

Complete Characterization of State Conversions by Work Extraction

Chung-Yun Hsieh^{1,*} and Manuel Gessner^{2,†}

¹*H. H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol BS8 1TL, United Kingdom*

²*Instituto de Física Corpuscular (IFIC), CSIC-Universitat de València and Departament de Física Teòrica, UV, Avinguda Vicent Andrés Estellés 19, E-46100 Burjassot (Valencia), Spain*

 (Received 5 April 2024; revised 9 March 2026; accepted 7 July 2026; published 12 August 2026)

We introduce a thermodynamic work-extraction task that describes the energy storage enhancement of quantum systems. This task induces majorizationlike conditions that provide a necessary and sufficient characterization of state conversions in general quantum resource theories. When applied to specific resources, these conditions reduce to the majorization conditions under unital channels and provide a thermodynamic version of Nielsen's theorem in entanglement theory. We show how this result establishes the first universal resource certification class based on thermodynamics, and how it can be employed to quantify general quantum resources based on work extraction.

DOI: [10.1103/4fls-5q8b](https://doi.org/10.1103/4fls-5q8b)

Introduction—Quantum advantages underpin the development of quantum science and technologies. Enabled by quantum resources, it is possible for certain quantum information tasks to surpass the performance of classical strategies that do not make use of these resources [1–3]. For instance, quantum teleportation [4], superdense coding [5], and sub-shot-noise interferometric precision with qubit probes [6,7] are possible only with entanglement as a resource [8]. Different levels of advantages in so-called device-independent quantum information tasks in cryptography and communication are enabled by utilizing quantum nonlocality [9,10], quantum steering [11–13], measurement incompatibility [14], and quantum complementarity [15]. Furthermore, certain quantum dynamical features are, in fact, indispensable resources for quantum memories [16–22], quantum communication [23–26], preserving and generating quantum phenomena [24,27–31], and non-equilibrium thermodynamics [24–26,32].

Most of the above-mentioned tasks and their quantum advantages are *resource-dependent*. For instance, we do not expect a nonequilibrium state (a thermodynamic resource called athermality [33–36]) to be also useful for teleportation. Also, a quantum dynamics that is able to generate entanglement does not necessarily have a strong ability to transmit information. A natural question is thus whether there is a single, *resource-independent* class of operational

tasks in which general quantum resources can provide advantages over resource-free ones. Performing such a class of tasks in the laboratory implies the ability to certify quantum resources operationally independent of the types of resources.

Remarkably, such *universal resource certification classes* (URCCs) do exist. Through a general approach called *quantum resource theories* [37], or simply *resource theories*, it has been proved that general quantum resources (with reasonable physical assumptions) can provide advantages in discrimination tasks [38–45], exclusion tasks [15,46,47], parameter-estimation tasks in metrology [48], and input-output games [49]. These URCCs tell us how to universally certify quantum resources via their operational advantages in the corresponding physical tasks [50], serving as vital interdisciplinary bridges.

Surprisingly, there is no known thermodynamic URCC. In thermodynamics, quantum signatures have been identified in quantum heat engines [52–57] and conservation laws [58–62] (see also Refs. [63–67]), and thermodynamic advantages have been demonstrated by using entanglement [58,68–71], coherence [72,73], steering [54–57], incompatibility [74], and quantum dynamics [24–27,30]. Recently, it has been shown that one can witness certain quantum properties by observing heat [75,76]. Yet, it remains unknown whether there is a *single* class of operational work-extraction tasks in thermodynamics that can certify general quantum resources. Any such URCC, once found, can reveal operational advantages of general quantum resources in extracting controllable energy.

There is a *stronger version* of the above question—one whose answer would directly imply the existence of such a thermodynamic URCC. This stronger question is *whether a class of work-extraction tasks exists that can completely characterize state conversions in general resource*

*Contact author: chung-yun.hsieh@bristol.ac.uk

†Contact author: manuel.gessner@uv.es

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

theories. Such a class, if it exists, can provide a full “ordering” for the resource contents of states that can be used to witness the resource [77]. Moreover, such a class is by construction fully operational as *work* is energy in a controlled, reusable form.

Here, we fully answer the above question in the positive. We introduce a work-extraction task describing quantum system energy storage enhancement [Eq. (5)]. Then, we show that it can induce majorizationlike conditions to *completely* determine the (one-shot) state conversion in general resource theories (Theorem 1). Interestingly, when we apply this result to the resource theory of informational nonequilibrium (see, e.g., Ref. [78]), we reproduce the well-known majorization conditions for state conversions under unital channels (Theorem 2). Finally, this result also allows us to obtain the first worklike thermodynamic URCC (Theorem 3) as well as to thermodynamically quantify general resources (Theorem 4).

An extension of our result beyond state resources, as well as its thermodynamic implications for steering and measurement incompatibility are discussed in a separate companion paper [79].

Quantum resource theories—We begin by fixing notation. A quantum system is described by a state ρ , i.e., a positive semidefinite operator with unit trace: $\rho \geq 0$ and $\text{tr}(\rho) = 1$. The general evolution of such a system is represented by a quantum channel, a completely positive, trace-preserving linear map $\mathcal{E}$ that sends an initial state ρ to $\mathcal{E}(\rho)$ [80].

To study diverse quantum properties on a common footing, one employs the framework of resource theories [37], which provide a unified formalism for characterizing and quantifying quantum resources. To formalize the notion of a resource, let R denote a quantum property of interest (e.g., entanglement or coherence). A resource theory of R is specified by a pair $(\mathcal{F}_R, \mathcal{O}_R)$. The set $\mathcal{F}_R$ consists of the *free states*, i.e., states that do not contain R . A state ρ is *resourceful* if $\rho \notin \mathcal{F}_R$. For example, for entanglement, $\mathcal{F}_R$ is the set of separable states.

The set $\mathcal{O}_R$ is a set of channels, whose members are called *allowed operations*. These are operations allowed to manipulate states. Formally, any channel $\mathcal{N}$ in $\mathcal{O}_R$ must satisfy

$$\mathcal{N}(\eta) \in \mathcal{F}_R \quad \forall \quad \eta \in \mathcal{F}_R; \quad (1)$$

namely, it cannot generate R from free inputs. Equation (1) is a necessary condition for any allowed operation to satisfy, while one may consider extra constraints on $\mathcal{O}_R$ depending on the physical context; hence, generally, $\mathcal{O}_R$ does not contain all channels satisfying Eq. (1). Different choices of $\mathcal{O}_R$ lead to different operational settings for resource manipulation. For instance, in the case of entanglement, setting $\mathcal{O}_R$ as the set of local operations with classical communication (LOCCs) or as the set of local operations with shared randomness leads to different characterizations.

Working hypotheses—In most cases, $\mathcal{O}_R$ carries the following properties: (1) *convexity* (classical mixing of states is allowed), (2) *compactness* (if a channel is arbitrarily close to an allowed operation, it is also allowed [81]), (3) *it contains the identity channel* (doing nothing is allowed), (4) *closedness under function composition* (sequentially applying two allowed operations is also allowed), and (5) *it contains state-preparation channels of free states* (preparing free states is allowed).

From now on, we assume that $\mathcal{O}_R$ describes a set of allowed operations with these properties. Notably, with this assumption, as detailed in Appendix A, the free set $\mathcal{F}_R$ must be convex and compact (in the topology induced by the one norm).

Conversion problem in resource theories—One of the central goals in arguably every resource theory is to understand the conversion under allowed operations. To formalize it, consider a given $\mathcal{O}_R$ and two states ρ and σ . If there exists an allowed operation $\mathcal{N} \in \mathcal{O}_R$ achieving $\mathcal{N}(\rho) = \sigma$, we write $\rho \xrightarrow{\mathcal{O}_R} \sigma$. The *conversion problem* refers to determining *under which conditions we have* $\rho \xrightarrow{\mathcal{O}_R} \sigma$. Conceptually, a complete answer to the conversion problem can tell us, necessarily and sufficiently, whether an evolution from one state to another via an allowed operation can ever be possible, just like the second law of thermodynamics (see, e.g., Refs. [84–88]). Moreover, such an answer may serve as a bridge to further demonstrate the resource’s advantage in specific operational tasks (e.g., as in Ref. [77]). Here, we aim to develop a fully thermodynamic way to answer conversion problems for general resource theories (Fig. 1). To this end, we need to introduce the following task.

