

Non-Abelian Fractional Chern Insulators and Competing States in Flat Moiré Bands

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Breakthrough experiments have recently realized fractional Chern insulators (FCIs) in moiré materials. However, all states observed are Abelian; the possible existence of more exotic non-Abelian FCIs remains controversial both experimentally and theoretically. Here, we investigate the competition between charge density wave (CDW) order, gapless composite fermion liquid (CFL), and non-Abelian Moore-Read states at half filling of a moiré band. Although ground-state (quasi)degeneracies and spectral flow are not sufficient for distinguishing between charge order and Moore-Read states, we find evidence using entanglement spectroscopy that both these states of matter can be realized with Coulomb interactions. By further analyzing the graviton excitations of Moore-Read states, we unveil that the ground states exhibit a mixed behavior of Pfaffian and anti-Pfaffian, despite the weak breaking of particle-hole symmetry. In a double twisted bilayer graphene model, transitions between these phases can be driven by the coupling strength between the layers: at weak coupling there is a CFL phase and at strong coupling a CDW order emerges. Remarkably, however, there is compelling evidence for a non-Abelian Moore-Read FCI phase at intermediate coupling.

DOI: 10.1103/43nq-ntqm

Introduction—Moiré materials [1–5] have provided a long-sought platform for realizing fractional Chern insulators (FCIs) [6]: recently theoretically predicted and subsequently experimentally observed both in graphene based heterostructures [7–13] and twisted transition metal dichalcogenide bilayers [14–19]. While remarkable in not requiring any external magnetic field [12,16–19] and opening up the prospects of high temperature anyon physics, all states observed thus far Abelian. Non-Abelian states, which would entail an even richer phenomenology and prospects for applications [20], remain elusive in experiments and their theoretical identification is subject to intense discussion.

Even in FCI toy models, where Abelian FCIs are ubiquitous [21–30], non-Abelian states are notoriously hard to stabilize. In fact, most studies of non-Abelian FCIs feature rather artificial multibody interactions [31–36], strategically chosen to mimic their multibody pseudopotential

parent Hamiltonians that may be derived from the operator product expansion of the pertinent conformal field theories [37–39]. With more realistic two-body interactions it is much harder to stabilize non-Abelian FCIs and only a few models are known to harbor such phases [40,41]. In the context of Landau levels there is evidence of the paradigmatic non-Abelian fractional quantum Hall states, namely the Moore-Read (MR) states [37], in a narrow range of parameter space that fortunately includes the second Landau level [42,43].

While clearly extraordinarily challenging to realize in a lattice setting and without a strong magnetic field, very recent experiments claiming to show signs of a fractional topological insulator in twisted semiconductor tMoTe₂ [44] have reignited interest in non-Abelian moiré based FCIs [45–53]. The evidence for non-Abelian FCIs in these systems is intriguing: quantum numbers and quasidegeneracies match those of the MR states [45,49], and band structures show similarities with the second Landau level [46]. However, charge density waves (CDWs) [54,55] and composite Fermi liquids (CFLs) [56,57] are fierce competitors to the non-Abelian FCIs. In fact, for some sample geometries all these may have the same quasidegeneracies and (momentum) quantum numbers. In particular, the CDWs may have the same ground-state quasidegeneracies and occur in the same momentum sectors for any sample size, making them very hard to distinguish from the sought-after MR FCIs. This is in particular pressing given the relatively small system sizes available to numerical

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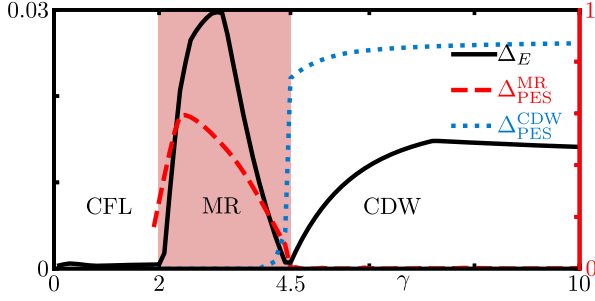


FIG. 1. Competition and phase transition between Moore-Read states and CDWs. Particle-cut entanglement spectrum gap Δ_{PES} for quasihole counting of the CDW (blue dotted line) and MR states (red dashed line), along with the energy difference between the MR ground states and the first excitation state (black solid line) Δ_E , as a function of the coupling strength γ (eV), respectively. Here we use a system size with 26-site clusters at half filling, and $N_A = 4$ for subsystem A in particle-cut entanglement spectrum calculations, and a bare Coulomb interaction.

simulation for which charge order may be less developed due to finite-size effects and/or in the case when the microscopic nature of the competing local order is not *a priori* obvious. In this case a more careful investigation of entanglement spectrum is a powerful method for distinguishing the competing phases [28].

