

## Critical Dynamics in Short-Range Quadratic Hamiltonians

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We investigate critical transport and the dynamical exponent through the spreading of an initially localized particle in quadratic Hamiltonians with short-range hopping in lattice dimension  $d_l$ . We consider critical dynamics that emerges when the Thouless time, i.e., the saturation time of the mean-squared displacement, approaches the typical Heisenberg time. We establish a relation,  $z = d_l/d_s$ , linking the critical dynamical exponent  $z$  to  $d_l$  and to the spectral fractal dimension  $d_s$ . This result has notable implications: it says that superdiffusive transport in  $d_l \geq 2$  and diffusive transport in  $d_l \geq 3$  cannot be critical in the sense defined above. Our findings clarify previous results on disordered and quasiperiodic models and, through Fibonacci potential models in two and three dimensions, provide nontrivial examples of critical dynamics in systems with  $d_l \neq 1$  and  $d_s \neq 1$ .

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**Introduction**—The transport of mass, energy, and other conserved quantities is a fundamental characteristic of physical systems [1–4], often described by the diffusion equation [5,6]. At the nanoscale, where quantum effects become significant, classical transport theories must be adjusted to account for quantum phenomena [7–10]. Interestingly, diffusion remains relevant even in quantum transport [9,11–25]; however, there exist notable exceptions. For instance, Google’s recent quantum simulator experiment [26] explored superdiffusive transport in integrable models [27–39] and their relation to the Kardar-Parisi-Zhang universality [40]. Another prominent example is slow subdiffusive dynamics in disordered [41–43] and quasiperiodic [42–44] systems. Thus, the diffusion often serves as the reference point in transport studies, while its absence often signals critical behavior, such as integrability [45,46] or localization [47]. However, not all cases of nondiffusive transport indicate criticality; some may simply result from long-lived transient effects. Understanding the mechanisms underlying critical dynamics is therefore essential.

In systems governed by quadratic Hamiltonians, the dynamics are often characterized by the spreading of a local excitation [9] through the mean-squared displacement  $\sigma^2(t)$  [9,48–52,52–59] and the associated dynamical exponent  $z$  [defined in Eqs. (3) and (4) below]. Despite extensive studies on quadratic systems, several key questions remain

unresolved: How should critical dynamics be defined, and is there a simple relation for the dynamical exponent  $z$ ? When do nondiffusive transport types, such as superdiffusion and subdiffusion, indicate critical dynamics rather than transient effects? Figure 1 highlights this distinction in Fibonacci models across one to three dimensions. For a fixed  $z$ , nontrivial dynamics may be considered critical since  $\sigma^2(t)$  saturates near the system’s longest timescale, the typical Heisenberg time  $t_H^{\text{typ}}$  [Figs. 1(b) and 1(d)].

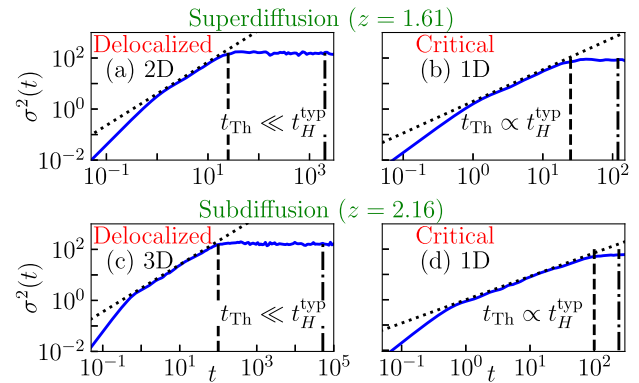


FIG. 1. The dynamics of the mean-squared displacement  $\sigma^2(t)$  [see Eq. (4)] in the separable Fibonacci models in dimensions one (1D), two (2D), and three (3D). We set  $L = 40$  and the potentials (a), (b)  $h = 2$ , (c), (d)  $h = 4$ ; see Eq. (5) and the text below. Dotted lines, which mark the slope of  $\sigma^2(t)$ , suggests that the dynamical exponent  $z$  is superdiffusive ( $1 < z < 2$ ) in (a), (b) and subdiffusive ( $z > 2$ ) in (c), (d). Vertical dashed line denotes the Thouless time  $t_{\text{Th}}$  and vertical dashed-dotted line denotes the typical Heisenberg time  $t_H^{\text{typ}}$ . In (b) and (d), the dynamics are critical since they fulfill proportionality in the sense of Eq. (1), while in (a) and (c) they are not.

Alternatively, they may be viewed as transient effects if the dynamics lead to complete wave-packet delocalization at times much shorter than  $t_H^{\text{typ}}$  [Figs. 1(a) and 1(c)].

In this Letter, we revisit the concept of critical transport. We examine the critical dynamics that satisfies the following condition: the saturation time of the wave-packet mean-squared displacement, referred to as the Thouless time  $t_{\text{Th}}$ , scales with the linear system size  $L$  in the same way as the typical Heisenberg time  $t_H^{\text{typ}}$ , i.e., at the inverse typical level spacing,

$$t_{\text{Th}}(L) \propto t_H^{\text{typ}}(L). \quad (1)$$

We then establish a simple relation for the dynamical exponent at criticality,

$$z = \frac{d_l}{d_s}, \quad (2)$$

where  $d_l$  is the lattice dimension and  $d_s$  is the spectral fractal dimension. Consequently, if one uses Eq. (1) as the definition of critical dynamics, superdiffusive transport ( $1 < z < 2$ ) cannot be critical in dimensions  $d_l \geq 2$ , and diffusive transport ( $z = 2$ ) cannot be critical in  $d_l \geq 3$ . We provide a simple but nontrivial example in which these predictions can be observed.

