

One-Loop QCD β Function as an Index

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In this Letter, we show that the one-loop QCD β function can be obtained from an index theorem on twistor space. This is achieved by recalling that the θ angle of self-dual gauge theory flows according to the one-loop β function. Rewriting self-dual gauge theory as a holomorphic theory on twistor space this flow can be computed as the anomaly to scale invariance. The one-loop Weyl anomaly coefficient $a-c$ can be recovered similarly.

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Introduction—The computation of the one-loop QCD β function [1,2] was a milestone in the study of quantum field theory. The initial computation was quite difficult, although it has since been streamlined by background field methods [3,4]. In this Letter we give an entirely new and very simple derivation which uses only algebro-geometric methods, namely twistor theory and the index theorem. We also give an index-theoretic computation of the one-loop Weyl anomaly coefficient $a-c$. The numerical factors that appear in the β function and Weyl anomaly emerge from the Bernoulli numbers in the index theorem.

θ angles in SDYM and QCD—Let us review the self-dual limit of Yang-Mills theory [5,6]. The fields of self-dual Yang-Mills theory include a gauge field A and an adjoint-valued anti-self-dual 2-form B . One can also couple to matter fields, but for now we will focus on the pure gauge theory. The action is

$$\int_{\mathbb{R}^4} \text{tr}(B \wedge F(A)_-) + \frac{i}{4\pi} \tau \int_{\mathbb{R}^4} \text{tr}(F(A) \wedge F(A)), \quad (1)$$

where τ is a complex parameter. We can view $\theta_{\text{SD}} = -2\pi i \tau$ as the θ angle of the self-dual theory. We will see momentarily that, upon deforming to full Yang-Mills theory, it differs in a crucial way from the standard θ angle. We work in Euclidean signature so that instantons appear as solutions to the classical equations of motion. In order for their contributions to be suppressed in the path integral we require that $\text{Im } \tau \geq 0$. In the usual way $\tau \sim \tau + 1$ on account of the integrality of the Chern character.

Perturbative QCD can be recovered as a deformation of self-dual QCD by adding the term $-(g_{\text{YM}}^2/8)\text{tr}(B^2)$ to the

Lagrangian. Integrating out B the action becomes

$$\frac{2}{g_{\text{YM}}^2} \int_{\mathbb{R}^4} \text{tr}(F(A)_- \wedge F(A)_-) + \frac{i}{4\pi} \tau \int_{\mathbb{R}^4} \text{tr}(F(A) \wedge F(A)). \quad (2)$$

Since $F(A)_- = \frac{1}{2}(F(A) - *F(A))$,

$$\begin{aligned} \text{tr}(F(A)_- \wedge F(A)_-) &= \frac{1}{2} \text{tr}(F(A) \wedge F(A)) \\ &\quad - \frac{1}{2} \text{tr}(F(A) \wedge *F(A)), \end{aligned} \quad (3)$$

and the action (2) is equivalent to

$$\begin{aligned} &-\frac{1}{g_{\text{YM}}^2} \int_{\mathbb{R}^4} \text{tr}(F(A) \wedge *F(A)) + \left(\frac{1}{g_{\text{YM}}^2} + \frac{i}{4\pi} \tau \right) \\ &\quad \times \int_{\mathbb{R}^4} \text{tr}(F(A) \wedge F(A)). \end{aligned} \quad (4)$$

(We are using the same conventions as [7], where the Yang-Mills action evaluates to $8\pi^2 k/g_{\text{YM}}^2$ on an instanton of charge k .)

Writing θ_{QCD} for the QCD θ angle in Euclidean signature, we have the relation

$$\tau = \frac{i}{2\pi} \theta_{\text{SD}} = \frac{i}{2\pi} \theta_{\text{QCD}} + \frac{4\pi i}{g_{\text{YM}}^2}. \quad (5)$$

Recalling that the Euclidean θ angle differs from its Lorentzian counterpart by a factor of i , τ can be identified with the complex coupling of QCD.

We know that, to leading order in θ_{QCD} , parity symmetry prevents θ_{QCD} from flowing under RG. Since g_{YM} definitely does flow, this tells us that θ_{SD} must also flow [8], and in such a way so as to leave θ_{QCD} unchanged.

More precisely, the variation of θ_{SD} with respect to the logarithm of the scale is

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$$\frac{\partial \theta_{\text{SD}}}{\partial \log \mu} = -\frac{16\pi^2}{g_{\text{YM}}^3} \frac{\partial g_{\text{YM}}}{\partial \log \mu} \quad (6)$$

Recall the standard formula for the one-loop β function

$$\frac{\partial g_{\text{YM}}}{\partial \log \mu} = -\frac{g_{\text{YM}}^3}{16\pi^2} b, \quad (7)$$

where

$$b = \frac{11}{3}N_c - \frac{2}{3}N_f - \frac{1}{6}N_s. \quad (8)$$

Here N_f, N_s are the number of fundamental Dirac fermions and fundamental complex scalars, respectively. The factors of $16\pi^2/g_{\text{YM}}^3$ cancel leaving

$$\frac{\partial \theta_{\text{SD}}}{\partial \log \mu} = b. \quad (9)$$

Hence, although the self-dual sector of QCD is perturbatively a conformal field theory, indeed it has no coupling, nonperturbatively the self-dual θ angle flows according to the one-loop QCD β function.

