

Conservative Black Hole Scattering at Fifth Post-Minkowskian and Second Self-Force Order

Mathias Driesse¹,^{*} Gustav Uhre Jakobsen^{1,2},^{*} Gustav Mogull^{1,2,3},^{*} Christoph Nega^{2,4},^{*}

Jan Plefka¹, Benjamin Sauer¹, and Johann Usovitsch¹

¹*Institut für Physik, Humboldt-Universität zu Berlin, 10099 Berlin, Germany*

²*Max Planck Institut für Gravitationsphysik (Albert Einstein Institut), 14476 Potsdam, Germany*

³*School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, United Kingdom*

⁴*Physik Department, Technische Universität München, 85748 Garching, Germany*



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Using the worldline quantum field theory formalism, we compute conservative contributions to the scattering angle and impulse for classical black hole scattering at fifth post-Minkowskian and second self-force order. This four-loop calculation involves nonplanar Feynman integrals and requires advanced integration-by-parts reduction, novel differential-equation strategies, and efficient boundary-integral algorithms to solve a system of hundreds of master integrals in four integral families on high-performance computing systems. The resulting function space includes multiple polylogarithms as well as iterated integrals with a $K3$ period, which generate a spurious velocity divergence at $v/c = \sqrt{8}/3$, $\gamma = 3$. This divergence is present in the potential region and must be canceled by contributions from the radiative memory region, while its dimensional-regularization pole should cancel against the radiative tail region. As the standard use of Feynman propagators fails to ensure this cancellation, we instead propose a “ $(\gamma - 3)$ ” conservative prescription that realizes both cancellations, leading to a physically sensible answer. All available low-velocity checks of our result against the post-Newtonian literature are satisfied.

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A decade after the first gravitational wave observation emitted by a binary black hole merger [1], the LIGO-Virgo-KAGRA Collaboration [1–3] now reports 218 detections of compact binary coalescences in our Universe [4]. In the coming decade, a third generation of ground- and space-based gravitational-wave detectors is scheduled to go online [5–7], which will dramatically increase the accuracy and frequency range of observations. This will open a new window into gravitational, astrophysical, nuclear, and fundamental physics. To fully exploit this observational potential, theoretical predictions for the dynamics and gravitational radiation of compact binaries must reach a comparable level of precision. This challenge has driven a wide effort comprising the perturbative schemes of post-Newtonian (PN) [8–11], post-Minkowskian (PM) [12–16], and gravitational self-force (SF) [17–20] expansions, in close synergy with numerical relativity [21–23]. Notably, techniques from perturbative quantum field theory and effective field theory have gained a leading role in these approaches, enabling increasingly high-order analytic

control over the classical two-body problem in general relativity [24].

The PM expansion is a weak-field expansion in powers of Newton’s constant (G), being the natural perturbative framework for the unbound scattering of two compact objects—black holes (BHs) or neutron stars (NSs)—or highly eccentric bound orbits [25–29]. As long as the separation of the two objects ($\sim|b|$) is large compared to their intrinsic sizes ($\sim Gm$), they are captured in an effective worldline theory of massive point particles coupled to gravity [30]. The key observables of the change of momentum (known as impulse), the scattering angle, and the far-field waveform have been systematically computed as loop corrections in this classical field theory including spin, tidal effects, and radiation reaction; see, e.g., Refs. [31–61] for worldline and [62–97] for amplitude based approaches. Both approaches have delivered the scattering angle and impulse up to fourth post-Minkowskian (4PM) order [40,41,56–58,86,87,90]. Employing the worldline quantum field theory (WQFT) formalism [16,34,49,98], this progress has culminated in the first determination of the conservative and dissipative fifth post-Minkowskian (5PM) contributions to the scattering angle and impulse at first self-force (1SF) order [24,99], i.e., the leading and subleading mass ratio contributions. State-of-the-art numerical computations have recently validated these high-order analytical PM predictions to an impressive degree [100,101].

