ only depends on the variables $\{x, z, \bar{z}\}$, as expected.

Conclusions—In this Letter we computed the three-loop soft anomalous dimension governing the singularities of QCD amplitudes with one heavy particle and any number of massless ones, summarized in Eq. (2). All ingredients except the most complicated one—the quadrupole function $\mathcal{F}_{1,3}^{(3)}(r_{ijQ}, r_{jkQ}, r_{ikQ})$ correlating the heavy particle with three massless ones—have been determined before, so powerful consistency checks are available. In particular, the lightlike limit of $\mathcal{F}_{1,3}^{(3)}$ reproduces the well-known massless quadrupole function $\mathcal{F}_{0,4}^{(3)}(\rho_{ijkq}, \rho_{kjiq})$ [36], as stated in Eq. (14). The latter also describes the three-particle collinear limit [104]. Furthermore, the two-particle collinear limits of $\mathcal{F}_{1,3}^{(3)}$ are independent of any kinematic variables, and are given by Eq. (12), in line with the constraints deduced from strict collinear factorization [63,104].

Beyond said kinematic limits, our result for $\mathcal{F}_{1,3}^{(3)}$ admits a highly constrained analytic structure in line with expectations: the symbol alphabet (11) directly relates to physical collinear singularities corresponding to $r_{uvQ} \rightarrow \{0, 1, \infty\}$. The only discontinuity accessible from the first sheet is the branch cut for $r_{uvQ} > 0$, a manifestation of the first-entry condition. The result respects the two Galois symmetries in Eq. (10). Finally, the three channels related by permuting lightlike velocities are all real in the Euclidean regime.

To obtain the new result for $\mathcal{F}_{1,3}^{(3)}$ we used a novel strategy, which retains the advantages of multiplicatively renormalizing a manifestly process-independent, gauge-invariant, and rescaling-invariant correlator of timelike semi-infinite Wilson lines, while taking full advantage of the simplification afforded by the lightlike limit through the use of the Method of Regions [65]. The most striking aspect of this simplification is the fact that each of the region integrals $\mathcal{Y}_{(ij;kQ)}^{[n_i, n_j, n_k]}$ we computed depends on just three rescaling-invariant ratios, the variables that remain finite in the limit, rather than the full set of six independent variables on which the correlator depends. This strategy opens the

way to compute the soft anomalous dimension at higher loop orders and for amplitudes involving more heavy particles at three loops.

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Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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