Our conventions are such that the path integral measure involves e^{-S} , where S is the action above. In the self-dual theory an instanton of charge k is weighted by $e^{-\theta_{\text{SD}}k}$ in the path integral. For an asymptotically free theory b is positive, so that instantons are exponentially suppressed in the UV.

That the one-loop measure on instanton moduli space is determined by the one-loop β function is well known [9,10]. In general it receives contributions from both massive and zero modes which conspire to reproduce (8). In supersymmetric theories the massive contributions cancel, and the one-loop coefficient can be obtained by counting zero modes. We will see that index-theoretic calculations on twistor space can recover the combined contribution in nonsupersymmetric theories.

Computing the flow of the θ angle using the index theorem—We will use the index theorem to give a different computation of the scale dependence of θ_{SD} . One-loop Feynman diagrams contribute [9–11] to the measure on the moduli space $\mathcal{M}_k^{\text{inst}}$ of instantons, and we will use the index theorem to compute how this determinant transforms under scaling of space-time.

The failure of the determinant to be scale invariant is an anomaly. Let us recall how anomalies for flavor symmetries can be computed using the index theorem, following [12]. Consider some free fermions on $\mathbb{R}^4$ with a G flavor symmetry. We can study the theory in the presence of a background G bundle by replacing the Dirac operator by the covariant Dirac operator. If $P \rightarrow X$ is a principal G bundle, then we get a family of theories parametrized by X .

The partition function of the theory is a section of a determinant line bundle on X . The curvature of the determinant line bundle can be computed using the family index theorem [13–15]; this curvature represents the anomaly.

We will use this technique to study the anomaly to scale invariance. Instead of considering a theory on $\mathbb{R}^4$ in the presence of a family of G bundles parametrized by some X , we will consider instead some family of manifolds parametrized by X .

We write $\mathbb{PT} = \mathcal{O}(1)^2 \rightarrow \mathbb{CP}^1$ for the twistor space of $\mathbb{R}^4$. Let X be some manifold equipped with a complex line bundle L , which we assume admits a square root. Given this data, we can build a family of twistor spaces $\mathbb{PT}_L \rightarrow X$, as the total space of the vector bundle

$$\mathcal{O}(1) \otimes L \oplus \mathcal{O}(1) \otimes L \rightarrow X \times \mathbb{CP}^1. \quad (10)$$

The moduli space of charge k instantons $\mathcal{M}_k^{\text{inst}}$ is the moduli of holomorphic bundles on $\mathbb{PT}$ with second Chern class k , framed at ∞ [16,17]. Let $\mathcal{M}_{k,L}^{\text{inst}} \rightarrow X$ be the corresponding family over X of moduli spaces of bundles on the fibers of $\mathbb{PT}_L \rightarrow X$.

The space of linearly embedded $\mathbb{CP}^1$ s in $\mathbb{PT}$ is complexified space-time. Correspondingly, we can build a family of complexified space-times from the family of twistor spaces $\mathbb{PT}_L$. This family of complexified space-times is simply $L \otimes \mathbb{C}^4$; it is clear that the scale is controlled by the line bundle L .

The determinant contribution to the measure on $\mathcal{M}_{k,L}^{\text{inst}}$ is a section of a determinant line bundle. We will find that the first Chern class of this line bundle, evaluated using the index theorem, is

$$bkc_1(L). \quad (11)$$

That is, the line bundle itself is isomorphic to $L^{\otimes bk}$. The anomaly to scale of the determinant of the $\bar{\partial}$ operator is bk , as desired.

Let us describe the index theory calculation we need to do. Writing

$$E \rightarrow \mathbb{PT}_L \times_X \mathcal{M}_{k,L}^{\text{inst}} \quad (12)$$

for the universal bundle, positive helicity gauge bosons live in the Dolbeault complex of twistor space valued in the bundle $\mathfrak{sl}(E)$. Negative helicity gluons live in $\mathfrak{sl}(E) \otimes K$, where $K = \mathcal{O}(-4) \otimes L^{-2}$ is the canonical bundle. Writing Π for a parity shift, positive helicity Dirac fermions live in $\Pi(E \oplus E^\vee) \otimes \mathbb{C}^{N_f} \otimes K^{1/4}$ and negative helicity Dirac fermions live in $\Pi(E \oplus E^\vee) \otimes \mathbb{C}^{N_f} \otimes K^{3/4}$. Fundamental complex scalars live in $(E \oplus E^\vee) \otimes K^{1/2} \otimes \mathbb{C}^{N_s}$. The total field content lives in the bundle

$$\begin{aligned}
 R = & \mathfrak{sl}(E) \otimes (\mathcal{O} \oplus K) \\
 & \oplus \Pi(E \oplus E^\vee) \otimes (K^{1/4} \oplus K^{3/4}) \otimes \mathbb{C}^{N_f} \\
 & \oplus (E \oplus E^\vee) \otimes K^{1/2} \otimes \mathbb{C}^{N_s}.
 \end{aligned} \quad (13)$$

