

Euler Buckling on Curved Surfaces

Shiheng Zhao (赵世恒)¹ and Pierre A. Haas^{2*}

¹Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, 01187 Dresden, Germany;
²Max Planck Institute of Molecular Cell Biology and Genetics, Pfotenhauerstraße 108, 01307 Dresden, Germany;
 and Center for Systems Biology Dresden, Pfotenhauerstraße 108, 01307 Dresden, Germany

(Received 4 April 2025; revised 31 July 2025; accepted 19 November 2025; published 10 December 2025)

Euler buckling epitomizes mechanical instabilities: an inextensible straight elastic line in the plane buckles under compression when the compressive force F reaches a critical value $F_* > 0$. But how does an elastic line buckle within a general curved surface? Here, we reveal that the classical instability changes fundamentally: by weakly nonlinear analysis of the buckling of an asymptotically short elastic line, we show that the critical force for the lowest buckling mode is $F_* = 0$ and discover a new bifurcation structure in which the modes of classical Euler buckling split into pairs. For long elastic lines, we numerically find an additional bifurcation by which the second of these new modes becomes the lowest mode and show that, at sufficiently large F , they snap discontinuously to higher end-to-end compression. Our results constitute the foundations for a class of buckling instabilities that arise within curved surfaces, for example when biological shape emerges in development.

DOI: 10.1103/63py-ph5s

In 1744, Euler [1] described the paradigm of a mechanical instability [2]: an elastic rod of length ℓ compressed by a force F applied to its ends buckles when F reaches a critical load $F_* \propto \ell^{-2}$. Two decades later, Lagrange [3] analyzed the higher buckling modes that become possible as F is increased further. The full nonlinear description of this instability was finally obtained in Ref. [4], which also reviewed the history of this fundamental mechanics problem. This nonlinear description is a culmination of two more centuries of work that saw the development of Euler's model of elasticity into the modern field theory of solid mechanics [5].

Much more recently, the importance of related mechanical instabilities for the emergence of biological shape has begun to be appreciated [6]. The best known instantiations of such instabilities are perhaps the gyrations of the brain [7–13] and the villi of the gut [14–17]; more exotic examples include the frill of the lizard *Chlamydosaurus* [18,19]. Still more recently, an additional class of morphogenetic instabilities was pointed out: these instabilities are believed to underlie the twisting phenotypes of germ-band extension in *Drosophila* mutants [20] or the symmetry breaking of its primordial hindgut [21], and they are

different from the former, three-dimensional instabilities of curved tissues because they are two-dimensional instabilities *within* the curved embryonic tissue of *Drosophila*. Classical Euler buckling is a planar instability, though, and the fundamental buckling problems within *general* surfaces have remained wide open for the two and a half centuries since Euler's work.

Here, we therefore analyze a minimal buckling instability within a curved surface (Fig. 1): we consider the

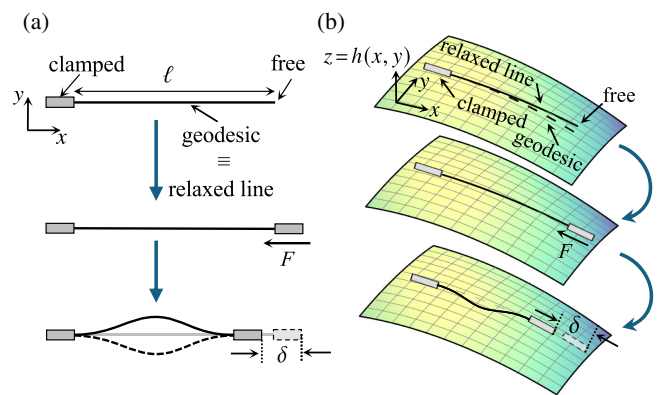


FIG. 1. Euler buckling in the plane and on a curved surface. (a) Classical Euler buckling in the plane. An inextensible elastic line of length ℓ , clamped at one end, relaxes into a straight, geodesic shape. Clamping of the other end and compression by a force F along the straight shape leads to buckling. The relative compression is δ . “Up” and “down” buckling (solid and dashed line in the bottom panel) is equivalent by symmetry. (b) Euler buckling on a curved surface $z = h(x, y)$. The relaxed elastic line differs, in general, from the “straight” geodesic, and the curvature of the surface selects either “up” or “down” buckling.

*Contact author: haas@pks.mpg.de

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Open access publication funded by the Max Planck Society.

buckling of a compressed inextensible elastic line of length ℓ on a smooth surface $z = h(x, y)$ embedded in three-dimensional Euclidean space with Cartesian coordinates (x, y, z) . The buckled shape minimizes the bending energy of the elastic line subject to the imposed compression and the constraint of inextensibility. We parametrize the elastic line by its arclength s , so that a point on it has coordinates $(x(s), y(s), z(s))$, with $z(s) = h(x(s), y(s))$. The Lagrangian of the problem is

$$\mathcal{L} = \frac{1}{2} \int_0^\ell \kappa(s)^2 ds - \int_0^\ell \lambda(s) [\alpha(s)^2 - 1] ds, \quad (1)$$

where $\kappa(s)^2 = x''(s)^2 + y''(s)^2 + z''(s)^2$ is the squared curvature of the elastic line and its stretch squared is $\alpha(s)^2 = x'(s)^2 + y'(s)^2 + z'(s)^2$, with dashes denoting differentiation with respect to s . The Lagrange multiplier function $\lambda(s)$ imposes inextensibility, $\alpha(s) \equiv 1$.