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Despite these advances, the last missing ingredient at 5PM order has been the subsubleading mass ratio or second self-force (2SF) order contributions to observables. While 5PM-2SF results for potential graviton modes were established in $\mathcal{N} = 8$ supergravity [102] and very recently in gravity [103] in an ordinary differential equation based formal PM expression, a complete 5PM-2SF order description has remained out of reach. Here, we compute conservative contributions to the 5PM-2SF scattering observables using WQFT. Our results characterize the prescription-dependent conservative 5PM scattering dynamics, providing a crucial step toward new analytic input for high-accuracy models tailored to the upcoming generation of gravitational-wave observatories, pending the transformation to the bound case [104–108].

Worldline quantum field theory—The two (nonspinning) BHs or NSs are modeled as point particles with trajectories $x_i^\mu(\tau)$. In proper time gauge $\dot{x}_i^2 = 1$,

$$S = - \sum_{i=1}^2 \frac{m_i}{2} \int d\tau g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu - \frac{1}{16\pi G} \int d^D x \sqrt{-g} R, \quad (1)$$

suppressing a gauge-fixing term S_{GF} . We use a mostly minus signature, dimensional regularization with $D = 4 - 2\epsilon$, and consider an expansion around the Minkowskian (G^0) background and straight-line trajectories:

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau), \quad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G} h_{\mu\nu}, \quad (2)$$

with the worldline deflections $z_i^\mu(\tau)$ and graviton field $h_{\mu\nu}(x)$. The initial data consist of the impact parameter $b^\mu = b_2^\mu - b_1^\mu$ and initial velocities v_1^μ, v_2^μ , with $b \cdot v_i = 0$, $v_i^2 = 1$ and $\gamma = v_1 \cdot v_2 = (1 - v^2)^{-1/2}$. We also introduce the parameter $x \in [0, 1]$ defined via $\gamma = (x + x^{-1})/2$ to rationalize expressions.

In the WQFT formalism, solutions to the classical equations of motion for the trajectory and metric are given by their tree-level one-point functions, $\langle x_i^\mu(\tau) \rangle$ and $\langle g_{\mu\nu}(x) \rangle$. Nevertheless, nontrivial Feynman loop integrals arise due to the hybrid nature of the WQFT: worldlines conserve only a single momentum component, as opposed to full D -dimensional momentum conservation in the bulk. The impulse of the first BH, Δp_1^μ , emerges from $\Delta p_1^\mu = \lim_{\omega \rightarrow 0} \omega^2 \langle z_1^\mu(\omega) \rangle$ in momentum (energy) space. At 5PM order, the impulse factorizes into SF sectors:

$$\Delta p_1^{(5)\mu} = m_1 m_2 \left(m_2^4 \Delta p_{\text{0SF}}^{(5)\mu} + m_1 m_2^3 \Delta p_{\text{1SF}}^{(5)\mu} + m_1^2 m_2^2 \Delta p_{\text{2SF}}^{(5)\mu} + m_1^3 m_2 \Delta p_{\text{1SF}}^{(5)\mu} + m_1^4 \Delta p_{\text{0SF}}^{(5)\mu} \right). \quad (3)$$

In this Letter we compute the unknown even-in-velocity conservative part of $\Delta p_{\text{2SF}}^{(5)\mu}$, while all other parts are known [24,99]. On a basis of b^μ , v_1^μ , and v_2^μ in the scattering plane, we compute the b component directly— v_i components are reconstructed using the on shell conditions $p_i^2 = (p_i + \Delta p_i)^2$ and momentum conservation $\Delta p_1^\mu = -\Delta p_2^\mu$.

Integrand—Generation of the WQFT integrand employs recursive diagrammatic techniques [16,24,47–49,56,57,99], producing a total of 651 Feynman diagrams. At the 5PM order, we insert up to six graviton vertices, whose complexity is reduced through an optimized nonlinear De Donder gauge [24]. A subtle point is how to define a *conservative* sector of the truly dissipative (in-in) problem in which $\Delta p_1^\mu + \Delta p_2^\mu \neq 0$ due to momentum (and angular momentum) being carried away by gravitational radiation. This has previously been achieved [32,33,37,47,56,58,99] by employing Feynman propagators (in-out) for the gravitons and retarded propagators on the worldline (in-in) [49,109], taking the part real and even in velocity v in the end. In this Letter we take a different approach, discussed as part of our boundary fixing—for now, we assume only that graviton propagators are individually time symmetric.