We are interested in the square root of the determinant of the $\bar{\partial}$ operator valued in R . We take the square root because the Dolbeault complex valued in R describes fields in the Batalin-Vilkovisky (BV) formalism, including antifields, and the path integral measure in the BV setting is a section the square root of the canonical bundle of the space of fields [18].

Let $T_{\mathbb{P}\mathbb{T}_L/X}$ be the tangent bundle pointing along the fibers of the map $\mathbb{P}\mathbb{T}_L \rightarrow X$. According to the Grothendieck-Riemann-Roch theorem [19–21] (a precursor of the family index theorem), the first Chern class of the determinant line is given by the integral

$$\frac{1}{2} \int_{\mathbb{P}\mathbb{T}_L} [\text{Td}(T_{\mathbb{P}\mathbb{T}_L/X}) \text{Ch}(R)]_8. \quad (14)$$

The factor of $\frac{1}{2}$ appears because we work in the BV formalism. The integral is taken along the fibers of the map $\mathbb{P}\mathbb{T}_L \times_X \mathcal{M}_{k,L}^{\text{inst}} \rightarrow \mathcal{M}_{k,L}^{\text{inst}}$, and $[\]_8$ indicates we only take the cohomology class of degree 8. Note that the integrand here is the equivariant Todd class and the equivariant Chern class with equivariant parameter $q = c_1(L)$; our computation can be viewed as being an equivariant index theory calculation, instead of a family index theory calculation. (That the equivariant and family index theorems are related is well understood, see for example [22].)

Let us compute the Todd class of $T_{\mathbb{P}\mathbb{T}_L/X}$. There is a short exact sequence

$$0 \rightarrow (\mathcal{O}(1) \otimes L)^2 \rightarrow T_{\mathbb{P}\mathbb{T}_L/X} \rightarrow \mathcal{O}(2) \rightarrow 0. \quad (15)$$

Therefore, the Todd class of $T_{\mathbb{P}\mathbb{T}_L/X}$ is

$$\text{Td}(T_{\mathbb{P}\mathbb{T}_L/X}) = \text{Td}(\mathcal{O}(1) \otimes L)^2 \text{Td}(\mathcal{O}(2)). \quad (16)$$

Let H be the first Chern class of $\mathcal{O}(1)$. For any line bundle S , recall that the Todd class is

$$\text{Td}(S) = 1 + \frac{c_1(S)}{2} + \frac{c_1(S)^2}{12} + \dots \quad (17)$$

Since $H^2 = 0$, and we are only interested in terms that are linear in $c_1(L)$, only the first three terms of the Todd class will play a role. We find that

$$\begin{aligned}
 \text{Td}(T_{\mathbb{P}\mathbb{T}_L/X}) &= \left(1 + \frac{1}{2}H + \frac{1}{2}c_1(L) + \frac{1}{6}c_1(L)H\right)^2 (1 + H) \\
 &= \left(1 + H + c_1(L) + \frac{5}{6}c_1(L)H\right)(1 + H) \\
 &= 1 + 2H + c_1(L) + \frac{11}{6}c_1(L)H.
 \end{aligned} \quad (18)$$

Here we have dropped all terms involving H^2 or $c_1(L)^2$.

Since R is a real vector bundle, it has only even Chern characters. Only the second and fourth Chern characters will play a role, as $\text{ch}_0(R)$ will multiply $\text{td}_4(T_{\mathbb{P}\mathbb{T}_L/X})$ which is zero. Because of this, the difference between $\mathfrak{sl}(E)$ and $E^\vee \otimes E$ does not matter. Further, terms in $\text{ch}_4(R)$ can only contribute if they are divisible by $c_1(L)$. To get a nonzero integral over $\mathbb{P}\mathbb{T}_L$, we also need only retain those terms in $\text{Ch}(R)$ which are divisible by $\text{ch}_2(E)$. Bearing this in mind, we can compute the contributions of each field to half of the Chern character of R .