Here we use entanglement spectroscopy to demonstrate that non-Abelian MR FCIs can indeed be realized in moiré systems. Our investigation also reveals that there is a fierce competition between all three aforementioned phases of matter and novel phase transitions may occur without changes in ground-states degeneracies. This is shown in Fig. 1: in a simple model of a double twisted bilayer graphene (TBG) [46], changing the interlayer coupling tunes the system from MR FCIs to low-entanglement states that exhibit the entanglement properties expected for CDWs. Both these phases exhibit the salient energetic characteristics of the MR states, yet are separated by an energy gap closing and with two telltale entanglement gaps opening and closing, respectively. Below we detail the setup followed by a description of evidence for the CDW (Fig. 2) and MR (Fig. 3) phases (as well as the transition between them in the End Matter). We also briefly comment on CFLs and put our results into a broader context.

Setup—We consider a valley and spin polarized Hamiltonian describing double sheets of TBG of the form [46]

$$H(\mathbf{r}) = \begin{pmatrix} u_1 & D_1^\dagger(\mathbf{r}) & 0 & 0 \\ D_1(\mathbf{r}) & u_2 & \gamma I & 0 \\ 0 & \gamma I & u_3 & D_2^\dagger(\mathbf{r}) \\ 0 & 0 & D_2(\mathbf{r}) & u_4 \end{pmatrix}. \quad (1)$$

Here, the two 2×2 diagonal blocks represent the two single-sheet TBGs with

$$D_i(\mathbf{r}) = \begin{pmatrix} -2ik_\theta^{-1}\bar{\partial} & \alpha U(\mathbf{r}) \\ \alpha U(-\mathbf{r}) & -2ik_\theta^{-1}\bar{\partial} \end{pmatrix}. \quad (2)$$

γ controls the interlayer coupling strength between the twisted bilayers, u_j is the gate potential, $k_\theta = 4\pi/3a_M$ with a_M the moiré length, $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$, and $U(\mathbf{r}) = e^{-iq_1\cdot\mathbf{r}} + e^{iq_2\cdot\mathbf{r}} + e^{-iq_3\cdot\mathbf{r}}$. Setting $\gamma = 0$ leads to two decoupled copies of chiral twisted bilayer graphene (cTBG) [9,58]. For the results present in the main text, we use the first magic angle for the cTBG ($\alpha \approx 0.586$) and $u_1 = 50$ meV, $u_2 = u_3 = u_4 = 0$ to separate the middle flat bands [see the band dispersion in Supplemental Material (SM) [59]].

It has been recently suggested that coupling multiple TBG layers might enable the flat band above the charge-neutrality point to mimic higher Landau level (LL) physics [46], which then offers a potential for realizing non-Abelian topological states at fractional band filling [42,43]. For this purpose, we now consider interactions in this band. To begin, we first assess the tendency toward valley polarization by projecting the interaction into the flat band above the charge-neutrality point, including both K and K' valleys. In the parameter regime of interest, we indeed find valley polarization across different system sizes at half band filling, where the low-lying energy states fully occupy at either K or K' valley (see SM [59]), which justifies focusing only on the flat band above the charge-neutrality point of the K valley. We also neglect the contribution of the band dispersion, which is well motivated in the strong interaction limit. While in realistic systems effects of band mixing may be substantial [60–62], we are here mainly interested in the fundamental question of realizing novel states within a given flat band. Throughout, we use the bare Coulomb interaction.

We focus on the Moore-Read states, a well-known non-Abelian topological phase that emerges at $1/2$ band filling, first proposed by Moore and Read [37]. Their momenta can be identified by considering the thin-torus (also known as root) configurations. For MR states, these configurations have two distinct types in Fock space, $11001100\cdots$ and $10101010\cdots$, respectively [63–66], which can be derived by considering the thin-torus limit of a three-body parent repulsion leading to the selection rule that there are no more than two particles on any four adjacent sites in the one-dimensional representation. Combining their center-of-mass translations, two different ground-state degeneracies appear: a sixfold degeneracy for an even number of electrons and a twofold degeneracy for an odd number of electrons (since only the $10101010\cdots$ states are compatible with the periodic boundary conditions). We emphasize that the competitors of the MR states may occur in the same momentum sectors.