Energy storage enhancement as a work-extraction task—In order to define a work-extraction task that is capable of certifying quantum resources, we focus on a finite-dimensional quantum system with Hamiltonian H . When this system reaches thermal equilibrium with a large bath in temperature $0 < T < \infty$, it will be described by the thermal state

$$\gamma = e^{-\frac{H}{k_B T}} / \text{tr}(e^{-\frac{H}{k_B T}}), \quad (2)$$

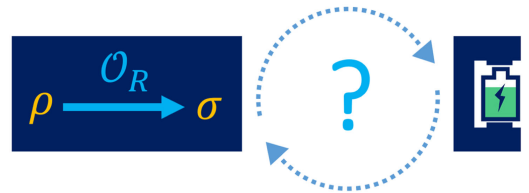


FIG. 1. Schematic illustration of the central question. We ask whether any class of work-extraction tasks can completely characterize state conversions in general resource theories. A suitable answer can offer a method to analyze a broad range of quantum effects via work extraction.

where k_B is the Boltzmann constant. When this system is prepared in a nonequilibrium state ρ , one can extract the following optimal amount of work from it [71,89,90]:

$$W(\rho, H) := (k_B T \ln 2) D(\rho \| \gamma), \quad (3)$$

where $D(\rho \| \sigma) := \text{tr}[\rho(\log_2 \rho - \log_2 \sigma)]$ is the quantum (Umegaki) relative entropy [91]. Notably, when the Hamiltonian is fully degenerate; i.e., $H = 0$, the above equation gives the optimal work extractable from ρ 's information content,

$$W_{\text{inf}}(\rho) := W(\rho, H = 0) = (k_B T \ln 2) D(\rho \| \mathbb{I}/d). \quad (4)$$

Inspired by Refs. [92–96] and the charging process of quantum batteries [69,97–103], we consider the figure of merit called *energy storage enhancement* of ρ with H ,

$$\Delta(\rho, H) := W(\rho, H) - W_{\text{inf}}(\rho). \quad (5)$$

Δ has a direct operational interpretation. If the system is initially prepared in a state ρ under a fully degenerate Hamiltonian, then $\Delta(\rho, H)$ is the change (possibly negative) in extractable work when the Hamiltonian is suddenly quenched to H without altering the state (see Fig. 2). Concretely, the extractable work with the initial fully degenerate Hamiltonian is $W_{\text{inf}}(\rho)$ [Eq. (4)], and after the quench to H it is $W(\rho, H)$ [Eq. (3)]. The difference, $\Delta(\rho, H)$, quantifies the additional amount of energy that can be stored through this process.

Completely characterizing state conversion by work extraction—Here, by using Δ , we can fully characterize conversion problems for general resource theories. To see this, for a given state ρ and Hamiltonian H , define

$$\Delta_{\mathcal{O}_R}(\rho, H) := \max_{\mathcal{N} \in \mathcal{O}_R} \Delta[\mathcal{N}(\rho), H], \quad (6)$$

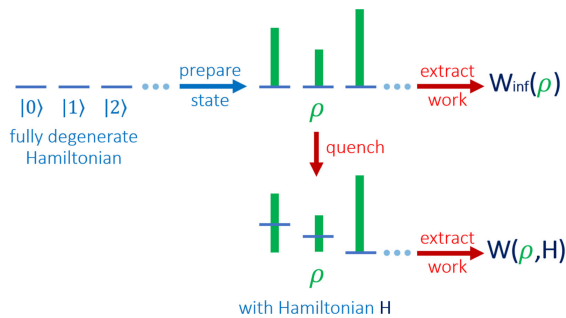


FIG. 2. Work-extraction task that describes energy storage enhancement. Each round starts with the same initial state ρ subject to a fully degenerate initial Hamiltonian. If we extract work with this setting, we obtain $W_{\text{inf}}(\rho)$ [Eq. (4)]. Alternatively, if we *quench* the Hamiltonian into a new one, H , and then perform work extraction, we obtain $W(\rho, H)$ [Eq. (3)]. The energy storage change $\Delta(\rho, H)$ is the difference between these two work values.

which is the highest value of Δ that can be reached by applying allowed operations to the state ρ . We thus call $\Delta_{\mathcal{O}_R}(\rho, H)$ the $\mathcal{O}_R$ -assisted energy storage enhancement of ρ with H . Then, we have the following result, serving as a thermodynamic, complete characterization of state conversion.

Theorem 1—Let $0 \leq \epsilon < \delta$ be fixed energy scales. Then $\rho \xrightarrow{\mathcal{O}_R} \sigma$ if and only if

$$\Delta_{\mathcal{O}_R}(\rho, H) \geq \Delta_{\mathcal{O}_R}(\sigma, H) \quad \forall \quad \epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}. \quad (7)$$

We detail the proof in Appendix B. Theorem 1 gives the first thermodynamic characterization, in a necessary and sufficient way, of general one-shot state conversion, i.e., it assumes access to a single copy of the state. Here, ϵ and δ are freely chosen energy scales to fit the relevant physical settings. Hence, a complete characterization can already be achieved by focusing on Hamiltonians with limited energy scales. It is interesting to note that Theorem 1 shows that the quantity $W(\rho, H)$ [Eq. (3)], which characterizes the *asymptotically* extractable work [71,89,90], also completely characterizes *one-shot* state conversion properties. We also note that although infinite conditions are necessary for general complete resource theories [104], *finite* conditions with practically feasible Hamiltonians may suffice for specific resources, as we shall demonstrate later.

Interestingly, Eq. (7) takes a form that is similar to the so-called majorization (see, e.g., Ref. [78]), as it implies that the state ρ “outperforms” (or at least equals) σ in terms of the figure of merit $\Delta_{\mathcal{O}_R}$ for every possible Hamiltonian in the given range. We will see in the next section that, when applied to the appropriate setting, Theorem 1 indeed reproduces the actual majorization condition.

Implication for informational thermodynamics and entanglement—We now apply our result to the resource theory of *informational nonequilibrium* (see Ref. [78] for review). For a d -dimensional system, the free set is $\mathcal{F}_R = \{\mathbb{I}/d\}$, corresponding to the thermal state [see Eq. (2)] of a fully degenerate Hamiltonian. In this case, with no energy differences, the thermodynamic contribution stems solely from the state’s information content. The allowed operations $\mathcal{O}_R$ are all energy-conserving closed system dynamics and their classical mixtures, i.e., all mixed unitary channels. We write “ $\rho \xrightarrow{\text{MU}} \sigma$ ” if there is one such channel converting ρ into σ . As we have a fixed system dimension, conversions under mixed unitary channels are equivalent to conversions under unital channels, denoted by “ $\rho \xrightarrow{\text{unital}} \sigma$ ” (see Lemma 10 in Ref. [78]). These state conversions can be characterized by the so-called majorization. Formally, for a given state ρ , let $\{\rho_i^\downarrow\}_{i=0}^{d-1}$ be its eigenvalues in the nonincreasing order, i.e., $\rho_i^\downarrow \geq \rho_{i+1}^\downarrow \quad \forall \quad i = 0, \dots, d-2$. Then, ρ is said to *majorize* σ , denoted by $\rho \succ \sigma$, if (see, e.g., Ref. [78])

$$\sum_{i=0}^k \rho_i^\downarrow \geq \sum_{i=0}^k \sigma_i^\downarrow \quad \forall \quad k = 0, \dots, d-2. \quad (8)$$

As detailed in Ref. [78], by applying Hardy-Littlewood-Pólya theorem [105] (see also Ref. [106]), we have the well-known result,

$$\rho \xrightarrow{\text{unital}} \sigma \quad \text{iff} \quad \rho \xrightarrow{\text{MU}} \sigma \quad \text{iff} \quad \rho \succ \sigma. \quad (9)$$

Namely, state conversions under allowed operations of informational nonequilibrium are fully captured by $d-1$ inequalities through majorization Eq. (8). Here, as an application, we will demonstrate that Theorem 1 reproduces the above result in this special case. Notably, as Theorem 1's proof relies on Sion's minimax theorem [107], rather than the Hardy-Littlewood-Pólya theorem, we thus uncover a novel proof for this well-known result.