Our diagrams organize themselves into 14 top-level sectors with respect to the integration-by-parts (IBP) identities, displayed in Fig. 1—these are indicative of the Feynman diagrams encountered at 2SF. A vital difference between 1SF and 2SF is that we cannot use partial fraction identities on worldline propagators to “planarize” the entire integrand [99]. Nevertheless, partial fractions are still used to resolve linear identities between worldline propagators. We require four separate integral families (P, NP1, NP2, NPM), explicit expressions are provided in the Supplemental Material [110]. The planar family (P) accounts for the top sectors [(a)–(g)] of Fig. 1, and all diagrams take a “Mondrian” structure without lines crossing each other. The first two nonplanar families (NP1, NP2) capture the possibility of one [(h)–(i)] or two [(j)] crossings of the graviton propagators.

The last nonplanar family (NPM), associated with top sectors [(k)–(m)], captures the effect of *nonlinear memory* [111–114]. Nonplanarity of these diagrams is carried by a vertex where three *active* gravitons (that may go on-shell) meet. The effect is to produce nonlinear corrections to the Einstein equation expanded around a Schwarzschild background. There is an S_3 permutation symmetry about this vertex, which we exploit next to shifts on the loop momenta in order to maximally simplify our integrand as preparation for IBP reduction.

Integration-by-parts reduction—The first step toward full integration was finding linear IBP identities between the elements of our four integral families, enabling reduction to a much smaller basis of master integrals. Our IBP reduction was performed using Kira 3.0 [115–118]. This was the computational bottleneck of the calculation, consuming

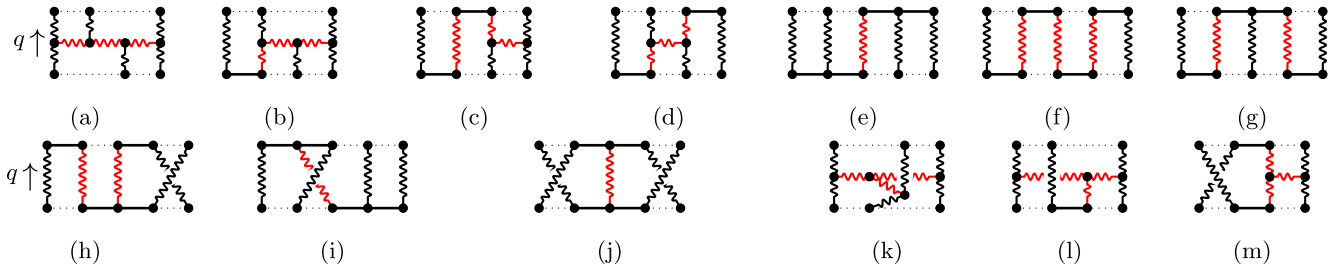


FIG. 1. The 14 top-level sectors of the four-loop 2SF integral families: planar P in the first row (a)–(g); in the second row nonplanar with one graviton crossing NP1 (h),(i), nonplanar with two graviton crossings NP2 (j), and nonplanars arising from nonlinear memory NPM (k)–(m), with an active three-graviton vertex. All integrals encountered in our calculation involve subsets of those appearing in one of these sectors. Solid lines denote worldline propagators, $1/\ell \cdot v_i$, wavy line graviton propagators, $1/\ell^2$, and the dotted lines may be interpreted as a cut worldline propagator, $\delta(\ell \cdot v_i)$. In that sense, the top-level sectors have 13 propagators, and active (red) graviton propagators may go on shell. Finally, q^μ is the total momentum transfer.

