

Tripling Fluctuations and Peaked Sound Speed in Fermionic Matter

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 (Received 6 December 2024; revised 7 May 2025; accepted 27 June 2025; published 23 July 2025)

A crossover involving three-fermion clusters is relevant to the hadron-quark crossover, which, if occurring in a neutron star, could naturally reproduce the dense-matter equation of state recently deduced from simultaneous observations of neutron-star masses and radii. To understand the crossover mechanism, we examine the role of *tripling* fluctuations induced by the formation of three-fermion clusters. The phase-shift representation of fluctuations manifests an interplay of bound and scattering states, leading to nonmonotonic momentum distributions of baryonlike clusters and peaked sound speed at finite densities. We demonstrate them by applying our approach to a nonrelativistic system of one-dimensional three-color fermions analogous to the hadron-quark matter.

DOI: [10.1103/4ywp-752m](https://doi.org/10.1103/4ywp-752m)

Introduction—The properties of dense matter above the normal nuclear density, once elusive due to the scarcity of empirical data, are now becoming accessible through recent observational advances. The long-awaited simultaneous determination of neutron star masses and radii indicates that the equation of state (EOS) of neutron star cores radically stiffens, i.e., pressure P grows rapidly as energy density ϵ increases [1,2]. The inferred core density can be high enough for baryons composed of three quarks to overlap. These together imply that a transition from baryonic matter into dense quark matter, if it indeed occurs as expected, should be smooth.

The quark-hadron crossover (QHC) scenario, originally focused on the symmetry aspects [3], was first realized in the EOS through a phenomenological interpolation between hadronic and quark matter [4], predicting radical stiffening in the crossover domain. A useful measure of the stiffening is the peak of sound speed, $c_s = (dP/d\epsilon)^{1/2}$. The microscopic reasoning, why crossover models not only avoid softening associated with phase transitions but also drive the radical stiffening, is still in debate. One possible explanation is given by models of quarkyonic matter [5,6] in which the quark Pauli blocking forbids baryons to occupy states at low momenta

and therefore make them relativistic, leading to the stiff EOS [7,8]. The resulting momentum distribution of baryons is highly nonmonotonic.

The previous quarkyonic matter models did not manifestly describe the formation of baryons in dense matter. In this Letter, we treat three-fermion correlations responsible for the baryon formations and show how the non-monotonic baryon distribution arises. Our analyses generalize studies in two-color or isospin quantum chromodynamics (QCD) in which the QHC have been established by lattice simulations [9,10]. For these theories, bosons condense at certain chemical potentials, and beyond which the transition proceeds smoothly in a similar way as the Bose-Einstein-condensate (BEC) to Bardeen-Cooper-Schrieffer (BCS) crossover [11–14] (see also reviews [15–18]) experimentally realized in ultracold atoms [19–21] and in superconductors [22–24]. In particular, the lattice simulations confirmed the sound-speed peak [9] and its qualitative behavior can be captured by simple models treating condensed bosons as mean fields [25,26].

For three-color QCD, the relevant hadrons are baryons and they must be included as *tripling* fluctuations induced by the formation of three-body bound states. The physics of tripling fluctuations is elusive compared to pairing fluctuations in the BEC-BCS crossover, but recently there has been some progress. In cold-atom physics, a smooth crossover from bound trimers to Cooper triples [27] has been discussed in three-component Fermi gases [28–30]. This scenario is found to be consistent with quantum Monte Carlo (QMC) simulations in a spatially one-dimensional system [31].

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Moreover, in one spatial dimension, a nonrelativistic three-component Fermi gas with three-body attraction involves asymptotic freedom and a trace anomaly [32] just like dense QCD matter [33].

In this Letter, we examine the role of tripling fluctuations by generalizing the pairing-fluctuation approach [13]. The phase-shift representation of the multibody propagator plays a central role in describing the nonmonotonic momentum distribution of baryons. To explore analytic insights we use a model of nonrelativistic three-color fermions in one spatial dimension. We show how the sound-speed peak is induced by tripling fluctuations in the crossover regime.

