

Quantum Mechanics Based on Real Numbers: A Consistent Description

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Complex numbers play a crucial role in quantum mechanics. However, their necessity remains debated: whether they are fundamental or merely convenient. Recently, it was shown that any real-number quantum theory satisfying certain postulates can be falsified with multipartite experiments. In this Letter we show that a physically motivated postulate about composite quantum systems allows us to construct quantum mechanics based on real numbers that reproduces predictions for all multipartite quantum experiments. Thus, we argue that real-valued quantum mechanics cannot be falsified, and therefore the use of complex numbers is a matter of convenience.

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Quantum mechanical predictions and measurement results are real valued, whereas the abstract quantum mechanical formalism typically relies on using complex numbers. It is therefore of fundamental interest to explore the implications of formulating quantum mechanics without complex numbers.

In addition to this intuitive argument, there are several mathematical ones that support the real formulation of quantum mechanics. Operations like time reversal that are antilinear in the complex formulation become linear in the real formulation [1]. Additionally, quantum logic's algebraic structures can consistently be mapped to real Hilbert spaces [2]. Furthermore, key properties like the ability to construct orthogonal bases and to define norms are preserved in real Hilbert spaces, suggesting that these spaces offer a consistent framework [3,4]. It is speculated that any fundamental physical theory can be constructed on any number field and the fundamental physical laws are invariant under the change of the number field [5], see, e.g., works on p -adic quantum mechanics [6,7].

The topic of a theory called “real quantum mechanics” (RQM) has been extensively studied. Two classes of RQM can be distinguished. First, the constructions from Refs. [8–12], which successfully predict experimental outcome statistics by including a binary real-valued system, often called a rebit in the literature [13]. These constructions are therefore equivalent to standard quantum mechanics,

referred to throughout this text as “complex quantum mechanics.” However, the rebit is “universal” in the sense of being accessible to all subsystems of a global composite state, even under spacelike separation. This property was interpreted as nonlocality of the rebit. Second, the models considered in Refs. [14–16], which use the tensor product postulate for composite systems. These models are mathematically inequivalent to standard quantum mechanics.

Recently, it has been shown in the context of multipartite Bell-type experiments [17–20] that the second class of formulations of quantum mechanics using real numbers, would be incompatible with the experimental outcome statistics, thus rendering them subject to experimental falsification. The corresponding experiments were carried out [21–23], and their results were indeed inconsistent with the second class of formulations of real-number quantum mechanics. A crucial postulate in the above articles is that the Hilbert space of a composite system is given by the tensor product of the Hilbert spaces of the individual subsystems. However, this mathematical postulate, though widely adopted, is too restrictive for defining a real-vector-space formulation of standard quantum mechanics. Here we will show that postulating a more fundamental physical requirement instead, leads to a real-valued theory which is equivalent to complex quantum mechanics. We postulate that, given two separate subsystems, an operation acting only on one subsystem does not have a measurable effect on the other one. In other words, a local operator must act trivially on all other subsystems. Note that the tensor product structure of complex quantum mechanics naturally fulfills our postulate (see Sec. E of Supplemental Material [24]).

In this Letter, we propose a real-valued construction for composite quantum systems. Note that its form is similar to the ones from the literature [8–10,12]. However, we introduce a mathematical structure and show that it does

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fulfill the above requirement of locality. Our construction is based on the notion of the composite Hilbert space as a real quotient space. Thus, it reflects certain equivalences of state vectors that are automatically contained in a complex formulation of quantum mechanics. As our construction is a one-to-one mapping of complex quantum mechanics to a real description, it is compatible with the outcomes of all quantum experiments, including multipartite Bell-type scenarios.