(i) The gauge boson bundle $\mathfrak{sl}(N_c) \otimes (\mathcal{O} \oplus K)$ contributes

$$2N_c(1 - 2H - c_1(L) + 4c_1(L)H)\text{ch}_2(E). \quad (19)$$

(ii) The Dirac fermion bundle $\Pi(E \oplus E^\vee) \otimes (K^{1/4} \oplus K^{3/4}) \otimes \mathbb{C}^{N_f}$ contributes

$$-2N_f \left(1 - 2H - c_1(L) + \frac{5}{2}c_1(L)H\right)\text{ch}_2(E). \quad (20)$$

(iii) The complex scalar bundle $(E \oplus E^\vee) \otimes K^{1/2} \otimes \mathbb{C}^{N_s}$ contributes

$$N_s(1 - 2H - c_1(L) + 2c_1(L)H)\text{ch}_2(E). \quad (21)$$

We need to multiply the sum of these three terms by the Todd class, retaining only those terms divisible by $c_1(L)H\text{ch}_2(E)$. The integral over $\mathbb{P}\mathbb{T}_L$ will send $H\text{ch}_2(E)$ to the instanton number k .

The result is

$$\begin{aligned}
 &\left(\frac{11}{6}(2N_c - 2N_f + N_s) - 4(2N_c - 2N_f + N_s)\right. \\
 &\quad \left.+ (8N_c - 5N_f + 2N_s)\right)c_1(L) \int_{\mathbb{P}\mathbb{T}_L} H\text{ch}_2(E) \\
 &= \left(\frac{11}{3}N_c - \frac{2}{3}N_f - \frac{1}{6}N_s\right)kc_1(L)
 \end{aligned} \quad (22)$$

matching precisely our expectation from Eq. (9).

Weyl anomaly on Gibbons-Hawking spaces—Gravitational counterparts of the above techniques can also be used to compute one-loop Weyl anomalies. Similarities between the twistorial anomalies of self-dual theories and

one-loop Weyl anomalies have previously been noted in [23]; here we will see that the latter arise as particular examples of the former [24]. As for the one-loop β function, the Weyl anomalies of supersymmetric theories can be recovered by counting space-time zero modes [27]. In contrast, the index theorem on twistor space is sensitive to the contributions of massive modes in nonsupersymmetric theories.

Consider self-dual QCD on the Einstein four-manifold (M, g) . Constant infinitesimal scale transformations act by $\delta g_{\mu\nu} = 2g_{\mu\nu}$, so the scale dependence of the partition function $\mathcal{Z}$ can be computed using

$$\int_M 2g^{\mu\nu} \frac{\delta \log \mathcal{Z}}{\delta g^{\mu\nu}} = - \int_M \text{vol}_g g^{\mu\nu} \langle T_{\mu\nu} \rangle, \quad (23)$$

where $T_{\mu\nu} = -2\delta S/\delta g^{\mu\nu}$ is the Hilbert stress-energy tensor. The right-hand side is determined by the one-loop Weyl anomaly

$$g^{\mu\nu} \langle T_{\mu\nu} \rangle = \frac{1}{(4\pi)^2} (cC^2 - aE + \xi\Delta S) - \frac{b}{2(4\pi)^2} \text{tr}(F^2), \quad (24)$$

where $C^2 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ for C the Weyl tensor, E is the Euler density, and S is the Ricci scalar [28,29]. For an Einstein manifold M these invariants simplify: $C^2 = E$ and $S = 0$, so the constant ξ drops out completely. Specializing to the case of self-dual QCD the anomaly coefficients are [30]

$$\begin{aligned} a &= \frac{1}{180} \left(31N_g + \frac{11}{2}N_f N_c + N_s N_c \right), \\ c &= \frac{1}{60} (6N_g + 3N_f N_c + N_s N_c) \end{aligned} \quad (25)$$

for $N_g = \dim \text{SU}(N_c) = N_c^2 - 1$ the number of gluons. One-loop exactness of self-dual QCD ensures there are no higher loop corrections.

Now suppose we further demand that (M, g) is a self-dual Einstein four-manifold, i.e., that both $\text{Ric} = 0$ and $C = *C$. Then

$$\delta \log \mathcal{Z} = -\frac{1}{8\pi^2} (c - a) \int_M \text{tr}_{T_M} (C \wedge C), \quad (26)$$

where we are interpreting C as a 2-form with values in endomorphisms of the tangent bundle. We recognize the right-hand side as $(c - a)p$ for p the Pontryagin class of M .

We will use index theory to give an independent computation of (26). In order to do so we restrict to a simple class self-dual Einstein four-manifolds: the Gibbons-Hawking metrics [31]. These are asymptotically locally Euclidean, that is, they resemble $\mathbb{R}^4/\Gamma$ at infinity for some finite subgroup $\Gamma \subset \text{SU}(2)$. In the particular case of the Gibbons-Hawking metrics Γ is the cyclic group of order