Let $\{|i\rangle\}_{i=0}^{d-1}$ be a fixed basis and $\rho^\downarrow := \sum_{i=0}^{d-1} \rho_i^\downarrow |i\rangle\langle i|$, which is a “reordered” version of the state ρ in this basis. For $k = 0, \dots, d-2$, consider Hamiltonians $H_k := \omega \sum_{i=0}^k |i\rangle\langle i|$, where $\omega > 0$ is a fixed energy scale. Each H_k has two different energy levels (i.e., 0 and $\omega > 0$) and $(d-k-1)$ -fold ground state degeneracy. Then, applying Theorem 1 gives the following.

Theorem 2—For two d -dimensional states ρ and σ , we have $\rho \xrightarrow{\text{unital}} \sigma$ if and only if $\rho \xrightarrow{\text{MU}} \sigma$ if and only if

$$\Delta(\rho^\downarrow, H_k) \geq \Delta(\sigma^\downarrow, H_k) \quad \forall \quad k = 0, \dots, d-2. \quad (10)$$

Moreover, the above statement implies Eq. (9).

See Appendix C for proof. This illustrates Theorem 1's applicability to specific resources. In fact, Theorem 2 goes beyond the existing notion of Eq. (9) by linking state conversions to a *work-extraction task* with a clear thermodynamic interpretation. The involved Hamiltonians describe simple systems whose energy scales can be adjusted arbitrarily to fit practical settings (see also Ref. [108]). Since Δ can be obtained by two work-extraction experiments (one for W_{inf} ; one for W), Theorem 2 suggests that, in a d -dimensional system, $2(d-1)$ *work-extraction experiments with simple Hamiltonians completely characterize state conversions for informational thermodynamics*. Hence, our results not only generalize the well-known result for unital channel conversions [Eq. (9)], but also provide an experimentally testable form for it via work extraction.

Next, we study the implications of Theorem 2 for entanglement theory. Suppose $|\psi\rangle, |\phi\rangle$ are bipartite pure states with equal local dimension d , and ψ, ϕ are their reduced states in the first subsystem. The well-known Nielsen's theorem [80,109] states that $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$ if and only if $\psi < \phi$, where “ $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$ ” means that one can convert $|\psi\rangle$ into $|\phi\rangle$ by LOCC. Then, Theorem 2 and

Nielsen's theorem jointly imply that $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$ if and only if

$$\Delta(\psi^\downarrow, H_k) \leq \Delta(\phi^\downarrow, H_k) \quad \forall \quad k = 0, \dots, d-2. \quad (11)$$

Hence, Theorem 2 uncovers a *thermodynamic form* of Nielsen's theorem: $2(d-1)$ different work-extraction experiments *performed in the local system* are enough to fully characterize pure state conversions under LOCC. Further investigation into our findings' applications to entanglement (e.g., the connection with Refs. [93,94]) and other specific quantum resources is beyond the scope of this work and is left for future research.

Witnessing and quantifying quantum resources by work extraction—We are now in the position to answer the question about the existence of a thermodynamic URCC. From Theorem 1 we can derive the following result.

Theorem 3—Let $0 \leq \epsilon < \delta$ be fixed energy scales. Then $\rho \notin \mathcal{F}_R$ if and only if there is a Hamiltonian $\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}$ such that

$$\Delta(\rho, H) > \max_{\eta \in \mathcal{F}_R} \Delta(\eta, H). \quad (12)$$

See Appendix D for the proof. Hence, the energy storage enhancement Δ alone can certify state resources, and every resourceful state can provide an advantage over free states in enhancing energy storage under a specific quenching process. Theorem 3 thus establishes the first protocol to universally certify general quantum resources via energy extraction, and thus a thermodynamic URCC.

In fact, apart from certifying quantum resources, we can further *quantify* them by work extraction via Δ . To do so, we introduce the following measure for some fixed energy scales $0 \leq \epsilon < \delta$ (whose dependence will be kept implicit):

$$\mathcal{M}_{\mathcal{O}_R}(\rho) := \max_{\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}} \left[\Delta_{\mathcal{O}_R}(\rho, H) - \max_{\eta \in \mathcal{F}_R} \Delta_{\mathcal{O}_R}(\eta, H) \right]. \quad (13)$$

Then, we have the following result.

Theorem 4—For all energy scales $0 \leq \epsilon < \delta$, the measure $\mathcal{M}_{\mathcal{O}_R}$ is a resource quantifier of R ; namely, it satisfies (1) $\mathcal{M}_{\mathcal{O}_R}(\rho) \geq 0$ and $\mathcal{M}_{\mathcal{O}_R}(\rho) = 0$ if and only if $\rho \in \mathcal{F}_R$ and (2) $\mathcal{M}_{\mathcal{O}_R}[\mathcal{N}(\rho)] \leq \mathcal{M}_{\mathcal{O}_R}(\rho) \quad \forall \quad \mathcal{N} \in \mathcal{O}_R$.

See Appendix E for its proof. We thus uncover the first thermodynamic, energetic way to *quantify* general state resources, which is a strictly stronger notion than certification.

Mean energy as a witness of quantum resources—As a corollary of our approach, one can observe that Theorems 1, 3, and 4 also hold by replacing $\Delta(\rho, H)$ with the mean energy $\text{tr}(\rho H)$. More specifically, Theorem 1's proof can be adapted to the mean energy by starting from Eq. (B3) in Appendix B (which thus also implies Theorems 3 and 4 for the mean energy). This means that

a resourceful state can attain a higher mean energy (for some Hamiltonian) than any free state. However, a state's mean energy may not directly equal its extractable work [which is more related to free energy; see also Eq. (B2) in Appendix B]. Consequently, attaining a higher mean energy is still insufficient to imply an operational advantage in work extraction. This is why we presented our main results in terms of Δ , which provides a clear operational picture.

Conclusions—We have introduced a thermodynamic work-extraction task that fully characterizes state conversions in arbitrary resource theories (Theorem 1). This framework yields the first thermodynamic universal certification class of quantum resources (Theorem 3) and provides a quantification of general resources based on work extraction (Theorem 4). Our approach further suggests that mean energy can also characterize state conversions and hence witness quantum resources. Consequently, together with Ref. [75], we conclude that thermodynamic energies—*heat flow*, *mean energy*, and *extractable work*—all can witness general quantum properties. This naturally raises the open question of whether *heat flow can also provide a complete set of monotones for general quantum resources*.

In the resource theory of informational nonequilibrium [78], our result recovers the majorization conditions for unital channel conversions and links them to work extraction (Theorem 2), while in entanglement theory it yields a thermodynamic interpretation of Nielsen's theorem. We conjecture that the same framework also reproduces thermomajorization in thermodynamics (see, e.g., Refs. [33,84–88]), a question we leave for future work.

Finally, for any resourceful state $\rho \notin \mathcal{F}_R$, by Theorem 3, there is a Hamiltonian $H > 0$ achieving $\Delta(\rho, H) > \max_{\eta \in \mathcal{F}_R} \Delta(\eta, H)$. By viewing the quenching as a “charging process,” Theorem 3 means that under a *fixed* charging process with some final Hamiltonian H , a system initially in ρ can be charged *strictly better* than one initially in any free state $\eta \in \mathcal{F}_R$. Namely, systems initially prepared in resourceful states can achieve larger energy storage enhancements under a fixed charging process. This thus raises the question of whether quantum batteries can be employed to characterize general state conversions via the standard ergotropy framework [103,110,111]. Extending our methods to ergotropic approaches, such as quantum battery capacity [103], and broader notions of work extraction [69,97–103] lies beyond the present scope and will be pursued in future work.