Charge density wave evidence—We first consider strong interlayer coupling in the double TBGs model. In this regime,

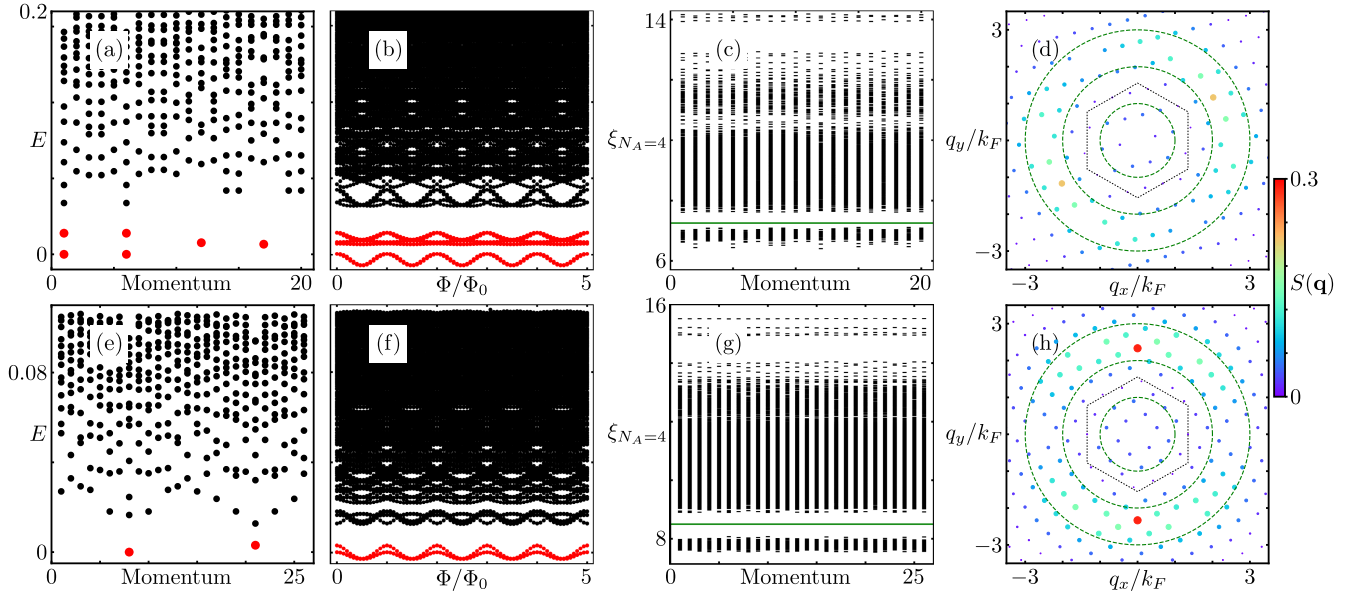


FIG. 2. Evidence for charge density waves. Panels (a)–(d) are the low-lying energy spectrum, spectral flow, particle-cut entanglement spectrum with $N_A = 4$, and the structure factor for the 20-site-cluster system at half filling (even number of electrons). Panels (e)–(h) present the corresponding results for the 26-site-cluster system at half filling (odd number of electrons). In the energy spectrum, the red dots indicate the ground states satisfying the center-of-mass momentum of Moore-Read states. In the entanglement spectrum, the number of states below the first entanglement gap (the green solid line) is 420 for 20-site cluster and 1430 for 26-site cluster, which adheres to the counting rule for CDWs. In the structure factor, the solid line indicates the moiré BZ and k_F represent the Fermi wave vector. For all plots, we employ $\gamma = 5.5$ eV and a bare Coulomb interaction.

we find two distinct types of ground-state degeneracies, as exemplified in Figs. 2(a) and 2(e) for $\gamma = 5.5$ eV. In our finite-size samples, both the fold of degeneracy and the ground-state momenta match the predictions by the aforementioned exclusion rule of the MR states. By applying an external magnetic flux, we further confirm the presence of an energy gap between the ground states and the excited states, below which the sixfold (even number of electrons) and twofold (odd number of electrons) quasidegenerate ground states flow into each other [see Figs. 2(b) and 2(d)]. The total quantum metric $\mathcal{G} \equiv (1/2\pi) \int_{\text{MBZ}} \text{tr} g(\mathbf{k}) d^2\mathbf{k}$ of the flat band, which is commonly used as an indicator of the closeness to a Landau level, indeed approaches 3—the value of the first Landau level (1LL), with increasing interlayer coupling [$g(\mathbf{k})$ is the quantum metric]. These findings, aligned with recent numerical observation in tMoTe₂ [45,47–50], seemingly suggest the existence of non-Abelian MR states with strong interlayer coupling.