$m \in \mathbb{Z}_{\geq 2}$. Explicitly,

$$g_{\text{GH}} = \frac{(d\tau + \vec{\omega}(\vec{x}) \cdot d\vec{x})^2}{V(\vec{x})} + V(\vec{x}) \|d\vec{x}\|^2, \quad (27)$$

where $\vec{x} \in \mathbb{R}^3$, $\tau \sim \tau + 4\pi$ are coordinates, and $\vec{\nabla} \times \vec{\omega}(\vec{x}) = \vec{\nabla} V(\vec{x})$ for

$$V(\vec{x}) = \sum_{n=1}^m \frac{1}{\|\vec{x} - \vec{x}_n\|}. \quad (28)$$

The metric depends on m points $\{\vec{x}_n\}_{n=1}^m \in \mathbb{R}^{3m}$, though acting by the Euclidean group on these parameters yields isometric geometries. Therefore the moduli space of Gibbons-Hawking metrics $\mathcal{M}^{\text{GH}}$ has dimension $3(m-2)$ (except in the case $m=2$ where it is one dimensional due to axisymmetry). The metric (27) is nonsingular as long as none of the centers $\vec{x}_n$ coincide. Constant scale transformations $s \in \mathbb{R}_{>0}$ acting by $g_{\text{GH}} \mapsto s^2 g_{\text{GH}}$ induce the action $\vec{x}_n \mapsto s^2 \vec{x}_n$ on $\mathcal{M}^{\text{GH}}$.

The advantage of restricting to Gibbons-Hawking metrics is that their twistor spaces have a simple description due to Hitchin [32]. Recall Penrose's nonlinear graviton [33]: it identifies complex four-manifolds equipped with self-dual Ricci-flat holomorphic metrics and complex threefolds that fiber holomorphically over $\mathbb{CP}^1$ with an $\mathcal{O}(2)$ valued symplectic form on the fibers. [The complex threefold is also required to admit a rational curve with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.] To recover a self-dual Ricci-flat Riemannian four-manifold the twistor space must also be equipped with an antiholomorphic involution σ acting as the antipodal map on the rational curve.

The twistor spaces of the Gibbons-Hawking metrics (27) are the hypersurfaces in $\mathcal{O}(m) \oplus \mathcal{O}(m) \oplus \mathcal{O}(2) \rightarrow \mathbb{CP}^1$ cut out by

$$XY = \prod_{n=1}^m (Z^2 - \vec{x}_n(\zeta)), \quad (29)$$

where the points in $\vec{x}_n \in \mathbb{R}^3$ have been identified with sections $\vec{x}_n(\zeta)$ of $\mathcal{O}(2) \rightarrow \mathbb{CP}^1$ equivariant under $(Z, \zeta) \mapsto (\bar{Z}/\bar{\zeta}^2, \zeta \mapsto -1/\bar{\zeta})$. The twistor spaces (29) admit an antiholomorphic involution

$$\sigma: (X, Y, Z, \zeta) \mapsto (\bar{X}/\bar{\zeta}^m, \bar{Y}/\bar{\zeta}^m, \bar{Z}/\bar{\zeta}^2, -1/\bar{\zeta}) \quad (30)$$

encoding the Riemannian slice. Forgetting the involution σ , the moduli space $\mathcal{M}^{\text{GH}}$ naturally complexifies to $\mathcal{M}_{\mathbb{C}}^{\text{GH}}$ where the points $\{\vec{x}_n\}_{n=1}^m$ are now permitted to take values in $\mathbb{C}^{3m}$. Sitting above this moduli space is a universal bundle of twistor spaces $\mathcal{PT} \rightarrow \mathcal{M}_{\mathbb{C}}^{\text{GH}}$. Scale transformations complexify to a $\mathbb{C}^\times$ action

$$(X, Y, Z, \zeta, \vec{x}_n) \mapsto (sX, sY, sZ, \zeta, s^2 \vec{x}_n) \quad (31)$$

on the universal bundle. We will now compute the anomaly to this $\mathbb{C}^\times$ action.

Computing the Weyl anomaly using the index theorem— Our strategy is to mimic the calculation in the section on “computing the flow of the θ angle using the index theorem.” We can define self-dual QCD at points of $\mathcal{M}_C^{\text{GH}}$ as a holomorphic theory on the associated twistor space. We will study a family of self-dual gauge theories on complexified Gibbons-Hawking spaces whose scale is controlled by the line bundle $L \rightarrow X$. We begin by forming the associated bundle $\mathcal{PT}_L = L \times_{\mathbb{C}^\times} \mathcal{PT} \rightarrow X$. This factors through the family of moduli spaces $\mathcal{M}_L^{\text{GH}} = L \times_{\mathbb{C}^\times} \mathcal{M}_C^{\text{GH}} \rightarrow X$. As in the case of instantons, we consider the square root of the determinant of the $\bar{\partial}$ operator on a modification of the bundle R (13): we replace the bundle E with the trivial $\mathbb{C}^{N_c}$ bundle over $\mathcal{PT}_L$, $\mathcal{O}$ by the structure sheaf of $\mathcal{PT}_L$ and take $K = \text{Det}(T_{\mathcal{PT}_L/\mathcal{M}_L^{\text{GH}}})^{-1}$.