Clustering fluctuations—Simply adding many-body cluster contributions causes questions concerning the double counting of constituent particles. To handle this problem we first present the thermodynamic potential characterized by the phase shift [34–37]. There is an important constraint on the asymptotic behaviors of the phase shift and we find how the contributions from composite states are canceled at high energy. Such cancellations are crucial to establish the dominance of constituent particles in thermodynamics at high densities.

We are interested in the N -body cluster contributions induced by the interaction V to the thermodynamic potential Ω . Subtracting a term which is already included for the single-particle self-energy, the N -body fluctuations can be written as

$$\delta\Omega_N = -T \sum_{\mathbf{K}, \omega_\ell} \text{tr}_N [\ln(1 - G_0 V) + G_0 V], \quad (1)$$

where G_0 is the bare N -body propagator characterized by the total Matsubara frequency $\omega_\ell = (2\ell + 1)\pi T$ with the temperature T , the center-of-mass momentum $\mathbf{K}$, the relative momenta $\mathbf{k}_1, \dots, \mathbf{k}_{N-1}$, and the relative frequencies $\omega_{l_1}, \dots, \omega_{l_{N-1}}$ (where we omitted the argument of G_0 for convenience). The trace is taken for all relative momenta and frequencies as $\text{tr}_N = \int_{k_1, \dots, k_{N-1}}$, where $\int_k = T \int d\mathbf{k} / (2\pi)^d \sum_{\omega_i}$. The Matsubara sum in Eq. (1) can be transformed into the integral representation by deforming the contour as

$$\delta\Omega_N = -T \sum_{\mathbf{K}} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \ln(1 + e^{-\omega/T}) \partial_\omega \text{Im} \Phi, \quad (2)$$

where $\Phi = \text{tr}_N [\ln(1 - G_0 V) + G_0 V]$. Here we consider the short-range interaction responsible for cluster formations, which can often be written in a separable form. This allows us to take the trace of G_0 in Φ independently as $\Phi = [\ln(1 - \mathcal{G}_0 V) + \mathcal{G}_0 V]$ with $\mathcal{G}_0 = \text{tr}_N [G_0]$.

Now we introduce the phase φ of propagators as $\mathcal{G}/\mathcal{G}_0 = |\mathcal{G}/\mathcal{G}_0| e^{i\varphi}$ with the trace $\mathcal{G}$ of the dressed N -body propagator $G = G_0 + G_0 V G$. φ is subject to a constraint that interactions cannot change the size of the state space. Any propagators must satisfy the sum rule

$$\int_{-\infty}^{\infty} d\omega \text{Im}(\mathcal{G} - \mathcal{G}_0) = - \int_{-\infty}^{\infty} d\omega \partial_\omega \ln(\mathcal{G}/\mathcal{G}_0) = 0, \quad (3)$$

which leads to $\varphi(\mathbf{K}, \infty) = \varphi(\mathbf{K}, -\infty) = 0$, with setting the phase at $\omega = -\infty$ to zero. Since we have $\text{Im} \Phi = -\varphi + |\mathcal{G}_0/\mathcal{G}| \sin \varphi$, Eq. (3) leads to

$$\text{Im} \Phi(\mathbf{K}, \omega \rightarrow \pm\infty) = 0. \quad (4)$$

These boundary conditions have important implications. The phase φ increases by π when ω passes the bound-state pole (and also the resonances). However, the increased phase eventually must be reduced back to the original value because of $\varphi(\mathbf{K}, \infty) = \varphi(\mathbf{K}, -\infty) = 0$. This indicates that $\partial_\omega \text{Im} \Phi(\mathbf{K}, \omega)$ must be negative in some interval of ω beyond the continuum threshold, and this negative contribution tends to cancel the positive bound-state contributions.