To define our buckling problem on this surface, we first paraphrase classical Euler buckling [Fig. 1(a)]: in the plane, an elastic line with one clamped end and one free end relaxes into a straight line ($\kappa = 0$). Clamping of the other end and compression, by a force F , along the straight, relaxed configuration by a relative amount δ leads to buckling. The “up” and “down” buckling modes are equivalent because of the symmetry of the plane. On a curved surface [Fig. 1(b)], however, it is known that the relaxed elastic line need not be “straight,” i.e., need not be a geodesic [22]. Clamping of the other end and compression along the relaxed shape will still lead to buckling, but it will turn out that the curvature of the surface selects, in general, either up or down buckling by breaking the symmetry of the two buckling modes.

To show that the curvature of the surface changes the buckling instability even more fundamentally, we will first solve the buckling of a short elastic line asymptotically, before studying long elastic lines numerically.

Asymptotic buckling of a short elastic line on a curved surface—We solve the buckling of a short elastic line by asymptotic expansion for small ℓ . Up to translation and rotation of the surface, its tangent plane at the origin coincides with the (x, y) plane, and we may clamp one end of the elastic line there in the x direction. Scaling lengths by ℓ , we therefore take

$$h(x, y) = \ell(Ax^2 + Bxy + Cy^2), \quad (2)$$

in which the parameters A, B, C describe the surface. By solving the Euler–Lagrange equations of Eq. (1), we compute the shape of the relaxed line order by order and hence solve the buckling problem. We relegate the details of these calculations to the End Matter and Supplemental Material [23]. We first consider $\delta = O(\ell^2)$. Writing $\delta = \ell^2 d_{(2)}$, we compute the compressive force F in the tangent plane to the surface. In the lowest buckling mode, we find

$$F = \frac{4\pi^2}{\ell^2} - \frac{1}{\ell} \left(\frac{4|AB|\pi}{\sqrt{d_{(2)}}} \right) + [2d_{(2)}\pi^2 + g(A, B, C)] + O(\ell), \quad (3)$$

where $g(A, B, C) = 3B^2 + 12AC - 8A^2$. Equation (2) is the leading term in a Taylor expansion of a general surface near the origin. In the Supplemental Material [23], we show that its corrections at order $O(\ell^2)$ actually translate to corrections to Eq. (3) even at order $O(1)$. We will not pursue these corrections further, however, because the essential physics turn out to be captured by the paraboloid surface defined by Eq. (2).

Equation (3) gives the force $F = F_1$ applied at the compressed end of the elastic line. Because of the curvature of the surface, the tension along the elastic line is not constant, which is why this differs from the force $F_0 \neq F_1$ required to clamp its other end. In the Supplemental Material [23], we find $F_1 - F_0 = O(\ell)$.

In the plane, $A = B = C = 0$, and this asymmetry disappears as expected and $F = 4\pi^2/\ell^2 + 2d_{(2)}\pi^2 + O(\ell)$.

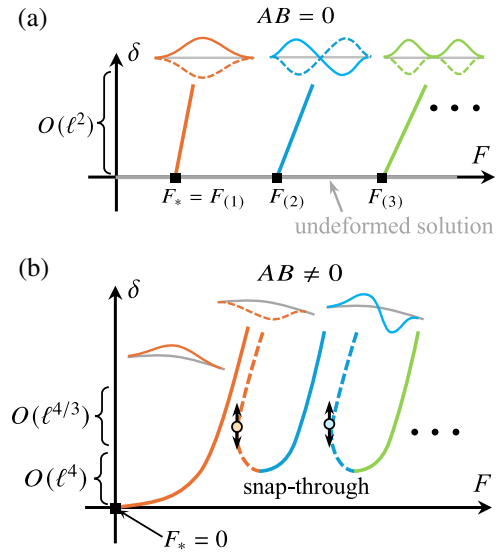


FIG. 2. Bifurcation diagrams for Euler buckling. (a) Bifurcation diagram of classical Euler buckling for $\ell \ll 1$: plot of the relative compression δ against the compressive force F , from asymptotic analysis for $\delta = O(\ell^2)$. The first buckling mode (inset) bifurcates off the undeformed solution $\delta = 0$ at the critical force $F_* = F_{(1)}$ for “up” or “down” buckling. Higher buckling modes (insets) have higher critical forces $F_{(2)}, F_{(3)}, \dots$. The same bifurcation diagram arises for “simple” curved surfaces [Eq. (2) with $AB = 0$]. (b) Corresponding bifurcation diagram for a general curved surface [Eq. (2) with $AB \neq 0$]: “Up” and “down” buckling modes split and the asymptotic analysis at order $O(\ell^2)$ breaks down. Asymptotic calculations for $\delta = O(\ell^4)$ show that $F_* = 0$ and that higher modes connect in pairs. Asymptotic calculations for $\delta = O(\ell^{4/3})$ shows that there are minima with $\partial F / \partial \delta = 0$ on the higher branches, at which the elastic line snaps to a different branch (arrows) if F is reduced.