The partition function of holomorphic BF theory on this family is a section of the determinant line bundle over $\mathcal{M}_L^{\text{GH}}$. Applying the Grothendieck-Riemann-Roch theorem the first Chern class of the determinant line is

$$\frac{1}{2} \int_{\mathcal{PT}_L} \left[\text{Td}(T_{\mathcal{PT}_L/\mathcal{M}_L^{\text{GH}}}) \text{Ch}(R) \right]_8. \quad (32)$$

Choosing H to have support at a point $\zeta \in \mathbb{CP}^1$ the integral localizes to the fiber $\pi^{-1}(\zeta) = M_{\text{GH}}(\zeta)$: the Gibbons-Hawking space in the complex structure corresponding to ζ . The normal bundle $\mathcal{N}$ can be identified with $T^{1,0}M_{\text{GH}}(\zeta)$, and its second Chern character is independent of ζ .

$$\begin{aligned} \text{ch}_2(T^{1,0}M_{\text{GH}}(\zeta)) &= -\frac{1}{2} c_2(T^{1,0}M_{\text{GH}}(\zeta) \oplus T^{0,1}M_{\text{GH}}(\zeta)) \\ &= -\frac{1}{2} c_2(T_C M_{\text{GH}}) = \frac{1}{2} p. \end{aligned} \quad (36)$$

Here M_{GH} is the Gibbons-Hawking space at a particular point of the moduli space $\mathcal{M}_L^{\text{GH}}$. We therefore find that the index evaluates to

$$-\frac{1}{360} (26N_g - 7N_f N_c - 4N_s N_c) c_1(L) p. \quad (37)$$

The integral is taken along the fibers of the map $\mathcal{PT}_L \rightarrow \mathcal{M}_L^{\text{GH}}$.

In order to evaluate (32) we exploit the fact that the twistor space of a self-dual Einstein manifold fibers holomorphically over $\mathbb{CP}^1$. We therefore have a holomorphic projection map $\pi: \mathcal{PT} \rightarrow \mathbb{CP}^1 \times \mathcal{M}_C^{\text{GH}}$, and differentiating yields the short exact sequence

$$0 \rightarrow \mathcal{N} \rightarrow T_{\mathcal{PT}/\mathcal{M}_C^{\text{GH}}} \rightarrow T_{\mathbb{CP}^1} \rightarrow 0, \quad (33)$$

where we abuse notation by writing $T_{\mathbb{CP}^1}$ for its pullback to $\mathcal{PT}$ and $\mathcal{N}$ denotes the normal bundle to the $\mathbb{CP}^1$ in the twistor space directions. From the complexified scaling action (31) this short exact sequence is $\mathbb{C}^\times$ equivariant if we assign $\mathcal{N}$ charge one. Forming the associated bundles gives the following short exact sequence over $\mathcal{PT}_L$

$$0 \rightarrow L \otimes \mathcal{N} \rightarrow T_{\mathcal{PT}_L/\mathcal{M}_L^{\text{GH}}} \rightarrow T_{\mathbb{CP}^1} \rightarrow 0, \quad (34)$$

allowing the index (32) to be evaluated by additivity; the computation is included in Supplemental Material [34]. We find the characteristic classes evaluate to

$$\begin{aligned} & -\frac{1}{12} \left(\frac{26}{15} (\text{rk}(\mathfrak{sl}(E)) - 2N_f \text{rk}(E) + N_s \text{rk}(E)) - 4(\text{rk}(\mathfrak{sl}(E)) - 2N_f \text{rk}(E) + N_s \text{rk}(E)) \right. \\ & \quad \left. + (4 \text{rk}(\mathfrak{sl}(E)) - 5N_f \text{rk}(E) + 2N_s \text{rk}(E)) \right) c_1(L) \int_{\mathcal{PT}_L} H \text{ch}_2(\mathcal{N}) \\ &= -\frac{1}{180} (26N_g - 7N_f N_c - 4N_s N_c) c_1(L) \int_{\mathcal{PT}_L} H \text{ch}_2(\mathcal{N}). \end{aligned} \quad (35)$$

Comparing to the a, c coefficients in Eq. (25) we recognize this as $(c - a)p$. Equivalently, the determinant line bundle is isomorphic $L^{\otimes(c-a)p}$ matching our expectation from Eq. (26).