*Preliminaries*—We consider systems described by quadratic fermionic Hamiltonians, i.e., by bilinear forms in creation and annihilation operators. The dynamics of a particle, or an excitation, are then governed by a lattice potential, and the mean-squared displacement  $\sigma_H^2(t)$  can be defined within a single-particle space. For a particle initially localized at site  $\mathbf{i}_0 = (i_0^1, \dots, i_0^{d_l})$  of a  $d_l$ -dimensional lattice,  $\sigma_H^2(t)$  is defined as

$$\sigma_H^2(t) = \sum_{\mathbf{i}} |\mathbf{i} - \mathbf{i}_0|^2 |\langle \mathbf{i} | e^{-i\hat{H}t} | \mathbf{i}_0 \rangle|^2, \quad (3)$$

where  $|\mathbf{i} - \mathbf{i}_0|$  is the Euclidean distance between the initial site  $\mathbf{i}_0$  and site  $\mathbf{i}$ , the sum runs over all lattice sites  $\mathbf{i}$ , and the time evolution is governed by the time-independent Hamiltonian  $\hat{H}$ . As we typically consider an ensemble of Hamiltonians, where for each realization we set the particle in the center of the lattice, we also define the Hamiltonian-averaged mean-squared displacement  $\sigma^2(t)$ . The time evolution of the latter is then used to define the dynamical exponent  $z$ ,

$$\sigma^2(t) = \langle \sigma_H^2(t) \rangle_H \propto t^{2/z}, \quad (4)$$

where  $\langle \dots \rangle_H$  denotes the average over Hamiltonian realizations.

We note that  $z$  in Eq. (4) is defined within a single-particle setup. The same  $z$  can be also measured from the dynamics of many-body states, e.g., the particle imbalance

of the domain wall state [13], the entanglement entropy [42,60,61], or the surface roughness [62–64].

The numerical results in this Letter are obtained for the Fibonacci models in 1D, 2D, and 3D lattices, defined as

$$\hat{H} = -J \sum_{\langle ij \rangle} (\hat{c}_i^\dagger \hat{c}_j + \hat{c}_j^\dagger \hat{c}_i) + \sum_{i=1}^V \epsilon_i \hat{n}_i, \quad (5)$$

where  $\hat{c}_j^\dagger (\hat{c}_j)$  are the fermionic creation (annihilation) operators at site  $j$ ,  $J \equiv 1$  is the hopping matrix element between nearest neighbor sites,  $\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i$  is the site occupation operator, and  $\epsilon_i$  is the on-site potential. The potential is separable and can be written as a sum,  $\epsilon_i = \sum_{l=1}^{d_l} \epsilon_{i^l}$ , where  $\epsilon_{i^l}$  is the potential value in direction  $l$ , with  $i^l \in \{1, \dots, L\}$  and  $V = L^{d_l}$  is the total number of lattice sites. The potential values in each direction are chosen by considering a randomly chosen subsequence of length  $L$  from the Fibonacci sequence of large length,  $L_F = 10^6$ . The Fibonacci sequence  $\epsilon_f$  for  $f \in \{1, \dots, L_F\}$  is generated from  $\epsilon_f = 2hG(fg) - h$ , where  $g = (\sqrt{5} - 1)/2$  and  $G(x) = [x + g] - [x]$ , with  $[x]$  being the integer part of  $x$  [13]. For  $d_l = 1$ , the system reduces to the usual 1D Fibonacci model [13,65,66].

As a technical remark, we note that the advantage of the separability of potentials in 2D (square lattice) and 3D (cubic lattice) is that the eigenenergy spectrum consists of a sum of eigenenergies of Fibonacci chains in each directions [67]; see End Matter for details. This allows us to reach spectra of systems of large linear system sizes,  $L = 10000$  in 2D and  $L = 500$  in 3D.

*Characteristic times*—We next define the two characteristic timescales relevant for our study, the Thouless and the Heisenberg time. The Thouless time  $t_{\text{Th}}$  is defined as the time when  $\sigma^2(t)$  saturates to a constant value, i.e., the time at which the particle reaches the lattice boundary. We mark  $t_{\text{Th}}$  by the vertical dashed lines in Fig. 1. For the Heisenberg time, we focus on its typical value. The typical Heisenberg time is defined as the inverse of the typical eigenvalue spacing times  $2\pi$ ,

$$t_H^{\text{typ}} = \frac{2\pi}{\delta E^{\text{typ}}} = \frac{2\pi}{\langle \exp(\langle \ln(E_\mu - E_{\mu-1}) \rangle_\mu) \rangle_H}, \quad (6)$$

where  $E_\mu$  are the eigenvalues of Hamiltonian  $\hat{H}$ ,  $\hat{H}|\mu\rangle = E_\mu|\mu\rangle$ , with  $\mu = 1, \dots, V$ . We note that while the averaging over Hamiltonian realizations can also be performed inside the exponent, i.e.,  $\langle \exp(\langle \ln(E_\mu - E_{\mu-1}) \rangle_\mu) \rangle_H \approx \exp(\langle \ln(E_\mu - E_{\mu-1}) \rangle_\mu)_H$ , it is essential to compute the typical value over  $\mu$  within a single Hamiltonian realization. In Eq. (6) we only consider nonzero energy gaps,  $E_\mu - E_{\mu-1} \neq 0$ , i.e., degeneracies (when exist) are excluded as they do not contribute nontrivially to the time evolution of the system. We average over all nonzero gaps, as they all

contribute to the time evolution. We mark  $t_H^{\text{typ}}$  by the vertical dashed-dotted lines in Fig. 1.