$\sim 3 \times 10^6$ core hours on a high-performance cluster—an order of magnitude larger than 1SF [24,99]. Yet this number masks the true complexity of the 2SF problem, which required two key additional technical improvements: (1) The use of symmetry relations in the IBP reduction specific to the conservative setup led to a reduction in the number of master integrals. For the P, NP1, and NP2 families, we let Kira generate these symmetry relations automatically. Symmetries of the NPM family were generated by hand and given as extra equations. (2) A careful choice of basis sped up the reduction by a factor of 10, thereby reducing the polynomial degree of the result. This “precanonical” basis is chosen as close to the canonical integrals as possible without introducing any algebraic or transcendental functions. When using finite-field reconstruction with FireFly [119,120], we observed a drastic reduction not only in the number of probes per prime field but also in the number of prime fields. In summary, we find the following numbers of master integrals (MIs) in our four families:

Family	P	NP1	NP2	NPM
No. of MIs	321	144	46	220

(4)

Differential equations—We compute the master integrals $\underline{I}(x; \epsilon) = (I_1, \dots, I_N)$ using differential equations (DEs) [121–123] in the kinematic variable x , i.e., $\partial_x \underline{I}(x; \epsilon) = B(x; \epsilon) \underline{I}(x; \epsilon)$, where the matrix $B(x; \epsilon)$ is derived using the IBP relations. To systematically calculate the expansion of the master integrals in the dimensional regulator ϵ , we transform the differential equation into canonical form [124], which is achieved by the rotation $\underline{J}(x; \epsilon) = T(x; \epsilon) \underline{I}(x; \epsilon)$. In the new basis, ϵ factorizes in the differential equation

$$\frac{\partial}{\partial x} \underline{J}(x; \epsilon) = \epsilon A(x) \underline{J}(x; \epsilon), \quad (5)$$

with the transformed matrix $\epsilon A(x) = T(BT^{-1} - \partial_x T^{-1})$. The structure of Eq. (5) enables us to write its solution in terms of iterated integrals.

To find the canonical basis $\underline{J}(x; \epsilon)$, we use the method developed in Refs. [125,126]. It is essential to understand the different geometries appearing in the individual sectors (groups of integrals that share the same propagators) of the set of master integrals. Two- and three-dimensional Calabi-Yau (CY) varieties are known to appear at the 5PM order [24,127–130]. These geometries are natural generalizations of elliptic curves to higher dimensions, and solve Einstein’s equations in the vacuum [131]. Traditionally, they have been used in the context of string theory compactifications [132,133], but have recently found applications in Feynman integrals for particle physics [134–140]. In our 5PM-2SF problem, we find two different Calabi-Yau varieties of dimension three and two, where the two-dimensional one is also called a $K3$ surface [141]—see Fig. 2. An analysis of these CY varieties, in particular their definitions via polynomial equations, has already been done by some of the present authors [128]. The periods $\varpi(x)$ of the CYs are determined by their Picard-Fuchs DEs $\mathcal{L}\varpi(x) = 0$. The corresponding operator for the $K3$ is given by ($\theta = x\partial_x$)

$$\mathcal{L}_{K3} = (1 - 34x^2 + x^4)\theta^3 - 6x^2(17 - x^2)\theta^2 - 12(3 - x)(3 + x)x^2\theta - 8(5 - x^2)x^2, \quad (6)$$

and is related to the Apéry operator [142,143]. The ϵ factorization of the pure CY3 and $K3$ sectors has been given in Ref. [126].

To compute a fully canonical system, we follow a similar strategy as outlined in our 5PM-1SF computation [24]. The increase in complexity of DEs compared with 1SF leads to increased block sizes up to 16×16 , cf. Fig. 2. For the smaller polylogarithmic diagonal blocks, we employ CANONICA [144]. For larger sectors, we use the Baikov representation of our integrals [145] to find a single