Surprisingly, upon a particle-cut entanglement spectrum (PES) analysis [28,67,68] (see the definition in SM [59]), we find that the ground states instead exhibit characteristics of the CDW. In Figs. 2(c) and 2(g), a narrow band is discernible at the bottom of the PES, separated from the higher entanglement levels by a visible gap. Remarkably, in both cases, the number of states below the gap is very different from the quasihole counting for non-Abelian MR states: there are many fewer eigenvalues below the gap. Instead, the observed counting matches a CDW order,

where the number of states below the gap is just the way of putting N_A particles into N_e specific positions [69]. Moreover, a visible gap at the counting of MR states is lacking, which provides corroborating evidence against such states. Therefore, even though system-size-dependent ground-state degeneracies and a gapped spectral flow are fulfilled, the presence of MR states is still not guaranteed due to the competing order.

Motivated by the PES findings, we further employ calculations of the ground-state structure factor $S(\mathbf{q})$, which is a typical method for validating the charge order [see Figs. 2(d) and 2(h)]. Indeed, we find pronounced peaks in $S(\mathbf{q})$ for both system sizes, in particular the one with odd number of electrons (26-site cluster). This supports the existence of charge order at strong interlayer coupling. Similar peaks in $S(\mathbf{q})$ together with the corresponding PES gap for CDW ground states persist for larger systems, as shown in SM [59].

Moore-Read states—Moving to intermediate coupling strength, we find compelling evidence for the true non-Abelian MR states. As shown in Figs. 3(a)–3(d) for $\gamma = 3.5$ eV, the low-lying energy spectrum for various system sizes exhibits correct ground-state degeneracies and center-of-mass momenta, as predicted by the exclusion rule for the MR states. Importantly, these ground states remain well isolated from the excited states by a visible gap, which persists as well upon inserting flux through the handles of the torus (see SM [59]).