While the Thouless time is usually interpreted as the longest physically relevant time scale, the Heisenberg time sets the upper bound for the accessible time scales in a finite system. Intuitively, the longest timescale in the dynamics of finite systems is associated with the small gap values, and hence we consider the typical rather than the average Heisenberg time. In quantum-chaotic systems and in localized systems, one usually encounters  $t_H^{\text{typ}} \propto V$  and hence the typical level spacing is proportional to the average level spacing. However, for systems affine to clustering of eigenvalues,  $t_H^{\text{typ}}$  may scale with a higher power of  $V$ . We parametrize this behavior as

$$t_H^{\text{typ}} = c_H V^n, \quad (7)$$

where  $c_H$  is a constant and  $n$  is a number that characterizes the level clustering. Hence, for  $n > 1$  the timescale of nontrivial dynamics becomes significantly longer than in systems with  $n = 1$ .

A known example for  $n \neq 1$  in Eq. (7) is the one-dimensional (1D) Aubry–André model, for which  $n = 2$  at criticality [68]. Another example are the separable 1D, 2D, and 3D Fibonacci models, for which the scaling of  $t_H^{\text{typ}}$  versus  $L$  are shown in Fig. 2. We find  $n = 1$  for the 2D and 3D systems at potential  $h = 0.5$  [see Fig. 2(a)], and  $n \approx 1$  for the 3D system at potential  $h = 5$ ; see Fig. 2(b). All other cases displayed in Fig. 2 exhibit  $n > 1$  as consequence of the level clustering in the spectrum.

**Spectral fractality**—We now show that the exponent  $n$  from Eq. (7) is connected to the fractal dimension  $d_s$  of the Hamiltonian spectrum [69–73]. We extract the latter using the box counting method. To this end, we transform the eigenenergies  $E_\mu$  as  $\epsilon_\mu = (E_\mu - E_\mu^{\min}) / (E_\mu^{\max} - E_\mu^{\min})$ , where  $E_\mu^{\max}$  is the maximal energy and  $E_\mu^{\min}$  is the minimal energy, such that the transformed eigenspectrum  $\epsilon_\mu$  spans

the interval  $[0, 1]$ . The latter is then divided into  $1/\epsilon$  boxes  $B$  of length  $\epsilon$ . For each Hamiltonian spectrum we define the scaling function of moment  $q$ ,

$$N_{s,H}^{(q)}(\epsilon) = \sum_{B \neq B_\emptyset} \left( \sum_{\epsilon_\mu \in B} 1 \right)^q, \quad (8)$$

where the first sum runs over nonempty boxes,  $B \neq B_\emptyset$ , and the second sum counts the number of levels in each box. The scaling function at  $q = 0$  can be interpreted as the number of boxes that contain at least one of the eigenenergies  $\epsilon_\mu$ . The averaged scaling function,

$$N_s^{(q)}(\epsilon) = \left\langle N_{s,H}^{(q)}(\epsilon) \right\rangle_H \propto \epsilon^{(q-1)d_s^{(q)}}, \quad (9)$$

defines the spectral fractal dimension  $d_s^{(q)}$ . Here, we mainly focus on the scaling function at  $q = 0$ ,  $N(\epsilon) \equiv N_s^{(0)}(\epsilon)$ , as it is connected to the typical Heisenberg time; see the discussion below. Its scaling can be expressed as

$$N(\epsilon) = c_N \left( \frac{1}{\epsilon} \right)^{d_s}, \quad (10)$$

where  $c_N$  is a constant and  $d_s \equiv d_s^{(0)}$ . We numerically test Eq. (10) in Fig. 4 of the End Matter, and we show that it indeed represents a meaningful ansatz to extract the spectral fractal dimension  $d_s$ .

Now, we argue that the exponents  $n$  in Eq. (7) and  $d_s$  in Eq. (10) are related as

$$n = \frac{1}{d_s}. \quad (11)$$

The argument for validity of Eq. (11) goes as follows. Since the typical level spacing  $\delta E^{\text{typ}}$  is related to its median, one expects that at  $\epsilon = \delta E^{\text{typ}} \propto (t_H^{\text{typ}})^{-1}$ , the number of occupied boxes  $N$  is proportional to  $V$  [one may approximate it as  $N(\epsilon = \delta E^{\text{typ}}) = V/m$ , with  $m = O(1)$ ]. Then, it follows from Eq. (10) that  $V^{1/d_s} \propto t_H^{\text{typ}}$ , which is identical to Eq. (7) assuming the relation  $n = 1/d_s$  in Eq. (11). We also tested Eq. (11) numerically in Figs. 2 and 4, in which we display the values of  $n$  and  $1/d_s$ , respectively, for the same system parameters. We find excellent agreement between the numerical values of  $n$  and  $1/d_s$ .