The convoluted integral of the phase with the thermal factor $\ln(1 + e^{-\omega/T})$ in Eq. (2) preferentially picks up the bound-state contributions at low temperatures, while at high temperature the scattering states also contribute and tend to cancel the bound-state contributions. Hence, in the high-temperature limit the thermodynamics should be dominated by single-particle contributions as expected. Such a mechanism plays a crucial role in understanding the microscopic mechanism of the hadron-quark crossover accompanied by tripling fluctuations associated with baryon formations. In this Letter we examine the occupation probability of baryonic states at finite density, and show how the above cancellation works differently for different $\mathbf{K}$.

Demonstration of tripling fluctuations—To see the role of clustering fluctuations beyond the two-body level, we consider a nonrelativistic three-color fermion with the color-singlet three-body attractive interaction in one spatial dimension, of which the dynamics is described by the Hamiltonian [32]

$$H = \sum_{p, \alpha} \xi_p c_{p, \alpha}^\dagger c_{p, \alpha} + V \sum_P B^\dagger(P) B(P), \quad (5)$$

where $\xi_p = p^2/(2m) - \mu$ is the kinetic energy of a fermion with mass m , measured from the chemical potential μ , $c_{p, \alpha}$ is the annihilation operator with color $\alpha = r, g, b$, and $B(P) = \sum_{q, k} c_{P/3-k+q/2, r} c_{P/3+k+q/2, g} c_{P/3-q, b}$ is the three-body operator. Here we consider the population-balanced case (without other degrees of freedom such as spins and flavors). The contact-type three-body coupling V is associated with the three-body binding energy $\mathcal{B}$ as [32]

$$\frac{1}{V(\Lambda)} = - \frac{m}{2\sqrt{3}\pi} \ln \left(\frac{\Lambda^2}{m\mathcal{B}} \right), \quad (6)$$

where Λ is the momentum cutoff and the coupling $V(\Lambda)$ is renormalized with $\mathcal{B}(>0)$ fixed (note that $\Lambda^2 \gg m\mathcal{B}$ is

used). The emergence of nonzero $\mathcal{B}$ is the consequence of quantum anomaly [32], and is expected to induce strong tripling fluctuations at finite densities.

Using Eq. (1) we obtain the thermodynamic potential as $\Omega = \Omega_{\text{HF}} + \delta\Omega_3$, where $\Omega_{\text{HF}} = -3T \sum_k \ln(1 + e^{-\xi_k^{\text{HF}}/T})$ with $\xi_k^{\text{HF}} = \xi_k + \Sigma_{\text{HF}}$ is the Hartree-Fock (HF) contribution. $\Sigma_{\text{HF}} = V(n_0/3)^2$ is the HF self-energy with the noninteracting fermion number density $n_0 = 3 \sum_k f(\xi_k)$ and $f(x) = 1/(e^{x/T} + 1)$ is the Fermi distribution function. The tripling fluctuation term is

$$\delta\Omega_3 = -T \sum_{K, i\omega_\ell} \left[\ln(1 - V\mathcal{G}_0) + V\mathcal{G}_0 \right], \quad (7)$$

where $\mathcal{G}_0 \equiv \mathcal{G}_0(K, i\omega_\ell)$ is the bare three-body propagator whose internal momenta and frequencies are traced out. Performing the analytical continuation of $\mathcal{G}_0$ to the real frequency $\omega + i\delta$ with infinitesimally small number δ , we obtain [29]

$$\mathcal{G}_0(K, \omega) = \sum_{k, q} \frac{(1 - f_1)(1 - f_2)(1 - f_3) + f_1 f_2 f_3}{\omega + i\delta - \xi_1 - \xi_2 - \xi_3}, \quad (8)$$

where $f_i = f(\xi_i)$ with single-particle energies $\xi_1 = \xi_{q/2+K/3+k}^{\text{HF}}$, $\xi_2 = \xi_{K/3-q}^{\text{HF}}$, and $\xi_3 = \xi_{q/2+K/3-k}^{\text{HF}}$. Λ is applied to the momentum summations in Eq. (8).