We propose the following postulates for both complex and real quantum mechanics. (P1) For every physical system A , there is a complex (real) Hilbert space $\mathcal{H}_A$ in which the state of the former is represented by a normalized vector $|\phi\rangle$, i.e., $\langle\phi|\phi\rangle = 1$. (P2) A measurement Π of system A corresponds to an ensemble of positive semi-definite operators $\{\Pi_r\}_r$ indexed by the measurement outcome r and acting on $\mathcal{H}_A$, obeying $\sum_r \Pi_r = \mathbb{1}_A$, where $\mathbb{1}_A$ denotes the identity. (P3) Born rule: If Π is measured while A is in the state $|\phi\rangle$, the probability of measuring outcome r is given by $P(r) = \langle\phi|\Pi_r|\phi\rangle$. (P4) Given two subsystems A and B , the operators used to describe measurements or transformations of system A , act trivially on B , and vice versa. In particular, this means that any two local operators O_A and O_B , acting on subsystems A and B , respectively, commute (i.e., $[O_A, O_B] = 0$) [25]. Note that postulates (P1)–(P3) are analogous to the ones in Ref. [17], and only (P4) is different. Further details on the definition of postulate (P4) can be found in Sec. E of Supplemental Material [24].

We will now show how these postulates allow a description of quantum mechanics using only real numbers. We start in complex quantum mechanics with a single quantum state. An abstract state $|\psi\rangle$ of a d -dimensional quantum system can be represented as $|\psi\rangle = (c_1, c_2, \dots, c_d)^T$ in a fixed basis, with $c_i \in \mathbb{C}$. It can also be represented as $|\psi\rangle = [\text{Re}(c_1), \dots, \text{Re}(c_d), \text{Im}(c_1), \dots, \text{Im}(c_d)]^T$. We will denote in the following $\text{Re}(|\psi\rangle) := [\text{Re}(c_1), \dots, \text{Re}(c_d)]^T$ and $\text{Im}(|\psi\rangle) := [\text{Im}(c_1), \dots, \text{Im}(c_d)]^T$. A mapping from the complex to the real representation of a (pure) state $|\psi\rangle \in \mathbb{C}^d$ (see also, e.g., Refs. [9,10,12,17]) can thus be written in the convenient form

$$\mathcal{S}: |\psi\rangle \mapsto \text{Re}(|\psi\rangle) \otimes |0\rangle_F + \text{Im}(|\psi\rangle) \otimes |1\rangle_F =: |\tilde{\psi}\rangle, \quad (1)$$

where the subscript F stands for “flag”: the two real degrees of freedom of the flag indicate the real and imaginary part of the original state vector. Here the real Hilbert space has the form $\mathcal{H} = \mathbb{R}^d \otimes \mathbb{R}_F^2$. We point out that even though Eq. (1) formally resembles the structure of an entangled state, $|\tilde{\psi}\rangle$ is a *single* quantum state and thus is not entangled. Throughout this Letter we will denote the real representation of states and operators with a tilde. The bra vector $\langle\tilde{\psi}|$ is defined as $|\tilde{\psi}\rangle^T$. As $\langle\tilde{\psi}|\tilde{\psi}\rangle = \langle\psi|\psi\rangle$ holds, normalization of $|\psi\rangle$ ensures normalization of $|\tilde{\psi}\rangle$, see (P1).

In quantum mechanics, a normalized state vector $|\psi\rangle$ is physically indistinguishable from, and thus equivalent to,

$e^{i\alpha}|\psi\rangle$, with $\alpha \in \mathbb{R}$; i.e., a global phase is undetectable. A global phase translates in the real representation, see Eq. (1), to the rotation $R_F(\alpha)$ by the angle α acting on the flag:

$$\begin{aligned} \mathcal{S}: e^{i\alpha}|\psi\rangle &\mapsto \text{Re}(|\psi\rangle) \otimes (\cos \alpha |0\rangle_F + \sin \alpha |1\rangle_F) \\ &\quad + \text{Im}(|\psi\rangle) \otimes (-\sin \alpha |0\rangle_F + \cos \alpha |1\rangle_F) \\ &= R_F(\alpha) \mathcal{S}(|\psi\rangle). \end{aligned} \quad (2)$$

Thus, the global phase or U(1) ambiguity in the definition of a complex quantum state vector corresponds to an SO(2) ambiguity in the real formulation. The following example shows that real quantum mechanics has certain features that are quite different from complex quantum mechanics: For $\alpha = \pi/2$, the real representations of $i|\psi\rangle$ and $|\psi\rangle$ are orthogonal to each other, even though these states are physically indistinguishable.