**Derivation of critical dynamical exponent**—We now turn to the derivation of the critical dynamical exponent  $z$  from Eq. (2). In the first step, we show that the Thouless time  $t_{\text{Th}}$  generally depends on the dynamical exponent  $z$ . For systems that are not localized (i.e., at  $z < \infty$ ), the mean displacement  $\sigma(t)$  approaches the upper bound  $\bar{\sigma}$  in the infinite-time limit, which is proportional to the linear system size,

$$\bar{\sigma} \equiv \lim_{t \rightarrow \infty} \sigma(t) \propto L. \quad (12)$$

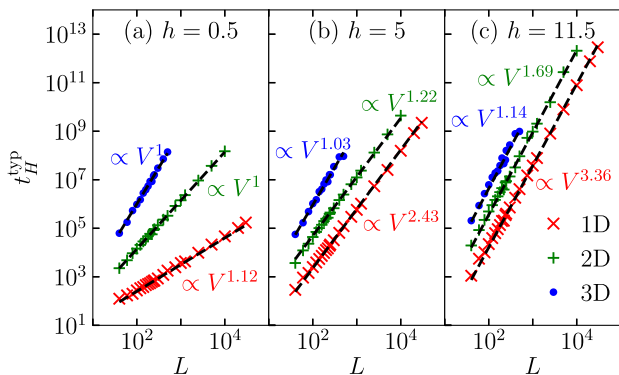


FIG. 2. Scaling of the typical Heisenberg time,  $t_H^{\text{typ}}$ , with linear systems size  $L$ , in the separable 1D, 2D, and 3D Fibonacci models. Results are shown for potentials (a)  $h = 0.5$ , (b)  $h = 5.0$  and (c)  $h = 11.5$ . Lines are fits to Eq. (7), with the values of  $n$  given in each panel, and  $V = L^{d_l}$  is the total number of lattice sites.

The upper bound  $\bar{\sigma}$  corresponds to the plateau of  $\sigma(t)$  in Fig. 1. We parametrize the Thouless time, i.e., the time at which the plateau  $\bar{\sigma}$  is reached, as  $t_{\text{Th}} \propto L^a$ , and the mean particle displacement, cf. Eq. (4), as  $\sigma(t) = c_\sigma t^{1/z}$ , where  $c_\sigma$  is a constant. Then, requiring that  $\sigma(t = t_{\text{Th}}) = c_\sigma t_{\text{Th}}^{1/z} \propto L$ , one gets  $a = z$  and hence

$$t_{\text{Th}} \propto L^z. \quad (13)$$

This is an expected result and it is consistent, e.g., with ballistic dynamics at  $z = 1$  and diffusion at  $z = 2$ .

The scaling of the typical Heisenberg time is, in contrast, governed by the spectral properties. Combining Eq. (7) with Eq. (11), its scaling with the linear system size  $L$  can be expressed as

$$t_H^{\text{typ}} \propto V_{d_s}^{-1} \propto L^{d_l/d_s}. \quad (14)$$

The two seemingly independent results in Eqs. (13) and (14) can be related at criticality when both times exhibit an identical dependence on  $L$ , as suggested by Eq. (1). Hence, the requirement for the scaling of the Thouless time to match the scaling of the typical Heisenberg time gives rise to the relation  $z = d_l/d_s$  in Eq. (2), which is the main result of this Letter.

We note that our notion of critical slowing down relies on the system's tendency to localize in real space. If localization is expected to occur in other physically relevant spaces, such as quasi-momentum space, the definition of Thouless time must be adjusted accordingly. This can be achieved by choosing a basis in which the saturation time of the mean-squared displacement is the largest. Then, the Thouless time can be interpreted as the longest relaxation time of the system.

A remarkable consequence of Eq. (2) is that it provides bounds on the critical dynamical exponent  $z$ , if one uses Eq. (1) as the definition of critical dynamics. Since we expect the fractal dimension  $d_s$  to be limited to the interval  $d_s \in [0, 1]$ , it follows that  $z \geq d_l$ . One can then argue that  $z^* = d_l$  represents the fastest critical dynamics. Hence, ballistic transport can only be critical in 1D, superdiffusive transport cannot be critical in dimensions 2 or higher, and diffusive transport cannot be critical in dimensions higher than 2.

We note that many previous works attempted for establishing a general connection between the dynamical exponent  $z$  and different versions of spectral fractal dimension; see, e.g., Refs. [66,67,74–116]. Our study suggests that the simple expression for  $z$ , cf. Eq. (2), indeed exists. However, it is limited to the regime of critical dynamics, which is given by a rather stringent criterion from Eq. (1).

Another notable consequence of the critical dynamics defined via Eq. (1) is the emergence of scale invariant principle [68,117–119]. The latter is obtained upon rescaling the time as  $t \rightarrow \tau = t/t_H^{\text{typ}}$ ; i.e., the dynamics

of  $\sigma$  becomes scale invariant in  $\tau$  when rescaled as  $\sigma \rightarrow \sigma' = \sigma/L$ . The scale invariance of  $\sigma'$  in  $\tau$  is similar to the scale invariance of the spectral form factor [120], the survival probability [68,117], and simple observables such as imbalance upon the corresponding rescaling in the  $y$  axis [118,119]. All these quantities are scale-invariant both for  $\tau < 1$  and  $\tau > 1$ , which hints on generality of the emergent scale invariance at criticality.