Acknowledgments—We thank Antonio Acín, Pharnam Bakhshinezhad, Shin-Liang Chen, Shao-Hua Hu, Huan-Yu Ku, Patryk Lipka-Bartosik, Matteo Lostaglio, Paul Skrzypczyk, Alexander Streltsov, and Benjamin Stratton for fruitful discussions and comments. C.-Y.H. acknowledges support from the Royal Society through Enhanced

Research Expenses (NFQI), the ERC Advanced Grant (FLQuant), and the Leverhulme Trust Early Career Fellowship (“Quantum complementarity: a novel resource for quantum science and technologies” with Grant No. ECF-2024-310). This work is supported by the Project No. PID2023-152724NA-I00, with funding from MCIU/AEI/10.13039/501100011033 and FSE+, by the Project No. CNS2024-154818 with funding by MICIU/AEI/10.13039/501100011033, by the Project No. RYC2021-031094-I, with funding from MCIN/AEI/10.13039/501100011033 and the European Union “NextGenerationEU” PRTR fund, by the Project No. CIPROM/2022/66 with funding by the Generalitat Valenciana, and by the Ministry of Economic Affairs and Digital Transformation of the Spanish Government through the QUANTUM ENIA Project call—QUANTUM SPAIN Project, by the European Union through the Recovery, Transformation and Resilience Plan—NextGenerationEU within the framework of the Digital Spain 2026 Agenda, and by the CSIC Interdisciplinary Thematic Platform (PTI+) on Quantum Technologies (PTI-QTEP+). This work is supported through the Project No. CEX2023-001292-S funded by MCIU/AEI.

Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

-
- [1] K. Kuroiwa, R. Takagi, G. Adesso, and H. Yamasaki, Every quantum helps: Operational advantage of quantum resources beyond convexity, *Phys. Rev. Lett.* **132**, 150201 (2024).
 - [2] K. Kuroiwa, R. Takagi, G. Adesso, and H. Yamasaki, Robustness- and weight-based resource measures without convexity restriction: Multicopy witness and operational advantage in static and dynamical quantum resource theories, *Phys. Rev. A* **109**, 042403 (2024).
 - [3] F. Meier and H. Yamasaki, Energy-consumption advantage of quantum computation, *PRX Energy* **4**, 023008 (2025).
 - [4] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, *Phys. Rev. Lett.* **70**, 1895 (1993).
 - [5] C. H. Bennett and S. J. Wiesner, Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, *Phys. Rev. Lett.* **69**, 2881 (1992).
 - [6] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Quantum metrology with nonclassical states of atomic ensembles, *Rev. Mod. Phys.* **90**, 035005 (2018).
 - [7] Z. Ren, W. Li, A. Smerzi, and M. Gessner, Metrological detection of multipartite entanglement from young diagrams, *Phys. Rev. Lett.* **126**, 080502 (2021).
 - [8] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* **81**, 865 (2009).

- [9] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, *Rev. Mod. Phys.* **86**, 419 (2014).
- [10] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-independent security of quantum cryptography against collective attacks, *Phys. Rev. Lett.* **98**, 230501 (2007).
- [11] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Quantum steering, *Rev. Mod. Phys.* **92**, 015001 (2020).
- [12] D. Cavalcanti and P. Skrzypczyk, Quantum steering: A review with focus on semidefinite programming, *Rep. Prog. Phys.* **80**, 024001 (2016).
- [13] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, *Phys. Rev. A* **85**, 010301(R) (2012).
- [14] O. Gühne, E. Haapasalo, T. Kraft, J.-P. Pellonpää, and R. Uola, Colloquium: Incompatible measurements in quantum information science, *Rev. Mod. Phys.* **95**, 011003 (2023).
- [15] C.-Y. Hsieh, R. Uola, and P. Skrzypczyk, Quantum complementarity: A novel resource for unambiguous exclusion and encryption, [arXiv:2309.11968](https://arxiv.org/abs/2309.11968).
- [16] D. Rosset, F. Buscemi, and Y.-C. Liang, Resource theory of quantum memories and their faithful verification with minimal assumptions, *Phys. Rev. X* **8**, 021033 (2018).
- [17] X. Yuan, Y. Liu, Q. Zhao, B. Regula, J. Thompson, and M. Gu, Universal and operational benchmarking of quantum memories, *npj Quantum Inf.* **7**, 108 (2021).
- [18] H.-Y. Ku, J. Kadlec, A. Černoch, M. T. Quintino, W. Zhou, K. Lemr, N. Lambert, A. Miranowicz, S.-L. Chen, F. Nori, and Y.-N. Chen, Quantifying quantumness of channels without entanglement, *PRX Quantum* **3**, 020338 (2022).
- [19] L. B. Vieira, H.-Y. Ku, and C. Budroni, Entanglement-breaking channels are a quantum memory resource, *Phys. Rev. Res.* **7**, 043281 (2025).
- [20] P. Abiuso, Verification of continuous-variable quantum memories, *Quantum Sci. Technol.* **9**, 01LT02 (2023).
- [21] V. Narasimhachar, J. Thompson, J. Ma, G. Gour, and M. Gu, Quantifying memory capacity as a quantum thermodynamic resource, *Phys. Rev. Lett.* **122**, 060601 (2019).
- [22] C.-Y. Hsieh, B. Stratton, C.-H. Wu, and H.-Y. Ku, Dynamical resource theory of incompatibility preservability, *Phys. Rev. A* **111**, 022422 (2025).
- [23] R. Takagi, K. Wang, and M. Hayashi, Application of the resource theory of channels to communication scenarios, *Phys. Rev. Lett.* **124**, 120502 (2020).
- [24] C.-Y. Hsieh, Communication, dynamical resource theory, and thermodynamics, *PRX Quantum* **2**, 020318 (2021).
- [25] C.-Y. Hsieh, Dynamical Landauer principle: Quantifying information transmission by thermodynamics, *Phys. Rev. Lett.* **134**, 050404 (2025).
- [26] C.-Y. Hsieh, Dynamical Landauer principle: Thermodynamic criteria of transmitting classical information, *Phys. Rev. A* **111**, 022207 (2025).
- [27] C.-Y. Hsieh, Resource preservability, *Quantum* **4**, 244 (2020).
- [28] Z.-W. Liu and A. Winter, Resource theories of quantum channels and the universal role of resource erasure, [arXiv:1904.04201](https://arxiv.org/abs/1904.04201).
- [29] Y. Liu and X. Yuan, Operational resource theory of quantum channels, *Phys. Rev. Res.* **2**, 012035 (2020).
- [30] C.-Y. Hsieh, M. Lostaglio, and A. Acín, Entanglement preserving local thermalization, *Phys. Rev. Res.* **2**, 013379 (2020).
- [31] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Measuring quantum coherence with entanglement, *Phys. Rev. Lett.* **115**, 020403 (2015).
- [32] B. Stratton, C.-Y. Hsieh, and P. Skrzypczyk, Dynamical resource theory of informational nonequilibrium preservability, *Phys. Rev. Lett.* **132**, 110202 (2024).
- [33] M. Lostaglio, An introductory review of the resource theory approach to thermodynamics, *Rep. Prog. Phys.* **82**, 114001 (2019).
- [34] P. Faist, J. Oppenheim, and R. Renner, Gibbs-preserving maps outperform thermal operations in the quantum regime, *New J. Phys.* **17**, 043003 (2015).
- [35] M. Horodecki, K. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), and U. Sen, Local information as a resource in distributed quantum systems, *Phys. Rev. Lett.* **90**, 100402 (2003).
- [36] M. Horodecki, P. Horodecki, and J. Oppenheim, Reversible transformations from pure to mixed states and the unique measure of information, *Phys. Rev. A* **67**, 062104 (2003).
- [37] E. Chitambar and G. Gour, Quantum resource theories, *Rev. Mod. Phys.* **91**, 025001 (2019).
- [38] M. Piani and J. Watrous, All entangled states are useful for channel discrimination, *Phys. Rev. Lett.* **102**, 250501 (2009).
- [39] R. Takagi, B. Regula, K. Bu, Z.-W. Liu, and G. Adesso, Operational advantage of quantum resources in subchannel discrimination, *Phys. Rev. Lett.* **122**, 140402 (2019).
- [40] P. Skrzypczyk and N. Linden, Robustness of measurement, discrimination games, and accessible information, *Phys. Rev. Lett.* **122**, 140403 (2019).
- [41] J. Bae, D. Chruściński, and M. Piani, More entanglement implies higher performance in channel discrimination tasks, *Phys. Rev. Lett.* **122**, 140404 (2019).
- [42] R. Takagi and B. Regula, General resource theories in quantum mechanics and beyond: Operational characterization via discrimination tasks, *Phys. Rev. X* **9**, 031053 (2019).
- [43] P. Skrzypczyk, I. Šupić, and D. Cavalcanti, All sets of incompatible measurements give an advantage in quantum state discrimination, *Phys. Rev. Lett.* **122**, 130403 (2019).
- [44] C.-Y. Hsieh, G. N. M. Tabia, Y.-C. Yin, and Y.-C. Liang, Resource marginal problems, *Quantum* **8**, 1353 (2024).
- [45] C.-Y. Hsieh, M. Lostaglio, and A. Acín, Quantum channel marginal problem, *Phys. Rev. Res.* **4**, 013249 (2022).
- [46] A. F. Ducuara and P. Skrzypczyk, Operational interpretation of weight-based resource quantifiers in convex quantum resource theories, *Phys. Rev. Lett.* **125**, 110401 (2020).
- [47] R. Uola, T. Bullock, T. Kraft, J.-P. Pellonpää, and N. Brunner, All quantum resources provide an advantage in exclusion tasks, *Phys. Rev. Lett.* **125**, 110402 (2020).
- [48] K. C. Tan, V. Narasimhachar, and B. Regula, Fisher information universally identifies quantum resources, *Phys. Rev. Lett.* **127**, 200402 (2021).