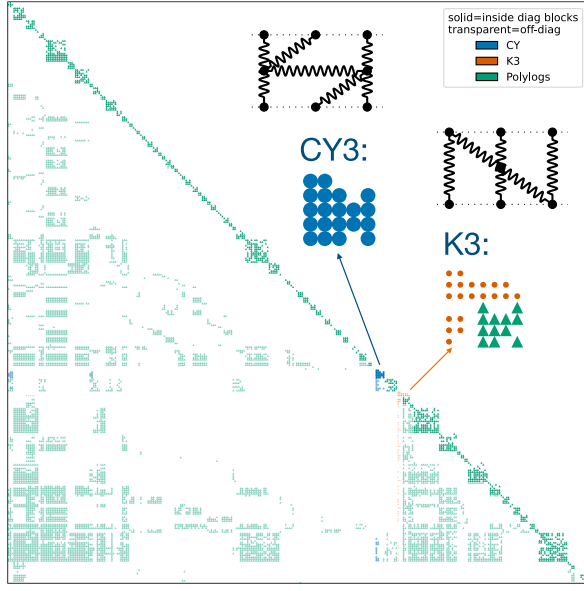


FIG. 2. Nonzero entries of the 321×321 DE matrix $A(x)$ of the planar family (P). The solid blocks on the diagonals determine the function spaces of CY3 (blue), K3 (orange), and polylogarithmic (green) type. The off diagonal entries are in corresponding lighter colors.

integral of uniform transcendental weight. Then we use INITIAL [146] to transform the whole diagonal block to canonical form. To transform the off diagonal blocks, we use in-house code based on FiniteFlow [147] and MultivariateApart [148]. For the couplings between polylogarithmic sectors and the CY sectors, we employ integrand analysis using the Baikov representation [145] and IBP-based searches for near-canonical integrals. Using successive transformations, these integrals are rotated into canonical form.

Interestingly, after IBP reduction we find that only the K3 sector contributes to our final result. The $\mathcal{L}_{K3}$ operator (6), and therefore the integrals in this sector, have a singular point at $x = 3 - 2\sqrt{2}$, or equivalently $\gamma = 3$, $v = \sqrt{8}/3$, which lies in the physical region $x \in [0, 1]$. We also needed to introduce the square root $r(x) = \sqrt{-1 + 34x^2 - x^4}$ in the canonicalization process, which shares the same singularity structure. Singularities at $\gamma = 3$ appear in partial results but must cancel in the observables.

Expansion by regions and divergences—After ϵ factorization we construct the solution for the master integrals in an ϵ expansion up to boundary integrals. We determine the boundaries in the small velocity limit ($x \rightarrow 1$, $\gamma \rightarrow 1$, $v \rightarrow 0$) using the *method of regions* [149–151]. This split into regions also plays a role in the separation of conservative and dissipative effects. The regions are characterized by scalings of loop momenta:

$$\ell^P = (\ell^0, \ell) \sim (v, 1), \quad \ell^R = (\ell^0, \ell) \sim (v, v). \quad (7)$$

We call these *potential* (P) or *radiative* (R) scalings respectively. Generally, all loop momenta may be potential, but only a few may also become radiative—the active gravitons, highlighted in red in Fig. 1.

In the first three families (P, NP1, and NP2) there are in total four propagators, D_{13} , D_{23} , D_{14} , D_{24} [defined in Eq. (18) of the Supplemental Material [110]], with active momenta although at most three can be present in the same diagram. In this Letter we include conservative contributions from regions with an even number of radiative gravitons (as was done at lower orders [32,37,47,56,99]). There are three regions: the potential region $PPPP$, the *tail* region composed of $RRPP$, $PPRR$, $RPRP$, and $PRPR$, and finally the *memory* region composed of $RPPR$ and $PRRP$ which is a novel 2SF feature—see Fig. 3. We use the names *tail* and *memory* as the corresponding regions include the tail effect [111,112,152] and memory effect respectively. The two RR regions, tail and memory, are not related by symmetries in contrast to their subregions.

For the nonplanar memory family NPM, the momenta of the propagators D_5^{NPM} , D_6^{NPM} , D_7^{NPM} [defined in Eq. (22) of the Supplemental Material [110]] are active. Only two of those three momenta are, however, linearly independent and give rise to two regions even in R — PP and RR —from which we include conservative contributions. The nonplanar RR region is of the memory type (and does not include any tail).