Below we set $f_i \rightarrow 0$ by neglecting the structural changes of a baryon caused by the statistical restriction on the intermediate quark states. This approximation is valid when K is sufficiently large so that quarks become free from the Pauli blocking effects [38]. At low K , this approximation is not necessarily valid, but the contributions from low K turn out to be suppressed anyway by the phase-shift effects. Then the integral in Eq. (8) can be carried out analytically as

$$\mathcal{G}_0(K, \omega) \simeq -\frac{m}{2\sqrt{3}\pi} \ln \left(\frac{-\tilde{\omega} - i\delta + \Lambda^2/m}{-\tilde{\omega} - i\delta} \right), \quad (9)$$

where we introduced $\tilde{\omega} = \omega + \tilde{\mu}_B - E_B^{\text{kin}}(K)$ with $E_B^{\text{kin}}(K) = K^2/2M_B \equiv K^2/6m$ and $\tilde{\mu}_B = 3\tilde{\mu} \equiv 3(\mu - \Sigma_{\text{HF}})$. After setting $\Lambda \rightarrow \infty$ in V and $\mathcal{G}_0$, we obtain the phase shift of the three-body propagator as

$$\varphi(\tilde{\omega}) = \pi\Theta(\tilde{\omega} + \mathcal{B})\Theta(-\tilde{\omega}) + \Theta(\tilde{\omega})\varphi_{\text{scatt}}(\tilde{\omega}), \quad (10)$$

where $\Theta(x)$ is the step function. In Eq. (10), the first term is the bound-state contribution and

$$\varphi_{\text{scatt}}(\tilde{\omega}) = \tan^{-1} \left[\frac{\pi}{\ln(|\tilde{\omega}|/\mathcal{B})} \right] \quad (11)$$

is the scattering-state contribution. $\varphi_{\text{scatt}}(\tilde{\omega})$ approaches π for $\tilde{\omega} \rightarrow 0_+$; passes $\pi/2$ for $\tilde{\omega} \rightarrow +\mathcal{B}$; and approaches zero

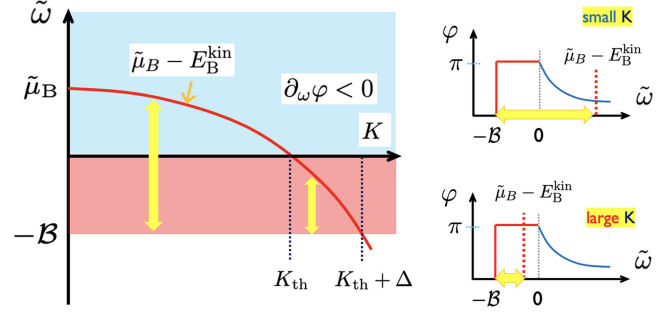


FIG. 1. Schematics of the phase shift φ at various ω and K . The momentum distribution of baryonlike clusters is obtained by the frequency integration of $\partial_\omega \varphi$ combined with the Fermi distribution of clusters. The integration domains (yellow arrows) are restricted by the Fermi edge of clusters (red solid curve in the left panel) at a given K . For a greater K , the domain with $\partial_\omega \varphi < 0$ (i.e., scattering state) becomes narrower so that the bound-state contributions on thermodynamic quantities survive from the cancellation with the scattering-state contributions. For a smaller K , the scattering-state contributions more strongly cancel the bound-state contributions so that the baryonic momentum distribution $f_B(K)$ is smaller, as we shall show below.

for $\tilde{\omega} \rightarrow +\infty$. Accordingly, $\varphi(\tilde{\omega})$ satisfies the boundary condition $\varphi(-\infty) = \varphi(\infty) = 0$. Taking the derivative leads to

$$\frac{1}{\pi} \partial_{\tilde{\omega}} \varphi(\tilde{\omega}) = \delta(\tilde{\omega} + \mathcal{B}) - \Theta(\tilde{\omega}) \frac{1}{\tilde{\omega} \ln^2(\tilde{\omega}/\mathcal{B}) + \pi^2}. \quad (12)$$