- [49] R. Uola, T. Kraft, and A. A. Abbott, Quantification of quantum dynamics with input-output games, *Phys. Rev. A* **101**, 052306 (2020).
- [50] See also Ref. [51] for spacelike separated resources.
- [51] D. Rosset, D. Schmid, and F. Buscemi, Type-independent characterization of spacelike separated resources, *Phys. Rev. Lett.* **125**, 210402 (2020).
- [52] R. Kosloff and T. Feldmann, Discrete four-stroke quantum heat engine exploring the origin of friction, *Phys. Rev. E* **65**, 055102 (2002).
- [53] T. Feldmann and R. Kosloff, Quantum lubrication: Suppression of friction in a first-principles four-stroke heat engine, *Phys. Rev. E* **73**, 025107 (2006).
- [54] W. Ji, Z. Chai, M. Wang, Y. Guo, X. Rong, F. Shi, C. Ren, Y. Wang, and J. Du, Spin quantum heat engine quantified by quantum steering, *Phys. Rev. Lett.* **128**, 090602 (2022).
- [55] K. Beyer, K. Luoma, and W. T. Strunz, Steering heat engines: A truly quantum Maxwell demon, *Phys. Rev. Lett.* **123**, 250606 (2019).
- [56] F.-J. Chan, Y.-T. Huang, J.-D. Lin, H.-Y. Ku, J.-S. Chen, H.-B. Chen, and Y.-N. Chen, Maxwell's two-demon engine under pure dephasing noise, *Phys. Rev. A* **106**, 052201 (2022).
- [57] T. Biswas, C. Datta, and L. P. García-Pintos, Quantum thermodynamic advantage in work extraction from steerable quantum correlations, *Phys. Rev. Lett.* **135**, 110402 (2025).
- [58] D. Jennings and T. Rudolph, Entanglement and the thermodynamic arrow of time, *Phys. Rev. E* **81**, 061130 (2010).
- [59] M. Lostaglio, D. Jennings, and T. Rudolph, Thermodynamic resource theories, non-commutativity and maximum entropy principles, *New J. Phys.* **19**, 043008 (2017).
- [60] S. Majidy, W. F. Braasch, A. Lasek, T. Upadhyaya, A. Kalev, and N. Yunger Halpern, Noncommuting conserved charges in quantum thermodynamics and beyond, *Nat. Rev. Phys.* **5**, 689 (2023).
- [61] N. Yunger Halpern, P. Faist, J. Oppenheim, and A. Winter, Microcanonical and resource-theoretic derivations of the thermal state of a quantum system with noncommuting charges, *Nat. Commun.* **7**, 12051 (2016).
- [62] Y. Guryanova, S. Popescu, A. J. Short, R. Silva, and P. Skrzypczyk, Thermodynamics of quantum systems with multiple conserved quantities, *Nat. Commun.* **7**, 12049 (2016).
- [63] M. Lostaglio, Certifying quantum signatures in thermodynamics and metrology via contextuality of quantum linear response, *Phys. Rev. Lett.* **125**, 230603 (2020).
- [64] A. Levy and M. Lostaglio, Quasiprobability distribution for heat fluctuations in the quantum regime, *PRX Quantum* **1**, 010309 (2020).
- [65] S. Puliyil, M. Banik, and M. Alimuddin, Thermodynamic signatures of genuinely multipartite entanglement, *Phys. Rev. Lett.* **129**, 070601 (2022).
- [66] T. Upadhyaya, W. F. Braasch, G. T. Landi, and N. Y. Halpern, Non-Abelian transport distinguishes three usually equivalent notions of entropy production, *PRX Quantum* **5**, 030355 (2024).
- [67] F. Centrone and M. Gessner, Breaking local quantum speed limits with steering, *Phys. Rev. Res.* **6**, L042067 (2024).
- [68] P. Lipka-Bartosik, G. F. D'otalevi, and P. Bakhshinezhad, Fundamental limits on anomalous energy flows in correlated quantum systems, *Phys. Rev. Lett.* **132**, 140402 (2024).
- [69] M. Perarnau-Llobet, K. V. Hovhannisyan, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, Extractable work from correlations, *Phys. Rev. X* **5**, 041011 (2015).
- [70] L. d. Rio, J. Åberg, R. Renner, O. Dahlsten, and V. Vedral, The thermodynamic meaning of negative entropy, *Nature (London)* **474**, 61 (2011).
- [71] P. Skrzypczyk, A. J. Short, and S. Popescu, Work extraction and thermodynamics for individual quantum systems, *Nat. Commun.* **5**, 4185 (2014).
- [72] N. Shiraishi and R. Takagi, Arbitrary amplification of quantum coherence in asymptotic and catalytic transformation, *Phys. Rev. Lett.* **132**, 180202 (2024).
- [73] K. Korzekwa, M. Lostaglio, J. Oppenheim, and D. Jennings, The extraction of work from quantum coherence, *New J. Phys.* **18**, 023045 (2016).
- [74] C.-Y. Hsieh and S.-L. Chen, Thermodynamic approach to quantifying incompatible instruments, *Phys. Rev. Lett.* **133**, 170401 (2024).
- [75] A. de Oliveira Junior, J. B. Brask, and P. Lipka-Bartosik, Heat as a witness of quantum properties, *Phys. Rev. Lett.* **134**, 050401 (2025).
- [76] R. A. Macêdo, A. de Oliveira Junior, N. E. Comar, L. L. Keller, J. B. Brask, L. C. Céleri, and R. Chaves, Every little thing heat does is magic, [arXiv:2604.08663](https://arxiv.org/abs/2604.08663).
- [77] F. Buscemi, All entangled quantum states are nonlocal, *Phys. Rev. Lett.* **108**, 200401 (2012).
- [78] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens, and N. Yunger Halpern, The resource theory of informational nonequilibrium in thermodynamics, *Phys. Rep.* **583**, 1 (2015).
- [79] C.-Y. Hsieh and M. Gessner, companion paper, General quantum resources provide advantages in work extraction tasks, [arXiv:2403.18753](https://arxiv.org/abs/2403.18753) [*Phys. Rev. A* (to be published)].
- [80] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, Cambridge, England, 2010).
- [81] We always assume $\mathcal{O}_R$ is compact in the topology induced by the diamond distance, i.e., the distance measure induced by diamond norm $\|\cdot\|_\diamond$. [82] (see also, e.g., Ref. [83]).
- [82] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).
- [83] B. Regula, R. Takagi, and M. Gu, Operational applications of the diamond norm and related measures in quantifying the non-physicality of quantum maps, *Quantum* **5**, 522 (2021).
- [84] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, The second laws of quantum thermodynamics, *Proc. Natl. Acad. Sci. U.S.A.* **112**, 3275 (2015).
- [85] P. Ćwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, Limitations on the evolution of quantum coherences: Towards fully quantum second laws of thermodynamics, *Phys. Rev. Lett.* **115**, 210403 (2015).