By taking $d_{(2)} \rightarrow 0$, we recover the critical force for classical Euler buckling, $F_* \sim 4\pi^2/\ell^2$. The stiffness of the buckled line is $\mu_* = \partial F/\partial(\ell\delta) \sim 2\pi^2/\ell^3$. Higher buckling modes for higher forces, described in the Supplemental Material [23], complete the bifurcation diagram [Fig. 2(a)].

In general, we cannot however take $d_{(2)} \rightarrow 0$ in Eq. (3), because it loses asymptoticity in this limit unless $AB = 0$. The latter includes the cases of (i) a surface for which the clamping direction is a principal axis ($B = 0$); and (ii) a surface that is flat in the direction of clamping ($A = 0$). To quadratic order, buckling on a sphere falls into case (i), because every direction is a principal direction on a sphere. To the same order, buckling of a line clamped along the axis of a cylinder is an example of case (ii). This illustrates that buckling on “simple” surfaces has $AB = 0$. We can then take $d_{(2)} \rightarrow 0$ in Eq. (3) to find $F_* \sim 4\pi^2/\ell^2 + g(A, B, C)$ and hence $\mu_* \sim 2\pi^2/\ell^3$; i.e., the critical buckling force is

merely shifted compared to the flat case, the stiffness remains unchanged from that case at leading order, and the bifurcation diagram of classical Euler buckling [Fig. 2(a)] is recovered.

The general case $AB \neq 0$ is much more interesting: The “up” and “down” buckling modes are no longer symmetric across the relaxed elastic line, as shown in the Supplemental Material [23]. We expect this to turn the pitchfork bifurcation of classical Euler buckling into an imperfect pitchfork [27], and hence the instability to become a “squeeze deformation” with $F_* = 0$ [Fig. 2(b)]. More precisely, the first two terms in Eq. (3) lose asymptoticity when $d_{(2)} = O(\ell^2)$. We therefore make the ansatz $\delta = \ell^4 d_{(4)}$, with $d_{(4)} = O(1)$. The calculations in the Supplemental Material [23] now lead to the result $F = \lambda_*^2/\ell^2 + O(\ell^{-1})$, in which λ_* is determined implicitly by the equation

$$\frac{d_{(4)}}{(AB)^2} = \frac{2\lambda_*[21\lambda_*^2 + 72 + (20\lambda_*^2 - 6)\cos\lambda_* + (4\lambda_*^2 - 66)\cos 2\lambda_*] - 3[(52\lambda_*^2 + 60)\sin\lambda_* + (19\lambda_*^2 - 30)\sin 2\lambda_*]}{90\lambda_*(\lambda_*\sin\lambda_* + 2\cos\lambda_* - 2)^2}. \quad (4)$$

In particular, $F \sim 175d_{(4)}/(AB\ell)^2$ for $d_{(4)} \ll 1$, which is consistent with $F_* = 0$, so $\mu_* \sim 175/[(AB)^2\ell^7] \propto \ell^{-7}$. Thus even the scaling of stiffness is modified from $\mu_* \propto \ell^{-3}$ of flat Euler buckling! For $d_{(4)}/(AB)^2 \gtrsim 0.21$, Eq. (4) is seen numerically to have multiple additional solutions for λ_* and hence F , which correspond to higher buckling modes [Fig. 2(b)]. At local minima of $d_{(4)}$ as a function of λ_* or F , pairs of different buckling modes merge. These higher modes do not therefore connect to $\delta = 0$, as they do in classical Euler buckling (Fig. 2). This has an important physical consequence: higher buckling modes of an elastic line on a flat surface can be excited, at least in principle, by fast compression beyond the critical force for these higher modes, but this is not possible on a general curved surface, where these higher modes are not small deformations of the straight shape.

Similarly, the second and third terms in Eq. (3) lose asymptoticity when $d_{(2)} = O(\ell^{-2/3}) \Leftrightarrow \delta = O(\ell^{4/3})$. With the ansatz $\delta = \ell^{4/3}d_{(4/3)}$, where $d_{(4/3)} = O(1)$, we find, in the Supplemental Material [23],

$$F = \frac{4\pi^2}{\ell^2} + \frac{1}{\ell^{2/3}} \left(2d_{(4/3)}\pi^2 \pm \frac{4|AB|\pi}{\sqrt{d_{(4/3)}}} \right) + O(1), \quad (5)$$

for the lowest (−) and next (+) buckling modes, which are equal if $AB = 0$. Thus, in the − mode, F continues to increase monotonically with compression, but there is a minimum, $\partial F/\partial\delta = 0$, on the + branch. If F is reduced on this branch, the elastic line snaps discontinuously a different branch at this minimum [Fig. 2(b)]. There is no analogous snap through between modes in classical

Euler buckling. Minima (and hence snap-through instabilities) arise for higher buckling modes, too, as shown in the Supplemental Material [23].