The sign of the scattering contribution is opposite to the bound state one, as mentioned earlier, leading to their cancellation (see also Fig. 1). Substituting this expression to $\delta\Omega_3$ reads

$$\begin{aligned} \delta\Omega_3 = & -T \sum_K \ln[1 + e^{-(\mathcal{B} + E_B^{\text{kin}} - \tilde{\mu}_B)/T}] \\ & + T \sum_K \int_0^\infty \frac{d\omega}{\omega} \frac{\ln[1 + e^{-(\omega + E_B^{\text{kin}} - \tilde{\mu}_B)/T}]}{\ln^2(\omega/\mathcal{B}) + \pi^2}. \end{aligned} \quad (13)$$

Using the thermodynamic identity $n = -\partial\Omega/\partial\mu$, where n is the fermion number density, we write $n = n_{\text{HF}} + n_{\text{fluc}}$ with $n_{\text{HF}} = -\partial\Omega_{\text{HF}}/\partial\mu$ and $n_{\text{fluc}} = -\partial\delta\Omega_3/\partial\mu$ given by (see also Supplemental Material [39])

$$n_{\text{HF}} = 3 \sum_k f_Q(k), \quad n_{\text{fluc}} = 3 \sum_K f_B(K). \quad (14)$$

The first term is the HF contribution as $f_Q(k) \equiv f(\xi_k^{\text{HF}})$ and the second one is the baryonlike fluctuation contributions given by

$$\begin{aligned} f_B(K) = & f(-\mathcal{B} + E_B^{\text{kin}} - \tilde{\mu}_B) \\ & - \int_0^\infty \frac{d\omega}{\omega} \frac{f(\omega + E_B^{\text{kin}} - \tilde{\mu}_B)}{\ln^2(\omega/\mathcal{B}) + \pi^2}, \end{aligned} \quad (15)$$

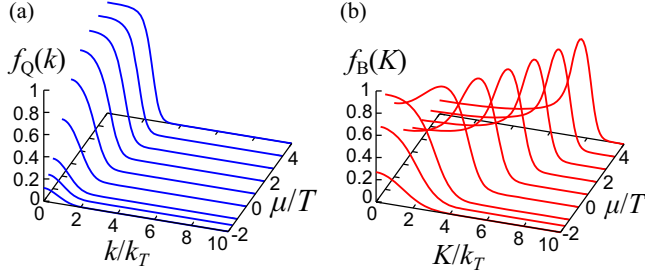


FIG. 2. The momentum distribution functions of (a) constituent (quarklike) fermions, $f_Q(k)$, and (b) (baryonlike) three-body states, $f_B(K)$, at $T/B = 0.2$. $k_T = \sqrt{2mT}$ is the momentum scale associated with the temperature T .

where the qualitative behavior of $f_B(K)$ is governed by the cancellation of bound and scattering state contributions as shown in Fig. 1. The numerical results for $f_Q(k)$ and $f_B(K)$ as functions of the momenta and μ/T are shown in Fig. 2. One can find a steplike distribution of $f_Q(k)$ and a momentum-shell structure of $f_B(K)$ at larger μ/T , which is consistent with the duality model [7]. The numerical results of n are also shown in Supplemental Material [39] and qualitatively agree with the QMC result [31].

To explore the analytic insights, we consider the low temperature limit ($T \rightarrow 0$). In the dilute limit such that $-\tilde{\mu}_B \leq 0$, the single-particle contributions vanish while baryonic contributions read $f_B^{T=0}(K)|_{\text{dilute}} = \Theta(\tilde{\mu}_B + B - E_B^{\text{kin}})$, where states are fully occupied up to $\sqrt{2M_B(\tilde{\mu}_B + B)}$. On the other hand, when $\tilde{\mu}_B \geq 0$, the single particle contributions become nonzero and at the same time the scattering channel in baryonic contributions gets open. Here we introduce the Fermi momentum $k_F = \sqrt{2m\tilde{\mu}}$ of quarklike fermions and the threshold momentum $K_{\text{th}} \equiv \sqrt{2M_B\tilde{\mu}_B} = 3k_F$, where the continuum for the baryonic channel is open. Using these, the quark distribution is $f_Q^{T=0}(k) = \Theta(k_F - k)$ and the baryon distribution is analytically obtained from Eq. (15) as