*Discussion—*We have evaluated the one-loop QCD β function using index-theoretic arguments on twistor space. Can higher-loop contributions to the β function also be obtained? In [26] one of the authors demonstrated a two-loop flow in the operator $\text{tr}(B^2)$ of the form

$$\frac{\partial}{\partial \log \mu} \text{tr}(B \wedge B) = -\frac{\lambda^2}{24\pi^4} \text{tr}(F \wedge F) \quad (38)$$

under the assumption that

$$\text{tr}_{\text{Ad} \oplus \Pi(\mathbb{F} \oplus \mathbb{F}^\vee)^{\oplus N_f} \oplus \mathbb{F}^{\oplus N_s}}(X^4) = \lambda^2 \text{tr}_F(X^2)^2. \quad (39)$$

Deforming self-dual QCD with this operator generates a two-loop contribution to the flow of the self-dual θ angle

$$\frac{\partial}{\partial \log \mu} \theta_{\text{SD}} = b - \frac{\lambda^2}{3} \frac{g_{\text{YM}}^2}{8\pi^2} \quad (40)$$

which, by the arguments of the section on “ θ angles in SDYM and QCD,” is compensated by a two-loop flow of the QCD coupling:

$$\frac{\partial}{\partial \log \mu} g_{\text{YM}} = -b \frac{g_{\text{YM}}^3}{16\pi^2} + \frac{2\lambda^2}{3} \frac{g_{\text{YM}}^5}{(16\pi^2)^2}. \quad (41)$$

Beyond two loops we expect higher-order corrections to the above generated by multiple insertions of $\text{tr}(B^2)$.

Equation (41) does not coincide with the familiar β function at two loops [40–43]. Indeed, $\lambda^2 = 0$ for $\mathcal{N} = 1$ supersymmetric Yang-Mills theory but the β function has a nonvanishing two-loop contribution. This discrepancy is accounted for by noting that $8\pi^2/g_{\text{YM}}^2$ is the real part of the “holomorphic” coupling θ_{SD} , which is one-loop exact in supersymmetric gauge theory [46,47]. Rescaling the gauge field in order to canonically normalize the gauge kinetic term generates an anomalous Jacobian responsible for higher-loop contributions to the flow of the canonical coupling.

We have seen that the linear combination of anomaly coefficients a - c can be computed via index theory. Can the two coefficients be recovered independently? This should follow from applying the arguments of the section on “computing the Weyl anomaly using the index theorem” on a self-dual Riemannian four-manifold with nonvanishing Ricci tensor. A natural asymptotically flat candidate is Burns space [48].

Indeed, that twistorial anomalies are sensitive to the a , c coefficients independently can be seen indirectly. Infinitesimal conformal symmetries of a self-dual conformal background lift to holomorphic vector fields on the corresponding twistor space. One-loop conformal anomalies therefore appear as (a subset of the) gravitational anomalies on twistor space. These are polynomials in the Chern classes of the tangent bundle of degree eight, i.e., a linear combination of c_1^4 , $c_2 c_1^2$, $c_3 c_1$, and c_2^2 . By evaluating the contribution from ordinary fields (conformally coupled scalars, fermions, gluons) and also from higher derivative fields appearing in the $\mathcal{N} = 4$ superconformal gravity supermultiplet (conformal gravitons, conformal gravitinos, antisymmetric tensors, higher derivative Weyl fermions, and conformally coupled dimension zero complex scalars) we find that the four classes naturally organize into

$$\begin{aligned} & \frac{1}{128} (16c_2 - 5c_1^2) c_1^2 a + \frac{1}{64} (3c_1^2 - 8c_2) c_1^2 c \\ & + \frac{1}{46080} (96c_2^2 - 80c_1^2 c_2 + 32c_1 c_3 + 15c_1^4) \chi \\ & + \frac{1}{64} (3c_1^3 - 16c_1 c_2 + 64c_3) c_1 \psi. \end{aligned} \quad (42)$$

Here a , c are the Weyl anomaly coefficients. The classes with coefficients χ , ψ do not represent space-time anomalies: χ is the coefficient of the twistorial anomaly in self-dual Einstein gravity which counts on-shell degrees of freedom weighted by parity [49] and ψ represents a novel twistorial anomaly in self-dual conformal gravity [50]. Values of the coefficients a , c , χ , and ψ are included in Supplemental Material [34]. In total we have evaluated the anomaly polynomial on eight different types of fields; four are needed to fix the classes multiplying a , c , and the remaining four are nontrivial checks on the validity of (42).

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- [1] H. D. Politzer, *Phys. Rev. Lett.* **30**, 1346 (1973).
 - [2] D. J. Gross and F. Wilczek, *Phys. Rev. D* **8**, 3633 (1973).
 - [3] L. F. Abbott, *Nucl. Phys.* **B185**, 189 (1981).
 - [4] L. F. Abbott, *Acta Phys. Pol. B* **13**, 33 (1982).
 - [5] G. Chalmers and W. Siegel, *Phys. Rev. D* **54**, 7628 (1996).
 - [6] A. Losev, I. Polyubin, and A. Rosly, *J. High Energy Phys.* **02** (2018) 041.
 - [7] T. Schäfer and E. V. Shuryak, *Rev. Mod. Phys.* **70**, 323 (1998).
 - [8] The flow of θ_{SD} was computed by a direct Feynman diagram analysis in [6].
 - [9] G. 't Hooft, *Phys. Rev. D* **14**, 3432 (1976); **18**, 2199(E) (1978).
 - [10] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, *Sov. Phys. Usp.* **25**, 195 (1982).
 - [11] L. S. Brown and D. B. Creamer, *Phys. Rev. D* **18**, 3695 (1978).
 - [12] N. K. Nielsen and B. Schroer, *Nucl. Phys.* **B127**, 493 (1977).
 - [13] M. F. Atiyah and I. M. Singer, *Ann. Math.* **93**, 119 (1971).
 - [14] D. Quillen, *Funct. Anal. Appl.* **19**, 31 (1985).
 - [15] J.-M. Bismut and D. S. Freed, *Commun. Math. Phys.* **106**, 159 (1986).
 - [16] R. S. Ward, *Phys. Lett.* **61A**, 81 (1977).