- [86] G. Gour, D. Jennings, F. Buscemi, R. Duan, and I. Marvian, Quantum majorization and a complete set of entropic conditions for quantum thermodynamics, *Nat. Commun.* **9**, 5352 (2018).
- [87] T. Theurer, E. Zanoni, C. Maria Scandolo, and G. Gour, Thermodynamic state convertibility is determined by qubit cooling and heating, *New J. Phys.* **25**, 123017 (2023).
- [88] G. Gour, Role of quantum coherence in thermodynamics, *PRX Quantum* **3**, 040323 (2022).
- [89] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource theory of quantum states out of thermal equilibrium, *Phys. Rev. Lett.* **111**, 250404 (2013).
- [90] Note that $W(\rho, H)$ is the average work that can be extracted in the limit of infinitely many copies of ρ [89], or in the limit of infinitely many steps in the protocol [71].
- [91] H. Umegaki, Conditional expectation in an operator algebra, IV (Entropy And Information), *Kodai Math. Sem. Rep.* **14**, 59 (1962).
- [92] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Thermodynamical approach to quantifying quantum correlations, *Phys. Rev. Lett.* **89**, 180402 (2002).
- [93] M. Alimuddin, T. Guha, and P. Parashar, Independence of work and entropy for equal-energetic finite quantum systems: Passive-state energy as an entanglement quantifier, *Phys. Rev. E* **102**, 012145 (2020).
- [94] M. Alimuddin, T. Guha, and P. Parashar, Bound on ergotropic gap for bipartite separable states, *Phys. Rev. A* **99**, 052320 (2019).
- [95] J. A. Vaccaro, F. Anselmi, H. M. Wiseman, and K. Jacobs, Tradeoff between extractable mechanical work, accessible entanglement, and ability to act as a reference system, under arbitrary superselection rules, *Phys. Rev. A* **77**, 032114 (2008).
- [96] H. Kwon, H. Jeong, D. Jennings, B. Yadin, and M. S. Kim, Clock–work trade-off relation for coherence in quantum thermodynamics, *Phys. Rev. Lett.* **120**, 150602 (2018).
- [97] S. Vinjanampathy and J. Anders, Quantum thermodynamics, *Contemp. Phys.* **57**, 545 (2016).
- [98] M. A. Ciampini, L. Mancino, A. Orioux, C. Vigliar, P. Mataloni, M. Paternostro, and M. Barbieri, Experimental extractable work-based multipartite separability criteria, *npj Quantum Inf.* **3**, 10 (2017).
- [99] G. Francica, J. Goold, F. Plastina, and M. Paternostro, Daemonic ergotropy: Enhanced work extraction from quantum correlations, *npj Quantum Inf.* **3**, 12 (2017).
- [100] G. M. Andolina, M. Keck, A. Mari, M. Campisi, V. Giovannetti, and M. Polini, Extractable work, the role of correlations, and asymptotic freedom in quantum batteries, *Phys. Rev. Lett.* **122**, 047702 (2019).
- [101] J. Monsel, M. Fellous-Asiani, B. Huard, and A. Auffèves, The energetic cost of work extraction, *Phys. Rev. Lett.* **124**, 130601 (2020).
- [102] T. Opatrny, A. Misra, and G. Kurizki, Work generation from thermal noise by quantum phase-sensitive observation, *Phys. Rev. Lett.* **127**, 040602 (2021).
- [103] X. Yang, Y.-H. Yang, M. Alimuddin, R. Salvia, S.-M. Fei, L.-M. Zhao, S. Nimmrichter, and M.-X. Luo, Battery capacity of energy-storing quantum systems, *Phys. Rev. Lett.* **131**, 030402 (2023).
- [104] C. Datta, R. Ganardi, T. V. Kondra, and A. Streltsov, Is there a finite complete set of monotones in any quantum resource theory?, *Phys. Rev. Lett.* **130**, 240204 (2023).
- [105] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed. (Cambridge University Press, Cambridge, England, 1988).
- [106] To see this, it suffices to combine Lemmas 6, 10, 16, and Definition 15 in Ref. [78].
- [107] M. Sion, On general minimax theorems, *Pac. J. Math.* **8**, 171 (1958).
- [108] M. Navascués and S. Popescu, How energy conservation limits our measurements, *Phys. Rev. Lett.* **112**, 140502 (2014).
- [109] M. A. Nielsen, Conditions for a class of entanglement transformations, *Phys. Rev. Lett.* **83**, 436 (1999).
- [110] A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, Maximal work extraction from finite quantum systems, *Europhys. Lett.* **67**, 565 (2004).
- [111] S. Tirone, R. Salvia, S. Chessa, and V. Giovannetti, Quantum work capacitances: Ultimate limits for energy extraction on noisy quantum batteries, *SciPost Phys.* **17**, 041 (2024).

End Matter

Appendix A: On the convexity and compactness of $\mathcal{F}_R$ —When the working hypotheses hold for $\mathcal{O}_R$, a convex mixture of two state-preparation channels of any two free states must also be in $\mathcal{O}_R$. Since allowed operations cannot output resourceful states from free inputs [i.e., Eq. (1)], this convex mixture must also be a state-preparation channel of some free state. This implies that $\mathcal{F}_R$ is convex.

To see $\mathcal{F}_R$'s compactness, suppose we have a limit point η of $\mathcal{F}_R$ in the topology induced by the one norm $\|\cdot\|_1$, where, for an operator P , its *one norm* reads $\|P\|_1 := \text{tr}(|P|)$ [80]. This means there is a countable sequence $\{\eta_n\}_{n=1}^\infty \subseteq \mathcal{F}_R$ achieving $\lim_{n \rightarrow \infty} \|\eta - \eta_n\|_1 = 0$. Let $\Lambda_n(\cdot) := \eta_n \text{tr}(\cdot)$

and $\Lambda_\infty(\cdot) := \eta \text{tr}(\cdot)$ be the corresponding state-preparation channels. Now, recall that, for any Hermitian-preserving map Λ with input system S , its *diamond norm* [82] is defined by

$$\|\Lambda\|_\diamond := \max \{ \|(\Lambda \otimes \mathcal{I})(\rho)\|_1 : \rho \text{ state in } SA \}, \quad (\text{A1})$$

where $\mathcal{I}$ is the identity map acting on an auxiliary system A with the same dimension as S (see, e.g., Ref. [83]). Then,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\Lambda_\infty - \Lambda_n\|_\diamond &= \lim_{n \rightarrow \infty} \max_{\rho_A} \|(\eta - \eta_n) \otimes \rho_A\|_1 \\ &= \lim_{n \rightarrow \infty} \|\eta - \eta_n\|_1 = 0, \end{aligned} \quad (\text{A2})$$

where ρ_A is a state living in the auxiliary system A . Consequently, Λ_∞ is a limit point of $\mathcal{O}_R$ in the topology induced by $\|\cdot\|_\diamond$. By the working hypotheses, $\mathcal{O}_R$ is closed in this topology, meaning that $\Lambda_\infty \in \mathcal{O}_R$. Since Λ_∞ is the state-preparation channel of η , and allowed operations cannot output resourceful states from free inputs [i.e., Eq. (1)], we conclude that $\eta \in \mathcal{F}_R$. Namely, $\mathcal{F}_R$ contains all its limit points in the topology induced by $\|\cdot\|_1$, meaning that it is closed in this topology. Because $\mathcal{F}_R$ is a subset of quantum states, it is bounded. Hence, it is bounded and closed in a finite-dimensional space, implying that it is indeed compact in the desired topology.