The physical indistinguishability of two real state vectors $|\tilde{\psi}\rangle$ and $|\tilde{\phi}\rangle = R_F(\alpha)|\tilde{\psi}\rangle$ is guaranteed if and only if the real measurement operators from (P2) lead to identical outcome probabilities, see (P3). Let us denote the map from a complex operator to its real representation by $\mathcal{T}$, and the real image of a complex operator Π by $\tilde{\Pi} = \mathcal{T}(\Pi)$. We require

$$\langle\tilde{\psi}|\tilde{\Pi}|\tilde{\psi}\rangle = \langle\tilde{\phi}|\tilde{\Pi}|\tilde{\phi}\rangle = \langle\tilde{\psi}|R_F(\alpha)^\dagger \tilde{\Pi} R_F(\alpha)|\tilde{\psi}\rangle \quad (3)$$

for any real vector $|\tilde{\psi}\rangle$ of the form given in Eq. (1) and for arbitrary $\alpha \in \mathbb{R}$. Hence, $\tilde{\Pi}$ must commute with $R_F(\alpha)$ for all α . Also requiring that $\mathcal{S}(\Pi|\psi\rangle) = \mathcal{T}(\Pi)\mathcal{S}(|\psi\rangle)$ implies that real operators have the form (for details see Sec. A of Supplemental Material [24])

$$\mathcal{T}(\Pi) = \text{Re}(\Pi) \otimes I_F + \text{Im}(\Pi) \otimes J_F, \quad (4)$$

where

$$I_F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (5)$$

and $\text{Re}(\Pi)$ [$\text{Im}(\Pi)$] denotes the matrix consisting of the real (imaginary) parts of the elements of Π in the basis used to define $\text{Re}(|\psi\rangle)$ and $\text{Im}(|\psi\rangle)$. Note that the restriction for the observables to the form given in Eq. (4) forbids nontrivial measurement of the flag subsystem only (e.g., the projective measurement $\{1 \otimes |0\rangle\langle 0|_F, 1 \otimes |1\rangle\langle 1|_F\}$). In this sense, the flag is not a directly accessible degree of freedom.

An alternative derivation of the map $\mathcal{T}$ given in Eq. (4) based on a quotient space argument is given in Sec. C and additional properties of $\mathcal{T}$ are in Sec. D of Supplemental Material [24].

The real representation in Eq. (4) preserves the algebraic structure, i.e., $\mathcal{T}(A_1 A_2) = \mathcal{T}(A_1)\mathcal{T}(A_2)$ for any (complex) observables A_1 and A_2 , and satisfies

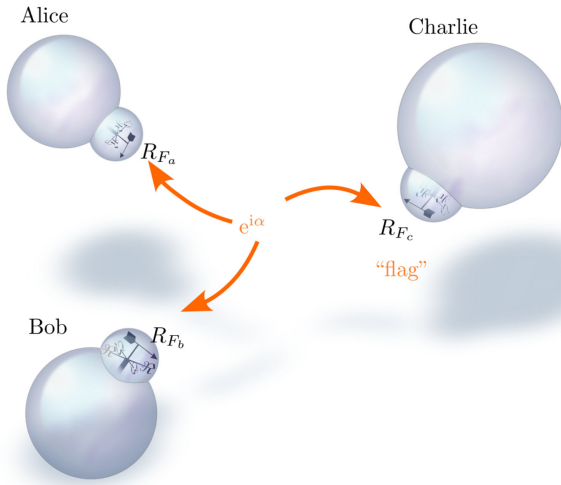


FIG. 1. A tripartite quantum system. The small spheres represent local two-dimensional flags, which are attached to each party to emulate the complex structure. There, a global phase factor $e^{i\alpha}$ can be arbitrarily split between the subsystems. The usual tensor product of real Hilbert spaces does not induce this ambiguity. However, the introduction of an equivalence relation on the multipartite flag state implements the same ambiguity: A global phase $e^{i\alpha}$ can be arbitrarily split between the local flag rotations R_{F_a} , R_{F_b} , and R_{F_c} . This formulation of quantum mechanics reproduces the predictions for all multipartite quantum experiments using only real numbers.

$$\mathcal{S}(|\psi\rangle)^T \mathcal{T}(A) \mathcal{S}(|\psi\rangle) = \langle \psi | A | \psi \rangle, \quad (6)$$

i.e., the maps $\mathcal{S}$ and $\mathcal{T}$ for vectors and operators leave all expectation values invariant.