Intrinsic and extrinsic buckling on curved surfaces—The bending energy in Eq. (1) involves the total curvature $\kappa(s)^2 = \kappa_g(s)^2 + \kappa_n(s)^2$ of the elastic line, where $\kappa_g(s)$ and $\kappa_n(s)$ are its geodesic and normal curvatures [24], respectively. It therefore penalises both bending of the line *in* the surface and its bending *with* the surface [28]. This is what one might term *extrinsic buckling* because it involves κ_n , which is an extrinsic property of the surface. By analogy, we introduce *intrinsic buckling* by replacing $\kappa \rightarrow \kappa_g$ in Eq. (1). In the latter case, the geodesic $\kappa_g(s) \equiv 0$ is trivially a minimum of the bending energy in Eq. (1), and so the relaxed elastic line is always a geodesic. With $\delta = \ell^2 d_{(2)}$, we obtain, in the Supplemental Material [23],

$$F = \frac{4\pi^2}{\ell^2} + (2d_{(2)}\pi^2 + K) + O(\ell), \quad (6)$$

where $K = 4AC - B^2$ is the Gaussian curvature [24] at the origin. Even for $AB \neq 0$, there is now no loss of asymptoticity as $d_{(2)} \rightarrow 0$, and we simply find $F_* \sim 4\pi^2/\ell^2 + K$. Thus, unlike extrinsic buckling, intrinsic buckling only shifts the buckling threshold of classical, flat Euler buckling. Harking back to the *Theorema egregium* [24], it is unsurprising that the shift is proportional to the Gaussian curvature (at leading order).

If $AB = 0$, then the entire problem is symmetric under reflection across the direction of clamping ($y \leftrightarrow -y$), so the symmetry of the modes of classical Euler buckling

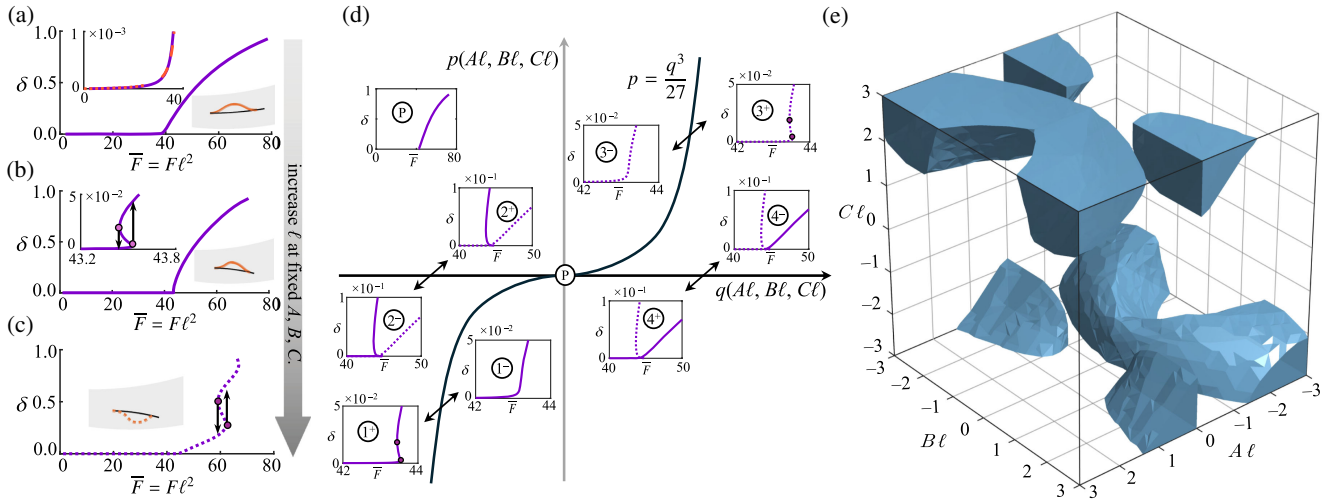


FIG. 3. Numerical Euler buckling on curved surfaces. (a) Force-compression diagram for the lowest buckling mode. Plot of the compression δ against the scaled force $\bar{F} = F\ell^2$ for a short elastic line: compression increases monotonically with compressive force. Inset: for $\delta \ll 1$, numerical results (solid line) agree with asymptotic calculations (dashed and dotted lines) for $\delta = O(\ell^2)$ and $\delta = O(\ell^4)$. (b) Analogous plot for a longer elastic line on the same surface. The lowest buckling mode remains the same (shape inset), but the line snaps discontinuously (arrows) as $\bar{F}$ is increased or decreased (inset). (c) For even longer elastic lines, the lowest buckling mode switches (shape inset). Again, there is a snap-through (arrows) as $\bar{F}$ varies. Parameter values: $A = B = C = 1$, (a) $\ell = 0.1$, (b) $\ell = 1.9$, (c) $\ell = 2$. (d) Euler buckling on curved surfaces as the unfolding of a pitchfork bifurcation [23,27] with normal parameters $p(A\ell, B\ell, C\ell), q(A\ell, B\ell, C\ell)$: the (F, δ) diagrams show realisations of the characteristic codimension-one bifurcations of the unfolding of a pitchfork bifurcation (P; $p = q = 0$), i.e., fold bifurcations (1, 3; $p = q^3/27 \neq 0$) and transcritical bifurcations (2, 4; $p = 0, q \neq 0$). See Supplemental Material [23] for parameter values. (e) Phase diagram of the snap-through instability in $(A\ell, B\ell, C\ell)$ parameter space: an elastic line snaps under compression within the shaded regions. See Supplemental Material [23] for additional visualizations of this plot.