This should be contrasted to rescaling  $t \rightarrow \tau' = t/t_{\text{Th}}$  and  $\sigma \rightarrow \sigma' = \sigma/L$ , known as Family-Vicsek scaling law [121]; see, e.g., Ref. [63]. The dynamics of  $\sigma'$  is indeed scale invariant, both before ( $\tau' < 1$ ) and after ( $\tau' > 1$ ) the Thouless time. However, since the collapse of  $\sigma'$  in  $\tau'$  is present both for the delocalized and critical dynamics, it cannot be used to detect criticality. Also, such time rescaling appears to be relevant for  $\sigma$  and related quantities, and not for other measures of the dynamics such as the spectral form factor.

*Numerical examples*—We now test our main result, Eq. (2), for the paradigmatic disordered and quasiperiodic quadratic models. Although many eigenstate and dynamical properties of these models were discussed before (see, e.g., the reviews of the Anderson [122], Aubry-André [123], and Fibonacci [124] models), our perspective of comparing the characteristic times to detect the criticality, Eq. (1), is new and it offers a principle behind the critical dynamics. We first argue that Eqs. (1) and (2) are consistent with the available results in the literature. Perhaps the most studied disordered model is the Anderson model [47], for which  $d_s = 1$  at criticality [66]. It was found that  $z = 3$  in a 3D lattice [125,126] and  $z = 5$  in a 5D lattice [126], consistent with Eq. (2). For the quasiperiodic models, most studies focused on the 1D Aubry-André model, i.e., at  $d_l = 1$  where it was found that  $1/z = d_s = 0.5$  at the critical point [65,66,78].

We note that, in these models, the definition of the critical dynamics, as in Eq. (1), is consistent with the usual picture of criticality [19,125,127–134], i.e., the critical dynamics are observed at the transition point from one phase to another; see the discussion in the End Matter.

Another well-studied quasi-periodic model is the 1D Fibonacci model, which we reinvestigate in Fig. 3. It is known [124] that the model is critical, in a sense of multifractal eigenstate properties, for any potential strength  $h$ . Then,  $1/z = d_s \in [0, 1]$  is a continuously varying function of  $h$  [65,66]. Both results are in agreement with Eq. (2).

To our knowledge, all previous studies of critical dynamics considered either  $d_l = 1$  and  $d_s \neq 1$ , or  $d_l \neq 1$  and  $d_s = 1$ . However, the derivation of Eq. (2) is not limited to these combinations. Here we fill the missing gap and, using the 2D and 3D Fibonacci models as examples, we show that there exist systems exhibiting critical dynamics with  $d_l \neq 1$  and  $d_s \neq 1$ . We note that other examples of non-trivial critical dynamics in higher dimensional quasiperiodic models may also exist

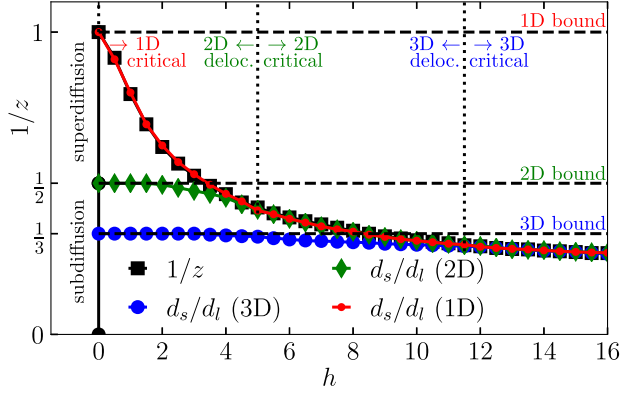


FIG. 3. Inverse dynamical exponent  $1/z$  and the ratio  $d_s/d_l$  of spectral fractal dimension  $d_s$  and the lattice dimension  $d_l$  as a function of the potential strength in 1D, 2D and 3D separable Fibonacci models.

[102,106,110,112,113,115,124,135–141] and should be studied in future work.

The separability of the potential in higher dimensions leads to the relation  $\sigma_{1D}^2(t) = \sigma_{2D}^2(t)/2 = \sigma_{3D}^2(t)/3$  for the same values of  $h$  (see the End Matter), and thus to the identical dynamical exponents  $z$  for the Fibonacci model in all dimensions. This property is illustrated in Fig. 1, where we compare  $\sigma^2(t)$  at  $h = 2$  in 2D and 1D [see Figs. 1(a) and 1(b)], and at  $h = 4$  in 1D and 3D [see Figs. 1(c) and 1(d)], observing identical  $z$  at fixed  $h$ . We hence extract  $z$  from the results in the 1D Fibonacci model at the largest size  $L = 30000$ . Results for  $1/z$  versus  $h$  are shown in Fig. 3. Starting with  $z = 1$  at  $h = 0$ , the system transits from the superdiffusive regime,  $1/2 < 1/z < 1$ , through the diffusive point at  $1/z = 1/2$ , to the subdiffusive regime,  $1/z < 1/2$ , in agreement with the results in Ref. [13].