Let us now derive the description of composite quantum systems in a real vector space. Here we analyze for simplicity the case of two two-dimensional subsystems labeled A and B . The structure of the states of composite systems with more than two subsystems and arbitrary dimension is detailed in Sec. B of Supplemental Material [24].

A straightforward attempt for the mapping from the complex to a real vector space would be to extend the map $\mathcal{S}$ to the tensor product $\mathcal{S}^{\otimes 2}$, i.e., $|\psi_A\rangle \otimes |\psi_B\rangle \mapsto \mathcal{S}(|\psi_A\rangle) \otimes \mathcal{S}(|\psi_B\rangle)$. However, it is obvious that the dimensions would not match: a vector in $\mathbb{C}_A^2 \otimes \mathbb{C}_B^2$ has $2 \times 2 \times 2 = 8$ real parameters, while a vector in $\mathbb{R}_A^4 \otimes \mathbb{R}_B^4$ has $4 \times 4 = 16$ parameters. In addition, remember the following important property of quantum mechanics: Given two independent subsystems, each of them can have an undetectable phase, see above. Because of linearity in each subsystem, this can also be interpreted as a global phase that is equal to the product of the individual phases. Vice versa, a global phase can be split up between the subsystems in an arbitrary way. Translating this property to a real representation, as schematically

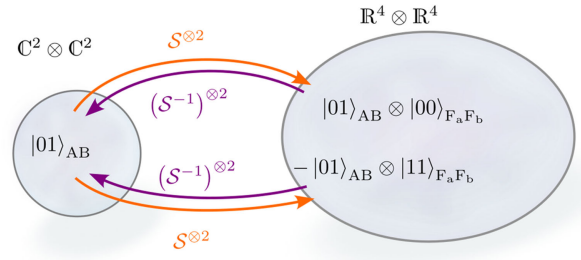


FIG. 2. Visualization of the ill-defined “map” $\mathcal{S}^{\otimes 2}$ (orange arrows). Since $|0\rangle_A \otimes |1\rangle_B = i|0\rangle_A \otimes (-i)|1\rangle_B$, one element in the domain would have more than one image. The tensored inverse map $(\mathcal{S}^{-1})^{\otimes 2}$ (purple arrows) is well defined but not invertible: Our example shows two elements in the domain which have the same image, and thus the kernel is nontrivial.

illustrated in Fig. 1, is the aim for the remainder of this Letter. The fact that the phase of a tensor product can be attributed to any of the subsystems is used in some quantum algorithms and has been referred to in the literature as “phase kickback” [26–29].

As an example for this phase ambiguity in tensor products, it holds that $|0\rangle_A \otimes |1\rangle_B = i|0\rangle_A \otimes (-i)|1\rangle_B$. However, the images of the left-hand side and the right-hand side of this equality, when applying $\mathcal{S}$ to each qubit individually, are different: $\mathcal{S}(|0\rangle_A) \otimes \mathcal{S}(|1\rangle_B) = |0\rangle_A \otimes |0\rangle_{F_a} \otimes |1\rangle_B \otimes |0\rangle_{F_b}$, which is not equal to $\mathcal{S}(i|0\rangle_A) \otimes \mathcal{S}(-i|1\rangle_B) = -|0\rangle_A \otimes |1\rangle_{F_a} \otimes |1\rangle_B \otimes |1\rangle_{F_b}$. Thus, tensoring the map $\mathcal{S}$ does not lead to a well-defined map. This is sketched in Fig. 2. There and in the following we will group the flag states together, as indicated by the labels $F_{a(b)}$. We will also omit the tensor product symbols between the systems A and B , and between the flags.