[Fig. 2(a)] is unbroken in the extrinsic and intrinsic cases. If $AB \neq 0$, then the geodesic is not symmetric under $y \leftrightarrow -y$ and neither is the relaxed elastic line in the extrinsic case. However, for intrinsic buckling, the two buckling modes do remain symmetric across the geodesic, which is not the case for the relaxed line in the extrinsic case [23]. This is the difference that explains why $F_* = 0$ in extrinsic buckling but $F_* > 0$ in intrinsic buckling (and provides mathematical motivation for the latter): The symmetry of the buckling problem is *not* broken by the asymmetry of the relaxed elastic line across the clamping direction, but by its deviations from the “straightness” of the geodesic and the consequent asymmetry of the buckling modes.

Numerical buckling of a long elastic line on a curved surface—We now extend our asymptotic results by solving the Euler-Lagrange equations of Eq. (1) numerically, as described in the End Matter, restricting again to the family of paraboloid surfaces defined by Eq. (2). For short elastic lines, the end-to-end compression δ increases monotonically with the applied compressive force F in the lowest (observed) buckling mode [Fig. 3(a)]. For small δ , these numerical results match the asymptotic calculation [Fig. 3(a), inset]. For longer lines, the buckling behaviour can change strikingly on the same surface [Figs. 3(b) and 3(c)]: there can now be snap-through even in the lowest mode so that, as F is increased or decreased, δ jumps

discontinuously where $\partial F / \partial \delta = 0$, across the inaccessible part of the solution branch with $\partial F / \partial \delta < 0$. Even more remarkably, the lowest buckling mode can also switch from “up” to “down” buckling (or vice versa): a long elastic line can buckle to a different side of its relaxed shape than [insets in Figs. 3(b) and 3(c)] than a short line on the same surface. In the Supplemental Material [23], we further illustrate that the forces at the clamped and compressed ends of these long elastic lines differ in general.

As already noted above, the curvature of the surface breaks the symmetries of the perfect pitchfork bifurcation of classical Euler buckling. Since the buckling problem has three independent parameters $(A\ell, B\ell, C\ell)$ up to rescaling, we expect to see the full unfolding of the pitchfork bifurcation [23,27] and, in particular, its characteristic codimension-1 bifurcations. Numerical examples [Fig. 3(d)] of these fold and transcritical bifurcations confirm this. This shows that the snap through in the unswitched lowest mode [Fig. 3(b)] is part of the unfolding of the pitchfork. However, we show in Supplemental Material [23] that the snap through in the switched lowest mode [Fig. 3(c)] is not part of this unfolding.

We observe snap through of long elastic lines numerically even on simple surfaces with $AB = 0$, which our asymptotic calculations showed not to display such behavior for short elastic lines. As illustrated in the Supplemental

Material [23], if $B = 0$, then the relaxed elastic line is straight ($y = 0$) and the buckling bifurcation is a perfect pitchfork by symmetry, so this confirms that the snap-through can result from a bifurcation separate from the unfolding pitchfork. If $A = 0$, then the relaxed elastic line is straight by symmetry, but, interestingly, the buckling bifurcation need not be a perfect pitchfork [23]. Again, this is associated with snap through [23].

More generally, we can determine numerically the points in $(A\ell, B\ell, C\ell)$ space at which an elastic line snaps under compression. The complex shape of this region of surface parameter space Fig. 3(e) obeys the symmetries of the buckling problem (End Matter), namely, $(A\ell, B\ell, C\ell) \leftrightarrow (A\ell, -B\ell, C\ell) \leftrightarrow (-A\ell, -B\ell, -C\ell)$. We emphasize that this surface is discontinuous across $A = 0$, with snap through possible at $A = 0$ as noted above.

In this Letter, we have shown that even the simplest buckling instability displays rich additional behavior on general curved surfaces absent from buckling on flat or “simple” curved surfaces. This wealth of behavior mirrors that seen for constrained elastic lines in the plane and on curved surfaces [26,29–38] and for related problems [39–42]. The instabilities *within* curved surfaces that we have studied here are complemented by the much more widely studied mechanical instabilities of curved surfaces [43–49]. In particular, the biophysical instabilities driving brain gyrification are also known to be affected by surface curvature [8,9].

Future work will have to explore the effect of global geometric constraints on the buckling instabilities within curved surfaces that we have studied: for example, how are these instabilities modified for long elastic lines that wind around a closed surface? The role of such global constraints has been highlighted for elastic filaments on spherical bubbles [26,37]. More mathematically, the effect of geometric singularities of the curved surface, such as folds or cusps, on these buckling instabilities remains an open problem.

By highlighting the new buckling behaviour on general surfaces that are, in particular, not spherical, our Letter emphasizes the importance of curvature differences for these instabilities. The role of curvature differences was also highlighted for the symmetry breaking of the *Drosophila* hindgut primordium, where they break a degeneracy to select the orientation of the shape of the primordium [21]. It is perhaps not surprising, though, that these curvature differences matter for morphogenesis: after all, to modify the oft-quoted adage, cows are not spherical!

Acknowledgments—This work was supported by the Max Planck Society. We thank Carl Modes, Jonas Neipel, and Stephan Grill for comments on the manuscript, and two anonymous reviewers for helpful comments.

Data availability—Code for numerical solution of the buckling problem is openly available at Ref. [50].