In Fig. 3, we then compare the ratio  $d_s/d_l$  to the inverse exponent  $1/z$ , and plot the results as a function of  $h$ . In 1D, we observe  $1/z \approx d_s/d_l$  for all values of  $h$ . This suggests that Eq. (2) is always valid and hence the entire parameter range of the 1D Fibonacci model exhibits critical dynamics [13]. In contrast, Eq. (2) in the 2D and 3D Fibonacci models is only valid above a certain threshold potential  $h^*$ . Determining  $h^*$  as the lowest  $h$  at which  $1/z$  and  $d_s/d_l$  match, we roughly estimate  $h^* \approx 5$  in 2D and  $h^* \approx 11.5$  in 3D. At  $h < h^*$ , the dynamics lead to a completed delocalization of the wave packet before the Heisenberg time is reached.

The emergence of a crossover scale  $h = h^*$  for critical dynamics may appear puzzling provided that the eigenstates of separable Fibonacci models are critical (fractal) in any dimension for all  $h$ ; see End Matter for details. Our results hence raise a fundamental question: are critical dynamics necessarily linked to fractal Hamiltonian eigenstates, or can one exist without the other? We argue that if Eq. (1) defines critical transport, then fractal Hamiltonian

eigenstates do not necessary require the dynamics to be critical. On the other hand, if one allows for critical dynamics to emerge beyond the validity of Eq. (1), then the relation  $z = d_l/d_s$  from Eq. (2) may not be generally valid at criticality. Here we propose the former interpretation of critical dynamics. In this picture, validity of Eq. (2) is the defining property of critical dynamics.

**Conclusions**—In this Letter, we introduced a novel perspective on critical dynamics in short-range quadratic Hamiltonians, grounded in the validity of the relation  $z = d_l/d_s$ . We provided nontrivial examples in separable Fibonacci models, where both  $d_s \neq 1$  and  $d_l \neq 1$ . In this interpretation, at criticality, one does not observe superdiffusion in dimensions two or higher, and diffusion in dimensions three and higher.

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**Data availability**—The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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## End Matter

*Appendix A: Models with separable potentials*—Due to separability of the potentials in the Fibonacci model [Eq. (5)], many properties in higher dimensions contain simple relations to the 1D model. Let us denote the eigenstates and eigenenergies of the 1D model as  $|\phi_n\rangle$ , and  $\varepsilon_n$ , respectively, with  $n = 1, \dots, L$ . Then, the eigenstates in, say, the 3D model are their products,  $|\psi_{nmp}\rangle = |\phi_n\rangle|\phi_m\rangle|\phi_p\rangle$ , and the eigenenergies are  $E_{nmp} = \varepsilon_n + \varepsilon_m + \varepsilon_p$ , with  $n, m, p = 1, \dots, L$ .

The structure of eigenstates and eigenenergies has also implications for the dynamics, since  $e^{-i\hat{H}t}|\psi_{nmp}\rangle = e^{-i(\varepsilon_n + \varepsilon_m + \varepsilon_p)t}|\psi_{nmp}\rangle$ . For example, the transition

probabilities  $|\langle i|e^{-i\hat{H}t}|\mathbf{i}_0\rangle|^2$ , which enter the expression for the mean-squared displacement in Eq. (3), become products of 1D transition probabilities. Then, one can express the mean-squared displacement in 3D as  $\sigma_{3D}^2(t) = \sigma_x^2(t) + \sigma_y^2(t) + \sigma_z^2(t)$ , i.e., by a sum of displacements in each direction. Assuming an identical structure of the potentials in each direction, we replace the displacement in each direction by  $\sigma_{1D}^2(t)$  and we obtain  $\sigma_{3D}^2(t) = 3\sigma_{1D}^2(t)$ . More generally, in  $d_I$  dimensions this relation is extended to  $\sigma_{dID}^2(t) = d_I\sigma_{1D}^2(t)$ .

Since the mean-squared displacements in various dimensions only differ by a time-independent multiplicative

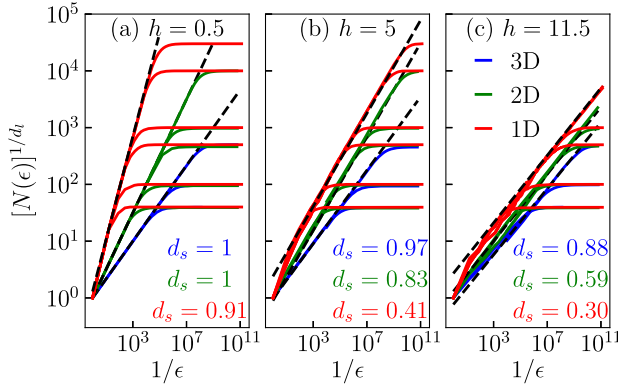


FIG. 4. Scaling of  $[N(\epsilon)]^{1/d_l}$  with the inverse box length  $1/\epsilon$ , see Eq. (10), in separable Fibonacci models of various dimension for (a)  $h = 0.5$ , (b)  $h = 5.0$ , and (c)  $h = 11.5$ . The linear sizes are  $L = 40, 100, 500, 1000, 10000, 30000$  for 1D lattice,  $L = 40, 100, 500, 1000, 10000$  for 2D lattice, and  $L = 40, 100, 500, 1000$  for the 3D lattice. The slopes of  $[N(\epsilon)]^{1/d_l}$ , given by the ratio  $d_s/d_l$ , are extracted from the fits (dashed lines). The extracted values of  $d_s$  should be compared to the values of  $n$  in Fig. 2, where the same model parameters were used. The numerical results confirm validity of the conjecture in Eq. (11) to a second digit.

prefactor  $d_l$ , they all give rise to the same dynamical exponent  $z$ ; see Eq. (4). Hence, the results for  $1/z$  in Fig. 3 in the main text applies to all dimensions under consideration.