Therefore, one has to follow a different approach. Instead of extending $\mathcal{S}$ to the tensor product, we extend its inverse $\mathcal{S}^{-1}$, which is well defined. The states $|01\rangle_{AB} \otimes |00\rangle_{F_a F_b}$ and $-|01\rangle_{AB} \otimes |11\rangle_{F_a F_b}$ have the same image under the map $(\mathcal{S}^{-1})^{\otimes 2}$ (see Fig. 2). These vectors correspond to the same physical state and are thus equivalent. Two vectors are equivalent if their difference belongs to the kernel $\ker[(\mathcal{S}^{-1})^{\otimes 2}]$ of the map $(\mathcal{S}^{-1})^{\otimes 2}$. It is easy to check that $\ker[(\mathcal{S}^{-1})^{\otimes 2}]$ is spanned by the vectors of the form $|\psi'\rangle_{AB} \otimes (|00\rangle_{F_a F_b} + |11\rangle_{F_a F_b})$ and $|\psi'\rangle_{AB} \otimes (|01\rangle_{F_a F_b} - |10\rangle_{F_a F_b})$, where $|\psi'\rangle_{AB} \in \mathbb{R}^2 \otimes \mathbb{R}^2$.

Consider then, instead of the tensor product $\mathbb{R}_A^4 \otimes \mathbb{R}_B^4$, the quotient space $\mathbb{R}_A^4 \otimes \mathbb{R}_B^4 / \ker[(\mathcal{S}^{-1})^{\otimes 2}]$, i.e., the space of the corresponding equivalence classes. This leads to a one-to-one map between the complex and the real representation of quantum mechanics. Then the equivalence classes $|\cdot\rangle_F$ associated with the flag basis states for the composite system are

$$\begin{aligned}
 |0\rangle_F &= |\psi_{\text{even}}^{(2)}\rangle_{F_a F_b} \\
 &\quad + \text{span}\{|00\rangle_{F_a F_b} + |11\rangle_{F_a F_b}, |01\rangle_{F_a F_b} - |10\rangle_{F_a F_b}\} \\
 |1\rangle_F &= |\psi_{\text{odd}}^{(2)}\rangle_{F_a F_b} \\
 &\quad + \text{span}\{|00\rangle_{F_a F_b} + |11\rangle_{F_a F_b}, |01\rangle_{F_a F_b} - |10\rangle_{F_a F_b}\},
 \end{aligned} \tag{7}$$

where $|\psi_{\text{even}}^{(2)}\rangle_{F_a F_b}$ and $|\psi_{\text{odd}}^{(2)}\rangle_{F_a F_b}$ are the canonical representatives of these equivalence classes (i.e., vectors orthogonal to both $|00\rangle_{F_a F_b} + |11\rangle_{F_a F_b}$ and $|01\rangle_{F_a F_b} - |10\rangle_{F_a F_b}$). Explicitly,

$$\begin{aligned}
 |\psi_{\text{even}}^{(2)}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle_{F_a F_b} - |11\rangle_{F_a F_b}) \quad \text{and} \\
 |\psi_{\text{odd}}^{(2)}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle_{F_a F_b} + |10\rangle_{F_a F_b}).
 \end{aligned} \tag{8}$$

Here the label even (odd) indicates an even (odd) number of entries 1 in each term of the superposition. Note that the vectors in Eq. (8) were also employed in Ref. [10]. Here we give a constructive explanation of them: They are the canonical representatives of equivalence classes. A related approach had been investigated in Ref. [9]; we refer to Sec. G of Supplemental Material [24] for a detailed comparison with our construction.

We point out that, even though the states in Eq. (8) formally resemble entanglement, they describe separable states. We define a pure state in real quantum mechanics as separable if and only if it is either a product state or equivalent to a product state. Otherwise, it is entangled. Thus, the states $|\psi_{\text{even}}^{(2)}\rangle$ and $|\psi_{\text{odd}}^{(2)}\rangle$ are in a superposition without being entangled, since they are equivalent to the product states $|00\rangle_{F_a F_b}$ and $|01\rangle_{F_a F_b}$, respectively. This is reminiscent of the symmetrized wave function of indistinguishable particles, which is in a superposition that also does not imply entanglement [30].