- [1] L. Euler, *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes Sive Solutio Problematis Isoperimetrici Latissimo Sensu Accepti* (Marcus Michaelis Bousquet et Socii, Lausanne & Geneva, 1744), pp. 267–268.
- [2] S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, 2nd ed. (McGraw-Hill, New York, 1985).
- [3] *Œuvres de Lagrange*, edited by M. J.-A. Serret (Gauthier-Villars, Paris, France, 1868), Vol. 2, pp. 125–170.
- [4] A. Goriely, R. Vandiver, and M. Destade, Nonlinear Euler buckling, *Proc. R. Soc. A* **464**, 3003 (2008).
- [5] R. W. Ogden, *Non-Linear Elastic Deformations* (Dover, New York, 1997).
- [6] C. M. Nelson, On buckling morphogenesis, *J. Biomech. Eng.* **138**, 021005 (2016).
- [7] D. P. Richman, R. M. Stewart, J. W. Hutchinson, and V. S. Caviness, Jr., Mechanical model of brain convolutional development, *Science* **189**, 18 (1975).
- [8] T. Tallinen, J. Y. Chung, J. S. Biggins, and L. Mahadevan, Gyrification from constrained cortical expansion, *Proc. Natl. Acad. Sci. U.S.A.* **111**, 12667 (2014).
- [9] S. Budday, P. Steinmann, A. Goriely, and E. Kuhl, Size and curvature regulate pattern selection in the mammalian brain, *Extreme Mech. Lett.* **4**, 193 (2015).
- [10] S. Budday, P. Steinmann, and E. Kuhl, The role of mechanics during brain development, *J. Mech. Phys. Solids* **72**, 75 (2014).
- [11] T. Tallinen, J. Y. Chung, F. Rousseau, N. Girard, J. Lefèvre, and L. Mahadevan, On the growth and form of cortical convolutions, *Nat. Phys.* **12**, 588 (2016).
- [12] S. Budday and P. Steinmann, On the influence of inhomogeneous stiffness and growth on mechanical instabilities in the developing brain, *Int. J. Solids Struct.* **132–133**, 31 (2018).
- [13] V. Balbi, M. Destade, and A. Goriely, Mechanics of human brain organoids, *Phys. Rev. E* **101**, 022403 (2020).
- [14] E. Hannezo, J. Prost, and J.-F. Joanny, Instabilities of monolayered epithelia: Shape and structure of villi and crypts, *Phys. Rev. Lett.* **107**, 078104 (2011).
- [15] T. Savin, N. A. Kurpios, A. E. Shyer, P. Florescu, H. Liang, L. Mahadevan, and C. J. Tabin, On the growth and form of the gut, *Nature (London)* **476**, 57 (2011).
- [16] A. E. Shyer, T. Tallinen, N. L. Nerurkar, Z. Wei, E. S. Gil, D. L. Kaplan, C. J. Tabin, and L. Mahadevan, Villification: How the gut gets its villi, *Science* **342**, 212 (2013).
- [17] H. K. Gill, S. Yin, J. C. Lawlor, T. R. Huycke, N. L. Nerurkar, C. J. Tabin, and L. Mahadevan, The developmental mechanics of divergent buckling patterns in the chick gut, *Proc. Natl. Acad. Sci. U.S.A.* **121**, e2310992121 (2024).
- [18] S. A. Montandon, A. Fofonjka, and M. C. Milinkovitch, Elastic instability during branchial ectoderm development causes folding of the Chlamydosaurus erectile frill, *eLife* **8**, e44455 (2019).
- [19] P. A. Haas, Morphogenesis: Mathematical models with frills, *eLife* **8**, e48520 (2019).
- [20] C. M. Smits, S. Dutta, V. Jain-Sharma, S. J. Streichan, and S. Y. Shvartsman, Maintaining symmetry during body axis elongation, *Curr. Biol.* **33**, 3536 (2023).
- [21] D. S. Alber, S. Zhao, A. O. Jacinto, E. F. Wieschaus, S. Y. Shvartsman, and P. A. Haas, A model for boundary-driven