*Appendix B: Numerical tests of Eq. (10)*—Here, we quantitatively study the scaling  $N(\epsilon) = c_N(1/\epsilon)^{d_s}$ , which is given by Eq. (10) of the main text. For a finite system,  $N(\epsilon)$  saturates to  $\lim_{\epsilon \rightarrow 0} N(\epsilon) \propto V$ , if there are no degeneracies in the spectrum, or to lower values, if degeneracies are present. Examples of the function  $N(\epsilon)$  are shown in Fig. 4 for the same model parameters and dimensions as in Fig. 2. We note that the comparison with systems across various dimensions is more convenient if one considers  $[N(\epsilon)]^{1/d_l}$ , as this quantity saturates at a linear system size,  $\lim_{\epsilon \rightarrow 0} [N(\epsilon)]^{1/d_l} \propto L$ , for all dimensions. We observe that the scaling in Eq. (10) describes very accurately the curves in Fig. 4, and the extracted values of  $d_s$  are given in each panel.

*Appendix C: Critical dynamical exponent and the correlation length exponent*—In the main text, we derived the critical dynamical exponent  $z$  in Eq. (2) from the definition of critical dynamics in Eq. (1). Here, we discuss an alternative perspective on critical dynamics for models showing a phase transition. In particular, we connect  $z$  to the exponent  $\nu$  that governs the divergence of correlation/localization length  $\xi_{\text{loc}}$  at the localization transition.

We follow Ref. [125], which established the relation between  $z$  and  $\nu$  for the Anderson model. Below, we then

show that these arguments can also be applied to the 1D Aubry-André model. The key assumption of Ref. [125] is that the mean-squared displacement has a scaling form,

$$\sigma^2(t) = Ct^{k_1} f[(W - W_c)t^{k_2}], \quad (\text{C1})$$

where  $f(x)$  is a scaling function,  $W$  is the disorder strength,  $W_c$  denotes the transition point, and  $k_1, k_2$  are exponents to be determined. The function  $f(x)$  is conjectured to exhibit different asymptotic limits, which we model as  $f(x \rightarrow -\infty) \propto |x|^u$  and  $f(x \rightarrow \infty) \propto x^v$ . At the transition point  $x = 0$ , the function  $f(0)$  is a nonzero constant, such that the exponent  $k_1$  can be linked to the critical dynamical exponent  $z$  via Eq. (4) as  $k_1 = 2/z$ .

We start with the  $d_l$ -dimensional Anderson model as discussed in Ref. [125]. In the delocalized phase of the Anderson model,  $W < W_c$ , diffusion is the dominant process at late times. Then, the mean-squared displacement can be expressed as  $\sigma^2(t) = 2d_l Dt$ , where  $D$  is the diffusion constant. It is proportional to conductivity, i.e., it scales as  $D \propto (W_c - W)^s$  [19,127], and thus

$$\sigma^2(t) \propto 2d_l(W_c - W)^s t. \quad (\text{C2})$$

In the ansatz from Eq. (C1), this corresponds to the limit  $f(x \rightarrow -\infty)$ , i.e., to the scaling of the form  $\sigma^2(t) \propto t^{k_1} [(W_c - W)t^{k_2}]^u$ . Comparing the latter to Eq. (C2), we obtain  $u = s$  and the constraint on the exponents,  $k_1 + sk_2 = 1$ . In the localized phase,  $W > W_c$ , the mean-squared displacement approaches at late times a time-independent profile that is proportional to the localization length,  $\xi_{\text{loc}} \propto (W - W_c)^{-\nu}$ , and thus

$$\sigma^2 \propto \xi_{\text{loc}}^2 \propto (W - W_c)^{-2\nu}. \quad (\text{C3})$$

In the ansatz from Eq. (C1), this corresponds to the limit  $f(x \rightarrow \infty)$ , i.e., to the scaling of the form  $\sigma^2(t) \propto t^{k_1} [(W - W_c)t^{k_2}]^v$ . Comparing the latter to Eq. (C3), we obtain  $v = -2\nu$  and hence another constraint on the exponents,  $k_1 - 2\nu k_2 = 0$ .

The constraints on the exponents  $k_1, k_2$  from Eq. (C1), which emerge from the analyses of asymptotic behaviors in Eqs. (C2) and (C3), give rise to the relations

$$k_1 = \frac{2}{z} = \frac{2\nu}{s + 2\nu}, \quad k_2 = \frac{1}{s + 2\nu}. \quad (\text{C4})$$

Using the Wegner's scaling relation  $s = (d_l - 2)\nu$  [127], we obtain  $z = d_l$ . This is consistent with the critical dynamical exponent  $z = d_l/d_s$  from Eq. (2), since  $d_s = 1$  in the Anderson model.