$$f_B^{T=0}(K) = \Theta(K_{\text{th}} + \Delta - K)\Theta(K - K_{\text{th}}) + \Theta(K_{\text{th}} - K) \left(\frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left[\frac{\ln \frac{K_{\text{th}}^2 - K^2}{2M_B B}}{\pi} \right] \right), \quad (16)$$

where we also introduced a function characterizing the thickness of the momentum shell $\Delta(K_{\text{th}}) \equiv \sqrt{K_{\text{th}}^2 + 2M_B B} - K_{\text{th}}$. When K approaches K_{th} , the distribution approaches the maximum, $f_B^{T=0}(K \rightarrow K_{\text{th}}) \rightarrow 1$. In dense limit, $K_{\text{th}}^2 \gg 2M_B B$, the distribution at $K = 0$ vanishes as $f_B^{T=0}(0) \rightarrow 0$.

In terms of the McLerran-Reddy model for quarkyonic matter [6], this result can be interpreted as the formations of the quark Fermi sphere up to $k = k_F$ and the momentum shell of baryonic excitations. While in the duality model [7]

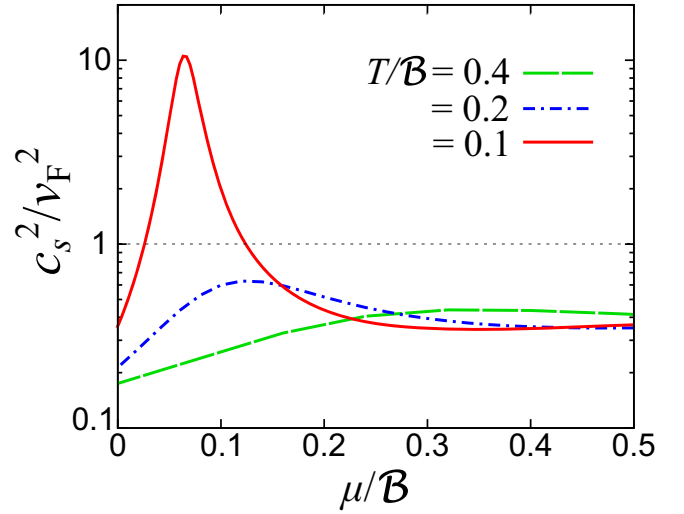


FIG. 3. Squared isothermal sound speed c_s^2/v_F^2 normalized by the Fermi velocity $v_F = \pi n/3m$ at different T . The horizontal dotted line represents the nonrelativistic conformal limit $c_s^2/v_F^2 = 1$.

the suppression of $f_B(K < 3k_F)$ is associated with the Pauli blocking effect of quarks, this suppression in the present case is caused by the negative scattering contributions that are necessary to cancel the doubly counted quark contributions in baryonic terms.

Sound speed—Let us turn to the squared isothermal sound speed c_s^2 . In the nonrelativistic system, it can be rewritten as $c_s^2 = (n/m)(\partial n/\partial \mu)_T^{-1}$, which becomes equivalent to the adiabatic sound speed at $T = 0$. As a measure of rapid stiffening of matter, we take the Fermi velocity $v_F = \pi n/3m$ as the natural baseline in the nonrelativistic model, instead of $c/\sqrt{d}$ (c : light velocity, d : spatial dimensions) used in relativistic models. One can see the peaked behavior of c_s^2/v_F^2 in the evolution of μ in Fig. 3 at different T/B . At high densities, c_s^2/v_F^2 slowly approaches unity from below.