Appendix B: Proof of Theorem 1—If $\mathcal{N}(\rho) = \sigma$ for some $\mathcal{N} \in \mathcal{O}_R$, then, by definition, $\Delta_{\mathcal{O}_R}(\rho, H) \geq \Delta_{\mathcal{O}_R}(\sigma, H) \forall H$. Hence, it suffices to prove the opposite direction. For ρ and σ , suppose we have Eq. (7). This means, for every $\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}$,

$$0 \leq \max_{\mathcal{N} \in \mathcal{O}_R} (\Delta[\mathcal{N}(\rho), H] - \Delta(\sigma, H)). \quad (\text{B1})$$

Now, note that, for any Hamiltonian H and state ρ in a d -dimensional system, we have the following formula:

$$\frac{\Delta(\rho, H)}{k_B T \ln 2} = \frac{\text{tr}(H\rho)}{k_B T \ln 2} + \log_2 \text{tr}\left(e^{-\frac{H}{k_B T}}\right) - \log_2 d. \quad (\text{B2})$$

Using the above formula in Eq. (B1), we obtain $0 \leq \max_{\mathcal{N} \in \mathcal{O}_R} \text{tr}([\mathcal{N}(\rho) - \sigma]H)$. Minimizing over all Hamiltonians H with $\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}$ gives

$$0 \leq \min_{\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}} \max_{\mathcal{N} \in \mathcal{O}_R} \text{tr}([\mathcal{N}(\rho) - \sigma]H). \quad (\text{B3})$$

Now, note that both $\mathcal{O}_R$ and $\{H | \epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}\}$ are compact and convex sets. Moreover, for fixed ρ and σ , define $f(\mathcal{N}, H) := \text{tr}([\mathcal{N}(\rho) - \sigma]H)$. Then, for any fixed $\mathcal{N}$, the function f is linear in H , meaning that it is continuous as well as quasiconvex in H . Similarly, for any fixed H , the function f is also linear in $\mathcal{N}$, meaning that it is continuous as well as quasiconcave in $\mathcal{N}$. The above implies that Sion's minimax theorem [107] can be applied to switch the order of the maximization and minimization, giving

$$0 \leq \max_{\mathcal{N} \in \mathcal{O}_R} \min_{\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}} \text{tr}([\mathcal{N}(\rho) - \sigma]H). \quad (\text{B4})$$

Define the function

$$f_{\min}(\mathcal{N}) := \min_{\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}} \text{tr}([\mathcal{N}(\rho) - \sigma]H). \quad (\text{B5})$$

For two channels $\mathcal{E}, \mathcal{L}$, without loss of generality, suppose $f_{\min}(\mathcal{E}) \geq f_{\min}(\mathcal{L})$. Then, we have (let $\tilde{H} = H/\delta$)

$$\begin{aligned} |f_{\min}(\mathcal{E}) - f_{\min}(\mathcal{L})| &\leq \delta \max_{(\epsilon/\delta)\mathbb{I} \leq \tilde{H} \leq \mathbb{I}} \text{tr}([\mathcal{E}(\rho) - \mathcal{L}(\rho)]\tilde{H}) \\ &\leq \delta \max_{0 \leq P \leq \mathbb{I}} \text{tr}([\mathcal{E}(\rho) - \mathcal{L}(\rho)]P) \\ &\leq \frac{\delta}{2} \|\mathcal{E} - \mathcal{L}\|_\diamond, \end{aligned} \quad (\text{B6})$$

meaning that $f_{\min}$ is Lipschitz continuous in the topology induced by the diamond norm $\|\cdot\|_\diamond$ [Eq. (A1)]. Here, we have used the trace distance formula $\|\rho - \sigma\|_1/2 = \max_{0 \leq P \leq \mathbb{I}} \text{tr}[(\rho - \sigma)P]$ [80]. Since $\mathcal{O}_R$ is compact in the same topology (by working hypothesis), there is $\mathcal{N}_* \in \mathcal{O}_R$ achieving $f_{\min}(\mathcal{N}_*) = \max_{\mathcal{N} \in \mathcal{O}_R} f_{\min}(\mathcal{N})$. Thus, Eq. (B4) leads to

$$0 \leq \min_{\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}} \text{tr}([\mathcal{N}_*(\rho) - \sigma]H), \quad (\text{B7})$$

which is equivalent to $0 \leq \mathcal{N}_*(\rho) - \sigma$. This means $\mathcal{N}_*(\rho) - \sigma$ is a traceless positive semidefinite operator, i.e., it is zero. Consequently, $\mathcal{N}_*(\rho) = \sigma$, which concludes the proof. ■

Appendix C: Proof of Theorem 2—Since there is always a unitary converting ρ to $\rho^\downarrow$, $\rho \xrightarrow{\text{MU}} \sigma$ if and only if $\rho^\downarrow \xrightarrow{\text{MU}} \sigma^\downarrow$, which holds if and only if (by setting $\epsilon = 0$ in Theorem 1)

$$\Delta_{\text{MU}}(\rho^\downarrow, H) \geq \Delta_{\text{MU}}(\sigma^\downarrow, H) \quad \forall 0 \leq H \leq \delta \mathbb{I}, \quad (\text{C1})$$

where we set “ $\mathcal{O}_R = \text{MU}$,” the set of all mixed unitary channels. Using Eq. (B2), we learn that Eq. (C1) holds if and only if $\max_{\mathcal{N} \in \text{MU}} \text{tr}[\mathcal{N}(\rho^\downarrow)H] \geq \max_{\mathcal{N} \in \text{MU}} \text{tr}[\mathcal{N}(\sigma^\downarrow)H]$. One can check that $\max_{\mathcal{N} \in \text{MU}} \text{tr}[\mathcal{N}(\rho^\downarrow)H] = \max_U \text{tr}(U\rho^\downarrow U^\dagger H) \forall H \geq 0$ (“ $\max_U$ ” denotes maximization over all unitaries U). Hence, Eq. (C1) holds if and only if

$$\max_U \text{tr}(U\rho^\downarrow U^\dagger H) \geq \max_U \text{tr}(U\sigma^\downarrow U^\dagger H) \quad \forall 0 \leq H \leq \delta \mathbb{I}. \quad (\text{C2})$$

Let $\{E_i^\downarrow(H)\}_i$ be H 's eigenenergies in the *nonincreasing* order; i.e., $E_i^\downarrow(H) \geq E_{i+1}^\downarrow(H) \forall i = 0, \dots, d-2$. In Lemma 1, we show that Eq. (C2) holds [and so does Eq. (C1)] if and only if

$$\sum_{i=0}^{d-1} (\rho_i^\downarrow - \sigma_i^\downarrow) E_i^\downarrow(H) \geq 0 \quad \forall 0 \leq H \leq \delta \mathbb{I}. \quad (\text{C3})$$

Now, recall that $H_k := \omega \sum_{i=0}^k |i\rangle\langle i|$ and $\{|i\rangle\}_i$ is the prefixed basis and $\omega > 0$. Using Eq. (B2) again, we conclude that Eq. (10) (i.e., Theorem 2's statement) holds if and only if

$$\omega \sum_{i=0}^k (\rho_i^\downarrow - \sigma_i^\downarrow) \geq 0 \quad \forall k = 0, \dots, d-2. \quad (\text{C4})$$

It remains to show that Eqs. (C3) and (C4) are equivalent. As Eq. (C3) implies Eq. (C4), it remains to show the converse. For a Hamiltonian $0 \leq H \leq \delta \mathbb{I}$, suppose $E_N^\downarrow(H) > 0$ is H 's smallest positive eigenenergy. Then, we have

$$\begin{aligned} & \sum_{i=0}^{d-1} (\rho_i^\downarrow - \sigma_i^\downarrow) E_i^\downarrow(H) \\ & \geq E_1^\downarrow(H) \sum_{i=0}^1 (\rho_i^\downarrow - \sigma_i^\downarrow) + \sum_{i=2}^{d-1} (\rho_i^\downarrow - \sigma_i^\downarrow) E_i^\downarrow(H) \\ & \geq E_2^\downarrow(H) \sum_{i=0}^2 (\rho_i^\downarrow - \sigma_i^\downarrow) + \sum_{i=3}^{d-1} (\rho_i^\downarrow - \sigma_i^\downarrow) E_i^\downarrow(H) \\ & \dots \geq E_N^\downarrow(H) \sum_{i=0}^N (\rho_i^\downarrow - \sigma_i^\downarrow) \geq 0. \end{aligned} \quad (\text{C5})$$

The second line is due to $\rho_0^\downarrow - \sigma_0^\downarrow \geq 0$ [setting $k = 0$ in Eq. (C4)] and $E_0^\downarrow \geq E_1^\downarrow$. The third line is due to $\sum_{i=0}^1 (\rho_i^\downarrow - \sigma_i^\downarrow) \geq 0$ [setting $k = 1$ in Eq. (C4)] and $E_1^\downarrow \geq E_2^\downarrow$. Applying this argument N times gives the last line, which is non-negative by setting $k = N$ in Eq. (C4). This shows that Eq. (C4) implies Eq. (C3). Finally, since $\omega > 0$, Eq. (C4) is equivalent to $\rho \succ \sigma$ [defined in Eq. (8)], leading to Eq. (9). ■

Finally, it remains to prove the following lemma.