We now extend these arguments to the 1D Aubry-André model. In the delocalized phase (delocalized in position space), the particle transport is ballistic. Since the velocity,

$v = \sigma/t$ , vanishes linearly at the transition point [133], we express the mean-squared displacement as

$$\sigma^2(t) \propto (W_c - W)^s t^2, \quad (\text{C5})$$

with  $s = 2$ . Invoking the ansatz from Eq. (C1) as considered above, one obtains the constraint  $k_1 + sk_2 = 2$ . In the localized phase, the asymptotic behavior is again described by Eq. (C3), however, with the localization length exponent  $\nu = 1$  [129,130], and the corresponding constraint is again  $k_1 - 2\nu k_2 = 0$ . When combined, the constraints give rise to the relations

$$k_1 = \frac{2}{z} = \frac{4\nu}{s + 2\nu}, \quad k_2 = \frac{2}{s + 2\nu}. \quad (\text{C6})$$

Using the values  $s = 2$  and  $\nu = 1$ , we obtain  $z = 2$ , which is consistent with the critical dynamical exponent  $z = d_l/d_s$  from Eq. (2), since  $d_l = 1$  and  $d_s = 0.5$  in the 1D Aubry-André model.

We note that one cannot apply the analysis carried out here to extract the critical dynamical exponent  $z$  in the 1D Fibonacci chain, since there is no localization-delocalization transition in this model. This suggests that the notion of critical dynamics, as invoked by Eqs. (1) and (2), is a more general definition that can be applied even in the absence of complete localization.

**Appendix D: Fractality of the eigenstates**—For the 1D Fibonacci model, it is known that for any potential  $h > 0$  the eigenstate fractal exponent  $\gamma$  satisfies  $\gamma < 1$  [124]; i.e., the eigenstates are (multi)fractal. We calculate  $\gamma$  via the averaged inverse participation ratio (IPR) that is for arbitrary dimension defined as

$$\text{IPR} = \langle V^{-1} \sum_{\mu} \sum_i |\langle i | \psi_{\mu} \rangle|^4 \rangle_H, \quad (\text{D1})$$

where  $|\psi_{\mu}\rangle$  denotes the Hamiltonian eigenstate, and  $\mu = 1, \dots, V$ . The eigenstate fractal exponent  $\gamma$  is then defined via the scaling of IPR from Eq. (D1) as  $\text{IPR} \propto V^{-\gamma} = L^{-\gamma d_l}$ . In the inset of Fig. 5, we plot  $\gamma$  versus  $h$

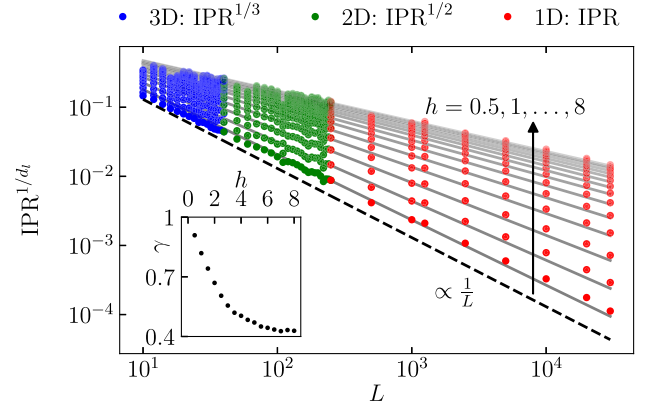


FIG. 5. The  $d_l$  root of the inverse participation ratio (IPR) in the corresponding  $d_l$  dimensional Fibonacci models. Results for  $\text{IPR}^{1/d_l}$  are shown on a log-log scale versus the linear system size  $L$ , for several values of the potential strength  $h = 0.5, 1, \dots, 8$ . The slope of the fitted curves gives the eigenstate fractal dimension  $\gamma$  [see Eq. (D2)], which is independent of  $d_l$  at a fixed value of  $h$ . Inset:  $\gamma$  versus  $h$  in the 1D model, showing that  $\gamma < 1$  for  $h > 0$ .

in the 1D Fibonacci model ( $d_l = 1$ ), and we observe that, indeed,  $\gamma < 1$  for all values of  $h > 0$ .

Using separability of potentials, discussed in Appendix A, one can show that the IPRs in higher dimensions are products of IPRs in one dimension,  $\text{IPR} = (\text{IPR}_{1D})^{d_l}$ . From this, it follows that the eigenstate fractal dimension  $\gamma$  is, at a fixed potential, identical in all dimensions, i.e.,  $\gamma_{1D} = \gamma_{2D} = \gamma_{3D}$ .

In the main panel of Fig. 5, we study the IPRs in 1D, 2D, and 3D Fibonacci models for various potential values  $h > 0$ . For the 2D and 3D lattices it is instructive to study the corresponding  $d_l$  root of the IPR, which is expected to scale as

$$\text{IPR}^{1/d_l} \propto L^{\gamma}. \quad (\text{D2})$$

This allows us to confirm in Fig. 5 that, indeed,  $\gamma$  is identical in all dimensions. Hence, the eigenstates of the separable Fibonacci models are fractal for all values of  $h$ .