As is known from the 4PM and 5PM-1SF orders, there is an intricate interplay between the various regions in the form of a cancellation of poles in the dimensional regulator ϵ , leading to terms $\propto \log v$ in the final finite result. Here, at our 5PM-2SF order a crucial novel feature arises: the potential region carries not only an ϵ pole, but also terms that diverge at $\gamma = 3$, which stem from the integrals associated with the K3 geometry. Such a singularity was also observed in the PN velocity expansion of the potential region in [102,103], which the authors attributed to the $\gamma = -1$ pole. This novel $\gamma = 3$ divergence does *not* occur in the tail region, while the memory region carries these velocity divergencies as well. Hence, in order to obtain a

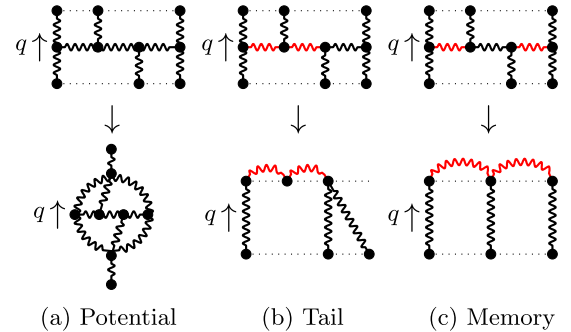


FIG. 3. Active gravitons that become radiative (red) determine the three regions for a planar top sector and yield the required boundary integrals. In the tail and memory region, the $i0^+$ prescription is crucial.

physically meaningful result we need a second intricate interplay between the regions to cancel divergencies.

Boundary integrals and the $i0^+$ prescription—The final step is to perform the boundary integrals in the discussed regions. It is here where the crucial $i0^+$ prescription for the graviton propagators makes its impact (next to assuming related propagator symmetries in the IBP reduction). We perform the required boundary integrals analytically at the static point $x = 1$. Depending on the candidate we use three strategies: (i) by transforming the integrals into a Feynman parameter representation, where direct integration in terms of hypergeometric functions in closed ϵ form is possible [24], (ii) by going to time [24] or frequency domain representations (see the Supplemental Material [110]), and (iii) by the “Feynman parameter integration through differential equations” strategy of [153] to compute the missing integrals numerically. From high-precision numerical data, we utilize the integer relation algorithm Partial Sum of Least Squares (PSLQ) [154] to reconstruct the analytic results.

The commonly used prescription for the conservative sector is Feynman $i0^+$ on the gravitons, together with taking the real and even part in v in the final result. The latter is equivalent to only picking an even number of R regions for the boundaries. This prescription led to a cancellation of ϵ poles between the regions and also matched the conservative results of the PN expansion in the low velocity limits at 3PM and 4PM orders. Yet, already at 5PM-1SF order [99] this prescription, while finite, fails to reproduce conservative terms, which are odd in v , stemming from the tail of tail [108].

At the 5PM-2SF order, the potential region is by construction not sensitive to the $i0^+$ prescription, and evaluating the tail region using Feynman propagators indeed cancels the ϵ pole coming from the potential region. Yet it leaves the novel singularity at $\gamma = 3$ intact that can only be cancelled from the memory region, which in turn should not introduce new ϵ poles. In fact, in the memory region there are only two boundary integrals appearing, depicted in Fig. 4. Intriguingly, *imposing* the cancellation of the $\gamma = 3$ singularity along with maintaining the ϵ -pole cancellation between the potential and tail regions determines their results up to a single undetermined coefficient c_M :

$$\begin{aligned} I_1^{(M)} &= \frac{1}{15(8\pi)^4 \epsilon} + \mathcal{O}(\epsilon^0), \\ I_2^{(M)} &= -\frac{5c_M}{6(8\pi)^4 \epsilon^2} + \mathcal{O}(\epsilon^{-1}). \end{aligned} \quad (8)$$

However, a calculation for the Feynman $i0^+$ prescription yields an ϵ^{-4} pole for $I_2^{(M)}$ which breaks finiteness of the final result through a ϵ^{-2} pole and a $(\gamma - 3)^{-15/2}$ divergence.