To understand the peaked behavior of c_s^2 , the most essential is the density susceptibility $\chi = (\partial n/\partial \mu)_T \equiv \chi_{\text{HF}} + \chi_{\text{fluc}}$, where $\chi_{\text{HF}} = (\partial n_{\text{HF}}/\partial \mu)_T$ and $\chi_{\text{fluc}} = (\partial n_{\text{fluc}}/\partial \mu)_T$ are the HF and fluctuation contributions, respectively. While χ_{HF} is always positive, χ_{fluc} can be negative due to the scattering-state contribution (see also Supplemental Material [39]). Intuitively, the strong suppression of $f_B(K)$ at small K in Fig. 2(b) indicates the decrease of n_{fluc} with increasing μ , namely, $(\partial n_{\text{fluc}}/\partial \mu)_T < 0$. The cancellation of χ_{HF} and χ_{fluc} leads to the enhancement of $c_s^2 \propto \chi^{-1}$. In this way, one can understand physics of clustering fluctuations in the cross-over involving the peaked sound speed, which allows for a schematic (but based on microscopic physics) viewpoint on the quarkyonic-matter EOS as shown in Supplemental Material [39].

Discussion—It is natural to ask what would happen if we extended our framework to the relativistic version, as the relativistic three-body spectra have been studied in Ref. [40]. Concerning the stiffening of matter, our findings should hold for relativistic models with higher dimensions, as the depletion mechanism of the baryon distribution is generic. Meanwhile, the height of the sound-speed peak is not universal. In relativistic models, the sound-speed peak is more significant in the direction of stiffening matter in the crossover regime more rapidly because of a larger disparity between the pressures of composite and constituent particles.

Moreover, to develop analytic insights, we have also neglected several effects at this moment. While we kept using the phase shift in vacuum, the phase shift can be modified in dense matter. For instance, there appear the Mott effect [38], which quenches the bound-state formation by the Pauli blocking [41,42], and also the Landau damping [43]. The chiral restoration, neglected in this work, is also important as it affects the quark masses and the vacuum energy. However, these effects do not spoil the main points of our discussion and can be studied in a parallel way. We expect that the inclusion of all these effects accelerates transitions from baryonic to quark matter and improves the estimation of the location, height, and width of the sound-speed peak.

The application of our framework to the confining theories deserves additional comments. Our calculated baryonic spectra are given by bound states plus continuum, and the latter is formed by free fermions. However, in confining phases, such deconfined fermions do not exist, so that the interpretation of the continuum is not obvious. Actually, hadronic spectra at high energy contain the continuum formed by resonances and hadronic scattering states. When hadronic states are sufficiently abundant at a given phase space, it is known that naive quark computations can reproduce such continuum (quark-hadron duality [44]). Phrasing it in dense matter context, one can expect such duality in the inner part of the baryon Fermi sea where states are highly occupied. Toward the Fermi surface, the phase space becomes sufficiently open for the purely baryonic picture to be valid. In this view, our parameter $\mathcal{B}$ should be related to the distance from the Fermi surface to the saturated inner part. We leave the explicit realization of these ideas for future work.

Summary—We have investigated the role of tripling fluctuations in the equilibrium properties of nonrelativistic three-color fermions in one spatial dimension, which can be regarded as an analog quantum simulator for the quarkyonic-type hadron-quark crossover in dense matter and is possibly accessible in future cold-atom experiments. Using the phase-shift approach to describe tripling fluctuations, we have elucidated how the momentum distributions of one-body and three-body fermion states change at finite densities. Our simplified approach is a minimal model that

not only reproduces the quarkyonic matter model, but also is qualitatively consistent with the QMC results. In the future, it will be interesting to apply our approach to systems closer to real QCD matter.

Acknowledgments—The authors thank Joaquín E. Drut, Yaqi Hou, and Eiji Nakano for useful discussions. This research was supported in part by Grants-in-Aid for Scientific Research provided by JSPS through Grants No. JP18H05406, No. JP22H01158, No. JP22K13981, No. JP23H01167, No. JP23K03377, No. JP23K25864, and No. JP25K01001.

Data availability—The data that support the findings of this Letter are not publicly available. The data are available from the authors upon reasonable request.

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