- tissue morphogenesis, *Proc. Natl. Acad. Sci. U.S.A.* **122**, e2505160122 (2025).
- [22] G. S. Manning, Relaxed elastic line on a curved surface, *Q. Appl. Math.* **45**, 515 (1987).
- [23] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/63py-ph5s>, which includes Refs. [24–27], for details of the asymptotic calculations and the numerical methods, for additional figures, and for a Supplemental Movie illustrating Fig. 3(e).
- [24] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice-Hall, Englewood Cliffs, NJ, 1976), Chap. 4, pp. 217–314.
- [25] R. E. Goldstein and S. A. Langer, Nonlinear dynamics of stiff polymers, *Phys. Rev. Lett.* **75**, 1094 (1995).
- [26] J. Guven and P. Vázquez-Montejo, Confinement of semi-flexible polymers, *Phys. Rev. E* **85**, 026603 (2012).
- [27] M. Golubitsky and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory, Volume I*, Applied Mathematical Sciences Vol. 51 (Springer, New York, 1985), Chap. 1, pp. 1–46.
- [28] More generally, an elastic line has different moduli B_n, B_g for bending with or within the surface, so its energy density is $[B_n \kappa_n(s)^2 + B_g \kappa_g(s)^2]/2$. Both extrinsic buckling ($B_n = B_g \neq 0$) and intrinsic buckling ($B_n = 0$) are special cases of this general energy density, which can be realised by a linked chain with links that are hinged so that they have different resistances to bending with and within the surface.
- [29] G. Arreaga, R. Capovilla, C. Chrysomalakos, and J. Guven, Area-constrained planar elastica, *Phys. Rev. E* **65**, 031801 (2002).
- [30] S. K. Veerapaneni, R. Raj, G. Biros, and P. K. Purohit, Analytical and numerical solutions for shapes of quiescent two-dimensional vesicles, *Int. J. Nonlinear Mech.* **44**, 257 (2009).
- [31] L. Giomi and L. Mahadevan, Minimal surfaces bounded by elastic lines, *Proc. R. Soc. A* **468**, 1851 (2012).
- [32] L. Giomi, Softly constrained films, *Soft Matter* **9**, 8121 (2013).
- [33] J. Hure and B. Audoly, Capillary buckling of a thin film adhering to a sphere, *J. Mech. Phys. Solids* **61**, 450 (2013).
- [34] G. Napoli and S. Turzi, Snap buckling of a confined thin elastic sheet, *Proc. R. Soc. A* **471**, 20150444 (2015).
- [35] F. Box, O. Kodio, D. O’Kiely, V. Cantelli, A. Goriely, and D. Vella, Dynamic buckling of an elastic ring in a soap film, *Phys. Rev. Lett.* **124**, 198003 (2020).
- [36] O. Kodio, A. Goriely, and D. Vella, Dynamic buckling of an inextensible elastic ring: Linear and nonlinear analyses, *Phys. Rev. E* **101**, 053002 (2020).
- [37] S. G. Prasath, J. Marthelot, R. Govindarajan, and N. Menon, Shapes of a filament on the surface of a bubble, *Proc. R. Soc. A* **477**, 20210353 (2021).
- [38] C. Shi, G. Zou, Z. Wu, M. Wang, X. Zhang, H. Gao, and X. Yi, Morphological transformations of vesicles with confined flexible filaments, *Proc. Natl. Acad. Sci. U.S.A.* **120**, e2300380120 (2023).
- [39] B. Audoly, N. Clauvelin, and S. Neukirch, Elastic knots, *Phys. Rev. Lett.* **99**, 164301 (2007).
- [40] P.-T. Brun, B. Audoly, N. M. Ribe, T. S. Eaves, and J. R. Lister, Liquid ropes: A geometrical model for thin viscous jet instabilities, *Phys. Rev. Lett.* **114**, 174501 (2015).
- [41] A. P. Zakharov and L. M. Pismen, Reshaping of a Janus ring, *Phys. Rev. E* **97**, 062705 (2018).
- [42] A. Zakharov, L. M. Pismen, and L. Ionov, Shape-morphing architectures actuated by Janus fibers, *Soft Matter* **16**, 2086 (2020).
- [43] B. Li, F. Jia, Y.-P. Cao, X.-Q. Feng, and H. Gao, Surface wrinkling patterns on a core-shell soft sphere, *Phys. Rev. Lett.* **106**, 234301 (2011).
- [44] N. Stoop, R. Lagrange, D. Terwagne, P. M. Reis, and J. Dunkel, Curvature-induced symmetry breaking determines elastic surface patterns, *Nat. Mater.* **14**, 337 (2015).
- [45] P. W. Miller, N. Stoop, and J. Dunkel, Geometry of wave propagation on active deformable surfaces, *Phys. Rev. Lett.* **120**, 268001 (2018).
- [46] Y. Tan, B. Hu, J. Song, Z. Chu, and W. Wu, Bioinspired multiscale wrinkling patterns on curved substrates: An overview, *Nano Micro Lett.* **12**, 101 (2020).
- [47] F. Xu, Y. Huang, S. Zhao, and X.-Q. Feng, Chiral topographic instability in shrinking spheres, *Nat. Comput. Sci.* **2**, 632 (2022).
- [48] I. Tobasco, Y. Timounay, D. Todorova, G. C. Leggat, J. D. Paulsen, and E. Katifori, Exact solutions for the wrinkle patterns of confined elastic shells, *Nat. Phys.* **18**, 1099 (2022).
- [49] T. Wang, Z. Dai, M. Potier-Ferry, and F. Xu, Curvature-regulated multiphase patterns in tori, *Phys. Rev. Lett.* **130**, 048201 (2023).
- [50] Code is available at [10.5281/zenodo.16639119](https://doi.org/10.5281/zenodo.16639119).

End Matter

Boundary conditions—Let $L = \kappa^2/2 - \lambda(\alpha^2 - 1)$ be the Lagrangian density in the Lagrangian $\mathcal{L}$ in Eq. (1). Scaling lengths by ℓ , the boundary terms in the variation $\delta\mathcal{L}$ of the Lagrangian with respect to x, y are $\mathcal{B} = \llbracket V_x \delta x + V_y \delta y + V_{x'} \delta x' + V_{y'} \delta y' \rrbracket_0^1$, where

$$\begin{aligned} V_x &= L_{,x} - \frac{dL_{,x''}}{ds}, & V_{x'} &= L_{,x''}, \\ V_y &= L_{,y} - \frac{dL_{,y''}}{ds}, & V_{y'} &= L_{,y''}, \end{aligned}$$

in which commata denote partial differentiation. The variations $\delta x, \delta y, \delta x', \delta y'$ are not independent because $(\alpha^2)_{,x} \delta x + (\alpha^2)_{,y} \delta y + (\alpha^2)_{,x'} \delta x' + (\alpha^2)_{,y'} \delta y' = 0$, which we obtain by varying the constraint $\alpha^2 = 1$. We use this relation to eliminate one of $\delta x, \delta y, \delta x', \delta y'$ from $\mathcal{B}$.