- [17] M. Atiyah, N. Hitchin, V. Drinfeld, and Y. I. Manin, *Phys. Lett.* **65A**, 185 (1978).
- [18] A. S. Schwarz, *Commun. Math. Phys.* **155**, 249 (1993).
- [19] A. Borel and J.-P. Serre, *Bull. Soc. Math. Fr.* **86**, 97 (1958).
- [20] F. Hirzebruch, A. Borel, and R. Schwarzenberger, *Topological Methods in Algebraic Geometry* (Springer Berlin-Heidelberg-New York, 1966), Vol. 175.
- [21] A. Grothendieck *et al.*, *Lecture Notes in Mathematics* (Springer-Verlag, Berlin-New York, 1971), Vol. 225.
- [22] D. S. Freed, *Surveys Diff. Geom.* **22**, 125 (2017).
- [23] G. Doran, R. Monteiro, and S. Wikeley, *J. High Energy Phys.* **07** (2024) 139.
- [24] For example, the portion of the quantum effective action of a 4D field theory determined by the trace anomaly can be described by a dimension zero scalar field coupling to curvatures [25]. This is reminiscent of the Green-Schwarz mechanism for twistorial anomalies [25][26].
- [25] R. J. Riegert, *Phys. Lett.* **134B**, 56 (1984).
- [26] K. J. Costello, [arXiv:2111.08879](https://arxiv.org/abs/2111.08879).
- [27] S. M. Christensen and M. J. Duff, *Nucl. Phys.* **B154**, 301 (1979).
- [28] D. M. Capper and M. J. Duff, *Nuovo Cimento A* **23**, 173 (1974).
- [29] M. J. Duff, *Nucl. Phys.* **B125**, 334 (1977).
- [30] M. J. Duff, *Classical Quantum Gravity* **11**, 1387 (1994).
- [31] G. W. Gibbons and S. W. Hawking, *Phys. Lett.* **78B**, 430 (1978).
- [32] N. J. Hitchin, *Math. Proc. Cambridge Philos. Soc.* **85**, 465 (1979).
- [33] R. Penrose, *Gen. Relativ. Gravit.* **7**, 31 (1976).
- [34] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.136.211601> for details of the index computation for the Weyl anomaly, and for the contributions of a range of massless fields to the gravitational anomaly on twistor space. Supplemental Material includes Refs. [35–39].
- [35] E. S. Fradkin and A. A. Tseytlin, *Nucl. Phys.* **B203**, 157 (1982).
- [36] E. S. Fradkin and A. A. Tseytlin, *Phys. Lett. B* **110**, 117 (1982); **126B** (1983).
- [37] E. S. Fradkin and A. A. Tseytlin, *Phys. Lett.* **134B**, 187 (1984).
- [38] E. Witten, *Commun. Math. Phys.* **252**, 189 (2004).
- [39] N. Berkovits and E. Witten, *J. High Energy Phys.* **08** (2004) 009.
- [40] W. E. Caswell, *Phys. Rev. Lett.* **33**, 244 (1974).
- [41] D. R. T. Jones, *Nucl. Phys.* **B75**, 531 (1974).
- [42] D. R. T. Jones, *Nucl. Phys.* **B87**, 127 (1975).
- [43] The two-loop flow of the canonical coupling also receives contributions from Yukawa couplings which are difficult to incorporate on twistor space [44,45].
- [44] D. R. T. Jones, *Phys. Rev. D* **25**, 581 (1982).
- [45] M. E. Machacek and M. T. Vaughn, *Nucl. Phys.* **B222**, 83 (1983).
- [46] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Phys. Lett.* **166B**, 329 (1986).
- [47] N. Arkani-Hamed and H. Murayama, *J. High Energy Phys.* **06** (2000) 030.
- [48] C. LeBrun, *J. Diff. Geom.* **34**, 223 (1991).
- [49] R. Bittleston, A. Sharma, and D. Skinner, *Commun. Math. Phys.* **403**, 1543 (2023).
- [50] Although we note that the twistorial anomaly coefficient χ is proportional to the one-loop vacuum energy [51]. It is not clear to us if ψ also has a space-time avatar.
- [51] L. Boyle and N. Turok, [arXiv:2110.06258](https://arxiv.org/abs/2110.06258).