Lemma 1—For ρ and $0 \leq H \leq \delta \mathbb{I}$, we have that

$$\max_U \text{tr}(U \rho^\downarrow U^\dagger H) = \sum_{i=0}^{d-1} \rho_i^\downarrow E_i^\downarrow(H). \quad (\text{C6})$$

Proof—For $0 \leq H \leq \delta \mathbb{I}$, let $\tilde{H} := \delta \mathbb{I} - H$. Then we have

$$\max_U \text{tr}(U \rho^\downarrow U^\dagger H) = \delta - \min_U \text{tr}(U \rho^\downarrow U^\dagger \tilde{H}). \quad (\text{C7})$$

Let us write $H = \sum_{i=0}^{d-1} E_i^\downarrow(H) |E_i\rangle \langle E_i|$, where $|E_i\rangle$ is H 's energy eigenstate with the eigenenergy $E_i^\downarrow(H)$. Hence, $\tilde{H} = \sum_{i=0}^{d-1} (\delta - E_i^\downarrow(H)) |E_i\rangle \langle E_i|$ with its eigenenergies in the nondecreasing order. Using the property of passive states [110] [see also Eqs. (1) to (3) in Ref. [69]], we obtain

$$\min_U \text{tr}(U \rho^\downarrow U^\dagger \tilde{H}) = \text{tr}(\tilde{\rho}_{\text{passive}} \tilde{H}), \quad (\text{C8})$$

where $\tilde{\rho}_{\text{passive}} := \sum_{i=0}^{d-1} \rho_i^\downarrow |E_i\rangle \langle E_i|$ is a so-called passive state with the Hamiltonian $\tilde{H}$ (i.e., diagonal in $\tilde{H}$'s eigenbasis with nonincreasing eigenvalues with respect

to $\tilde{H}$'s eigenenergies). Together with Eq. (C7), we have $\max_U \text{tr}(U \rho^\downarrow U^\dagger H) = \text{tr}(\tilde{\rho}_{\text{passive}} H) = \sum_{i=0}^{d-1} \rho_i^\downarrow E_i^\downarrow(H)$. ■

Appendix D: Proof of Theorem 3—Define the set

$$\mathcal{O}_R^{\min} := \text{Cov}\{\text{state-preparation of } \mathcal{F}_R \text{ \& identity}\}, \quad (\text{D1})$$

which is the set containing the identity channel, all state-preparation channels that prepare some free state, and their convex mixture. Since, in our approach, $\mathcal{F}_R$ is always convex and compact (see Appendix A), $\mathcal{O}_R^{\min}$ is a set satisfying the working hypothesis [to see its compactness, it suffices to consider the continuous function $f(p, \eta) := p\eta \text{tr}(\cdot) + (1-p)\mathcal{I}(\cdot)$ and note that it maps the compact set $[0, 1] \times \mathcal{F}_R$ onto $\mathcal{O}_R^{\min}$]. Consider energy scales $0 \leq \epsilon < \delta$ and a given $\rho \notin \mathcal{F}_R$. For a given free state $\eta_* \in \mathcal{F}_R$, since no channel in $\mathcal{O}_R^{\min}$ can convert it to ρ , Theorem 1 implies that there is a Hamiltonian $\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}$ achieving $\Delta_{\mathcal{O}_R^{\min}}(\rho, H) > \Delta_{\mathcal{O}_R^{\min}}(\eta_*, H)$. Now, the structure of $\mathcal{O}_R^{\min}$ ensures $\Delta_{\mathcal{O}_R^{\min}}(\eta_*, H) = \max_{\eta \in \mathcal{F}_R} \Delta(\eta, H)$. Hence,

$$\Delta_{\mathcal{O}_R^{\min}}(\rho, H) > \max_{\eta \in \mathcal{F}_R} \Delta(\eta, H). \quad (\text{D2})$$

Finally, using Eq. (B2), we have

$$\begin{aligned} \Delta_{\mathcal{O}_R^{\min}}(\rho, H) &= \max_{\substack{0 \leq p \leq 1; \\ \eta \in \mathcal{F}_R}} (p \Delta(\rho, H) + (1-p) \Delta(\eta, H)) \\ &= \max_{0 \leq p \leq 1} (p \Delta(\rho, H) + (1-p) \max_{\eta \in \mathcal{F}_R} \Delta(\eta, H)). \end{aligned} \quad (\text{D3})$$

Suppose p_* achieves the above maximum. Combining with Eq. (D2), we obtain $p_* \max_{\eta \in \mathcal{F}_R} \Delta(\eta, H) < p_* \Delta(\rho, H)$. This thus means we must have $p_* > 0$. Dividing p_* on both sides concludes the proof. ■

As a remark, such results are usually proved by the hyperplane separation theorem (see, e.g., Ref. [42]). Here, we prove it by Sion's minimax theorem (via Theorem 1).

Appendix E: Proof of Theorem 4—Since $\mathcal{O}_R$ contains all state-preparation channels of free states (from the working hypothesis), Eq. (6) implies that $\Delta_{\mathcal{O}_R}(\kappa, H) = \max_{\eta \in \mathcal{F}_R} \Delta(\eta, H) \quad \forall \epsilon \mathbb{I} \leq H \leq \delta \mathbb{I} \quad \text{and} \quad \forall \kappa \in \mathcal{F}_R$, meaning that $\mathcal{M}_{\mathcal{O}_R}(\eta) = 0 \quad \forall \eta \in \mathcal{F}_R$. Also, for $\rho \notin \mathcal{F}_R$, Theorem 3 implies that there is $\epsilon \mathbb{I} \leq H \leq \delta \mathbb{I}$ achieving $\max_{\eta \in \mathcal{F}_R} \Delta_{\mathcal{O}_R}(\eta, H) < \Delta_{\mathcal{O}_R}(\rho, H)$. Hence, $\mathcal{M}_{\mathcal{O}_R}(\rho) > 0 \quad \forall \rho \notin \mathcal{F}_R$. Finally, to show that $\mathcal{M}_{\mathcal{O}_R}$ is nonincreasing under allowed operations, note that $\Delta_{\mathcal{O}_R}[\mathcal{N}(\rho), H] \leq \Delta_{\mathcal{O}_R}(\rho, H) \quad \forall \rho \text{ and } \forall \mathcal{N} \in \mathcal{O}_R$. Hence,

$$\begin{aligned}
 \mathcal{M}_{\mathcal{O}_R}[\mathcal{N}(\rho)] &:= \max_{\epsilon\mathbb{I} \leq H \leq \delta\mathbb{I}} \left[\Delta_{\mathcal{O}_R}[\mathcal{N}(\rho), H] - \max_{\eta \in \mathcal{F}_R} \Delta_{\mathcal{O}_R}(\eta, H) \right] \\
 &\leq \max_{\epsilon\mathbb{I} \leq H \leq \delta\mathbb{I}} \left[\Delta_{\mathcal{O}_R}(\rho, H) - \max_{\eta \in \mathcal{F}_R} \Delta_{\mathcal{O}_R}(\eta, H) \right] = \mathcal{M}_{\mathcal{O}_R}(\rho)
 \end{aligned} \tag{E1}$$

which concludes the proof. ■