Let $\mathbf{r}(s) = (x(s), y(s), z(s))$ denote the position vector. At the clamped end $s = 0$, we impose $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{r}'(0) = \hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is the unit vector in the x -direction, which are compatible with the constraints because of the choice of surface made in Eq. (2). These conditions give

$$x = y = 0, \quad x' = 1, \quad y' = 0 \quad \text{at } s = 0.$$

At the free end $s = 1$ of the relaxed elastic line, we impose the natural boundary conditions for $\mathcal{B}$. We thus require

$$\frac{V_x}{(\alpha^2)_{,x}} = \frac{V_y}{(\alpha^2)_{,y}} = \frac{V_{x'}}{(\alpha^2)_{,x'}} = \frac{V_{y'}}{(\alpha^2)_{,y'}} \quad \text{at } s = 1.$$

The remaining boundary conditions on the relaxed elastic line are provided by the constraint $\alpha^2 = 1$ and its derivatives, as explained in the Supplemental Material [23].

Let $\mathbf{R}(s) = (X(s), Y(s), Z(s))$ correspond to this relaxed line. At the compressed end of the elastic line, we then impose $\mathbf{r}(1) = \mathbf{R}(1 - \delta)$ and $\mathbf{r}'(1) \parallel \mathbf{R}'(1 - \delta)$. Because the latter are unit vectors by inextensibility, they are in fact equal, so $\mathbf{r}'(1) = \mathbf{R}'(1 - \delta)$. The boundary conditions that ensure $\mathcal{B} = 0$ are therefore

$$\begin{aligned} x(1) &= X(1 - \delta), & y(1) &= Y(1 - \delta), \\ x'(1) &= X'(1 - \delta), & y'(1) &= Y'(1 - \delta). \end{aligned}$$

Forces and moments at the ends of the line—The forces $\mathbf{F} = (F_x, F_y, F_z)$ applied at the compressed or clamped ends of the elastic line are tangent to

$z = h(x, y)$, because any out-of-plane component is canceled by reaction forces confining the line to this surface. Its unit normal is $\mathbf{n} \parallel (-h_x, -h_y, 1)$. Thus $F_z = F_x h_{,x} + F_y h_{,y}$. Similarly, the moments $\mathbf{M} = m\mathbf{n}$ at each of the two ends are normal to $z = h(x, y)$.

The infinitesimal work done by these forces and moments applied at the two ends of the line is $\delta\mathcal{W} = F_x \delta x + F_y \delta y + F_z \delta z + m \delta\theta$, where, by variation of the constraint $z = h(x, y)$, $\delta z = h_{,x} \delta x + h_{,y} \delta y$, and where $\delta\theta$ is the infinitesimal rotation in the tangent plane to $z = h(x, y)$ corresponding to the displacement $\delta\mathbf{r} = (\delta x, \delta y, \delta z)$, i.e., the angle between $\mathbf{r}'$ and the projection of $\mathbf{r}' + \delta\mathbf{r}'$ onto said tangent plane. This determines $\delta\theta$ in terms of $\delta x, \delta y, \delta x', \delta y'$.

The principle of virtual work requires $\delta\mathcal{L} + \delta\mathcal{W} = 0$. Eliminating one of $\delta x, \delta y, \delta x', \delta y'$ as above finally gives three simultaneous equations for F_x, F_y, m at each end of the elastic line. The resulting expressions for F_x, F_y, F_z are given in Supplemental Material [23].

Asymptotic calculations—The full details of the asymptotic calculations are given in the Supplemental Material [23].

Numerical methods—We solve the Euler–Lagrange equations of Eq. (1) numerically using the `bvp5c` solver of MATLAB (The MathWorks, Inc.), relying on their first integral $\lambda(s) = \lambda_0 + 3\kappa(s)^2/4$, which is related to the tangential force balance on the elastic line [25,26], and where λ_0 is a constant to be determined [23]. This first integral differs from Eq. (A5) in Ref. [26] by a factor of 1/2 because of a corresponding difference in the definition of the Lagrangian.

Symmetries of the buckling problem—The physics of buckling are invariant under symmetries of space. In particular, the instabilities in Fig. 3 are unaffected by rotations or reflections $(x, y, z) \mapsto (x', y', z')$ that map the buckling problem for (A, B, C) onto that for some (A', B', C') , i.e., that leave the clamping direction invariant, $x' = x$, and map $z = Ax^2 + Bxy + Cy^2$ onto $z' = A'x'^2 + B'x'y' + Cy'^2$. These are precisely (i) $x' = x, y' = y, z' = -z$ (reflection in $x = y = 0$); (ii) $x' = x, y' = -y, z' = z$ (reflection in $x = z = 0$); and (iii) $x' = x, y' = -y, z' = -z$ (rotation through π about $x = 0$), which is the product of transformations (i) and (ii). Now, for transformation (i), $(A', B', C') = (-A, -B, -C)$ and, for transformation (ii), $(A', B', C') = (A, -B, C)$.