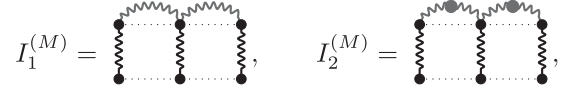


FIG. 4. The only two boundary integrals contributing in the memory region, here red dots indicate squared graviton propagators.

Hence, we need to redefine the conservative prescription in such a fashion that the following three conditions are met: (i) cancellation of dimensional regulator poles, (ii) cancellation of $(\gamma - 3)$ divergence, and (iii) time reflection symmetry. Such a prescription would provide Eq. (9) with a given c_M . We plot the contributions of the potential, tail and memory regions to the scattering angle for $c_M = 1$ in Fig. 5. Interestingly, we observe that evaluating $I_{1,2}^{(M)}$ with retarded propagators pointing toward the middle point provides such a prescription—that we term “ $\gamma - 3$ ”—and leads to the value $c_M = 1$, together with a finite impulse; see the Supplemental Material [110] for details. Yet we acknowledge that the physical motivation for this prescription is opaque and may not capture all conservative effects. We leave it to future work to elucidate this important question.

Function space—The final function space includes the $K3$ period $\varpi_{K3}(x)$, its derivative $\varpi'_{K3}(x)$, and up to 3 times iterated integrals:

$$\mathcal{I}[\varphi_1, \dots, \varphi_n; x] = \int_1^x dx' \varphi_1(x') \mathcal{I}[\varphi_2, \dots, \varphi_n; x'], \quad (9)$$

where the integration kernels $\varphi_i(x)$ are selected from the set

$$\left\{ \frac{1}{x}, \frac{1}{1+x^2}, \frac{1+x}{x(x-1)}, \frac{-1+x}{x(x+1)}, \frac{1-x^2}{x+x^2+x^3}, \frac{1-x^2}{x-x^2+x^3}, \frac{-1+x^2}{x+x^3}, \frac{(1+x^2)\varpi_{K3}(x)}{x}, \frac{1}{x r(x)\varpi_{K3}(x)} \right\} \quad (10)$$

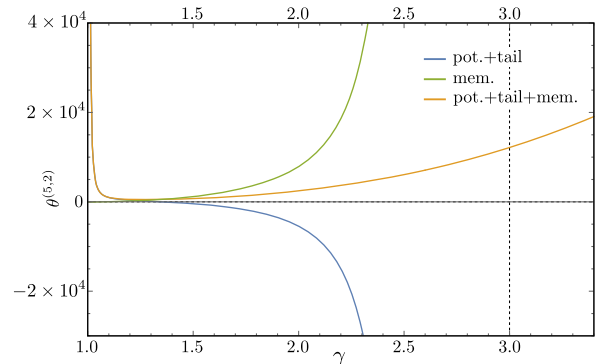


FIG. 5. The 5PM-2SF contribution to the scattering angle, $\theta_{\text{cons}}^{(5,2)}(\gamma)$: Potential and memory contributions both diverge for $\gamma \rightarrow 3$. These divergences cancel for the full result if one uses Eq. (9) irrespective of the value of c_M , which is set to 1 for this plot.

with the square root $r(x) = \sqrt{-1 + 34x^2 - x^4}$. The normalized $K3$ period is chosen to have an even velocity expansion $(1 + x^2)\varpi_{K3}(x) = 1 + (15v^2/32) + (351v^4/1024) + \mathcal{O}(v^6)$. Even though we encounter CY3 varieties in the DEs, the integrals associated with this geometry drop out after IBP reduction of the full integrand in D dimensions and do not contribute to the final result.