Thus, the elements of the whole quotient space $\mathbb{R}_A^4 \otimes \mathbb{R}_B^4 / \ker[(\mathcal{S}^{-1})^{\otimes 2}]$ can be conveniently expressed as Eq. (1), with $|0\rangle_F$ and $|1\rangle_F$ now replaced by the flag equivalence classes $|0\rangle_F$ and $|1\rangle_F$ or, equivalently, by the canonical representatives $|\psi_{\text{even}}^{(2)}\rangle$ and $|\psi_{\text{odd}}^{(2)}\rangle$; see Sec. A of Supplemental Material [24] for details. Since we did not specify the “nature” of the vectors $|0\rangle_F$ and $|1\rangle_F$, Eq. (1) and all its consequences, in particular Eq. (6), are valid also for composite systems, thus proving that the presented real-valued formulation of quantum mechanics reproduces the predictions of the complex quantum mechanics for all experiments, also multipartite ones. An explicit real-valued formulation of the experiment suggested in Ref. [17] is given in Sec. H of Supplemental Material [24].

Remember that our central argument is the physically motivated postulate (P4). In order to show how our

construction guarantees the local action of operators, consider the tensor product of two operators of the form given in Eq. (4): Tensor products $I_{F_a} \otimes I_{F_b}$, $I_{F_a} \otimes J_{F_b}$, $J_{F_a} \otimes I_{F_b}$, and $J_{F_a} \otimes J_{F_b}$ of the flag operators appear. As can be easily seen, $I_{F_a} \otimes I_{F_b}$ and $-J_{F_a} \otimes J_{F_b}$ act as the identity operator I_F on the basis states (equivalence classes) $|0\rangle_F$ and $|1\rangle_F$, see Eq. (7), of the flag of the composite system. Analogously, $I_{F_a} \otimes J_{F_b}$ and $J_{F_a} \otimes I_{F_b}$ act as J_F on these basis states [see also Eq. (51) of [24]] [31]. Hence, the embedding of the local actions on subsystems into the operators for the composite system is well defined. The postulate of local action (P4) is satisfied because the underlying structure is still the tensor product of the Hilbert spaces, though with an additional equivalence relation. [The satisfaction of (P4) in real and complex quantum mechanics is analyzed in more detail in Sec. E of Supplemental Material [24].] The reduced density operator in the real-valued formalism is still given via the (real) partial trace, which is derived in Sec. F of Supplemental Material [24].

In summary, we propose a formulation of quantum mechanics based on real numbers, which leads to exactly the same predictions for experimental outcomes as complex quantum mechanics [see Eq. (6) and discussion below]. Thus, we conclude that both theories are equivalent in this sense. In order to arrive at this formulation, we show that the tensor product postulate does not capture the ambiguity in assigning the global phase of a product state to its subsystems [26–29]. Instead, we argue that a postulate based on *physical* properties for composite systems—namely, that a local operation on one subsystem acts trivially on the other—is well motivated and does lead to a theory that corresponds to complex quantum mechanics. The above-mentioned phase ambiguity is ensured by employing the quotient space of the real tensor product. Moreover, in Sec. A of Supplemental Material [24] we show that, under certain natural conditions related to representations of $\text{SO}(2)$ rotations, our construction not only fulfills the required physical postulate but is also unique, up to Hilbert space isomorphisms.

Note that a similar real representation has been discussed in the literature before [8–10,12]—however, without linking it to the role of the tensor product postulate. We mention that a tensor product in complex quantum mechanics can also be formulated as a quotient space [32,33] and fulfills our physically motivated postulate. Let us point out other recent papers about equivalent real-valued formulations of quantum mechanics [34,35].

As stated in Ref. [36], the results of Refs. [17–20] are best understood not as a claim about the impossibility of a real-valued Hilbert space representation of quantum theory in principle, “but rather as a claim about the possibility of experimentally adjudicating between standard quantum theory and an alternative theory—a foil theory—known

as real-amplitude quantum theory” (which includes the real-valued version of the tensor product postulate).

In conclusion, complex numbers are not necessary to describe quantum mechanics—but they are certainly very useful.

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Data availability—No data were created or analyzed in this study.

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