Results—The scattering angle θ_{cons} follows from the impulse using $|\Delta p_{i,\text{cons}}^\mu| = 2p_\infty \sin(\theta_{\text{cons}}/2)$. Here $p_\infty = m_1 m_2 \sqrt{\gamma^2 - 1}/E$, the total (conserved) energy is $E = M\sqrt{1 + 2\nu(\gamma - 1)}$, and the total mass is $M = m_1 + m_2$, with $\nu = m_1 m_2 / M^2$ the symmetric mass ratio. The scattering angle is given in PM expansion as

$$\theta_{\text{cons}} = \frac{E}{M} \sum_{m \geq 1} \sum_{s=0}^{\lfloor \frac{m-1}{2} \rfloor} \left(\frac{GM}{|b|} \right)^m \nu^s \theta_{\text{cons}}^{(m,s)}(\gamma), \quad (11)$$

where s counts the SF order. Our main result is the 5PM-2SF contribution

$$\theta_{\text{cons}}^{(5,2)} = \sum_{k=1}^{36} c_k(\gamma) f_k(\gamma), \quad (12)$$

where $f_k(\gamma)$ are the 36 linear combinations of iterated integrals discussed in the previous section and $c_k(\gamma)$ are polynomials in γ and $\sqrt{\gamma^2 - 1} = \gamma v$; see the Tables in the Supplemental Material [110]. We present all our analytical results in the accompanying Zenodo [155] submission, thereby tagging the regions and also providing explicit PN expansions up to v^{500} for numerical evaluations.

Checks—Our 5PM angle agrees in the low-velocity $v = \sqrt{\gamma^2 - 1}/\gamma \rightarrow 0$ limit with the scattering angle up to 4PN order [156–160]. Up to 5PN order with the memory contribution using (8) in square brackets we find

$$\begin{aligned} \theta_{\text{cons}}^{(5,2)} = & \frac{2}{5v^6} + \frac{53}{5v^4} + \left(\frac{45341}{360} - \frac{41\pi^2}{12} \right) \frac{1}{v^2} + \frac{46629199}{15120} \\ & + \left[\frac{64c_M}{5} - \frac{11456}{135} \right] - \frac{221597\pi^2}{720} + \frac{2816\log[2v]}{45} \\ & + \mathcal{O}(v^2). \end{aligned} \quad (13)$$

Our $\mathcal{O}(v^0)$ result confirms a conjecture on the π^2 5PN contributions being purely potential [160,161], while our rational tail contribution at this order agrees with the 5PN tail contribution of Ref. [162] based on Ref. [112]. Restricting to the potential region we agree with the very recent low-velocity expanded result for the scattering angle of Ref. [103]. Finally, the discontinuity of the scattering angle is related to the radiated energy at 1 order lower in the PM expansion:

$$\frac{\theta_{\text{cons}}(-\gamma_- + i\epsilon) - \theta_{\text{cons}}(-\gamma_- - i\epsilon)}{2i\pi} = GE \frac{\partial E_{\text{rad}}|_{\text{odd-in-v}}}{\partial L} \quad (14)$$

where $\gamma_- = \gamma - 1$, and the total angular momentum is $L = p_\infty |b|$ [37,57,58,106,163].

Conclusions—In this Letter, we have computed conservative contributions to the momentum impulse and scattering angle in the nonspinning gravitational two-body problem at 5PM (G^5) and 2SF order using WQFT. The complexity of this extraordinarily challenging computation was an order of magnitude larger than the 1SF conservative computation [99], and could only be tackled by greatly innovating our integration strategies. The emerging function space is richer than that of the 1SF case (which involved only polylogarithms up to weight 3) due to the emergence of a $K3$ period.

A crucial discovery of our Letter was a spurious divergence existing at $v/c = \sqrt{8}/3$, i.e., $\gamma = 3$, which needs to cancel between the potential and memory regions. Yet the present prescription for extracting conservative data using Feynman propagators, previously adopted at 5PM-1SF order [99], *fails* to provide a finite result. Enforcing cancellation leads to a novel γ -3 prescription that we propose. This situation calls for a better definition of the conservative sector in BH scattering, perhaps relying on the operator $\hat{N} = -i \log \hat{S}$ [81,90,164–168]. Clearly, we should next turn to the fully dissipative 5PM-2SF computation, as here there is an unambiguous understanding of the world-line formalism using retarded propagators throughout Refs. [49,109].

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Data availability—All explicit analytical results of this article are openly available in a computer readable format [155].

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