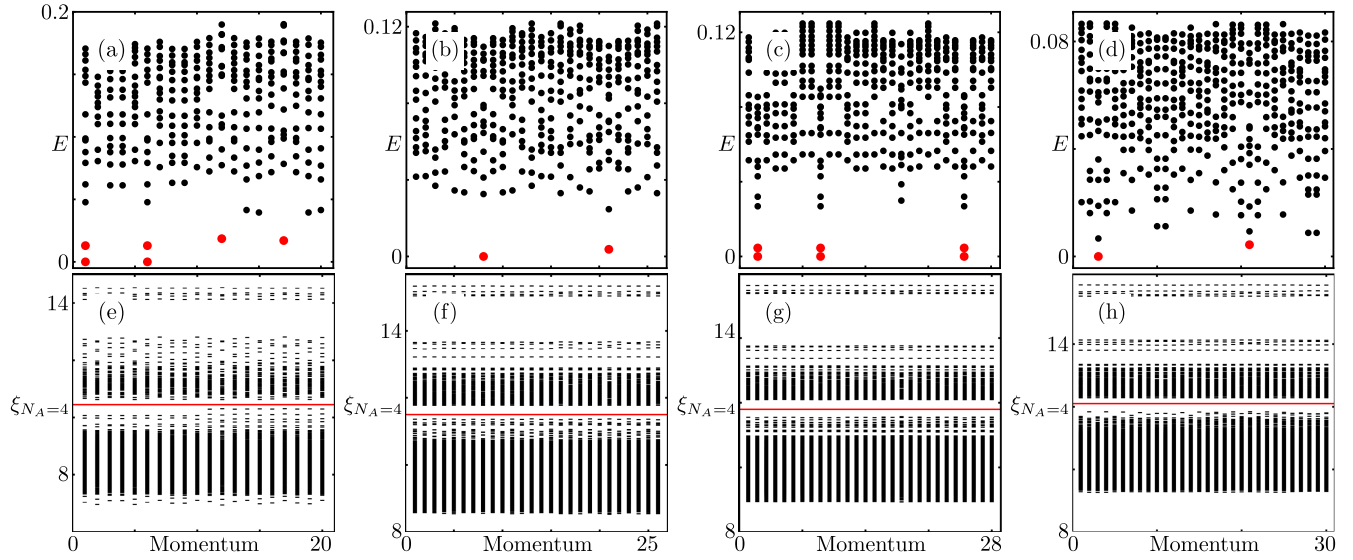


FIG. 3. Evidence for Moore-Read states. Panels (a)–(d) are the low-lying energy spectrum for system with 20-, 26-, 28-, and 30-site clusters, respectively. Here, the red dots represent the Moore-Read states. Panels (e)–(h) are their corresponding particle-cut entanglement spectrum for $N_A = 4$. The number of states below the first entanglement gap (the red solid line) is 3965, 13338, 18571, and 25185, matching the quasihole counting for Moore-Read states. For all plots, we use $\gamma = 3.5$ eV.

Next, to eliminate the possibility of charge orders, we again examine the PES of the sixfold ground states (for even number of electrons) and the twofold ground states (for odd number of electrons). In Figs. 3(e)–3(h), a clear entanglement gap consistently appears between the bottom wide band and the middle narrower band. Remarkably, the number of states below the gap precisely matches the quasihole counting of the MR states. The persistence of the PES gap across various system sizes and different N_A serves as a hallmark of the robustness of the MR states in moiré minibands. In fact, such strong evidence for the MR states does not even exist in the 1LL.

Now we conclude that the MR states only appear at intermediate coupling strength instead of at (infinitely) large γ , which was often thought as the optimal point to capture the 1LL physics as $\mathcal{G} = 3$. We attribute this discrepancy to the role of quantum metric fluctuation in determining the underlying physics. The 1LL has both $\mathcal{G} = 3$ and uniform quantum metric. By contrast, a Chern band cannot possess both. A compromise between approaching $\mathcal{G} = 3$ and minimizing quantum metric fluctuations must be made in a Chern band to realize the 1LL physics. In line with this, we indeed find both the minimal metric fluctuation and $\mathcal{G}$ not too far from 3 at intermediate coupling, where the MR states are favored (see SM [59]). We note that although the minimum of quantum metric fluctuation lies within the MR state regime, it is difficult to establish the correspondence between the phase transition point—identified via the energy spectrum or PES—and a strict threshold in quantum metric and its fluctuations, as these quantities are purely single-particle indicators and

cannot fully determine the many-body physics, which also depends on interaction details.

After confirming the existence of MR states in the moiré miniband, a natural question arises: are these non-Abelian MR states Pfaffian or anti-Pfaffian? These two possibilities are particle-hole conjugate of each other and share the same quasihole excitation statistics, making it necessary to discuss their distinctions in the context of moiré systems. In LLs, the presence of particle-hole symmetry gives rise to cat states of Pfaffian and anti-Pfaffian. Then, a total of 12 ground states are expected in the thermodynamic limit. In the case of the double TBG model, the absence of magnetic translation symmetry leads to particle-hole symmetry breaking. Under the particle-hole transformation $c_{\mathbf{k}}^\dagger \rightarrow d_{\mathbf{k}}$, a single-hole kinetic energy term $H_{\text{single-hole}} = \sum_{\mathbf{k}} E_h(\mathbf{k}) n(\mathbf{k})$ with $E_h(\mathbf{k}) = \sum_{\mathbf{k}' \in \text{MBZ}} (V_{\mathbf{k}\mathbf{k}'} + V_{\mathbf{k}\mathbf{k}'\mathbf{k}'} - V_{\mathbf{k}\mathbf{k}'\mathbf{k}'} - V_{\mathbf{k}'\mathbf{k}\mathbf{k}'})$ appears. In general, the hole dispersion $E_h(\mathbf{k})$ varies with $\mathbf{k}$ for moiré flatbands (see SM [59], which includes Ref. [70]). Furthermore, the occupation number of ground states is not constant as well (which usually exhibits a center-of-mass dependence). Together, these factors suggest that the single-hole kinetic term leads to an energy difference between Pfaffian and anti-Pfaffian states; thus, one of them may be selected out. However, as nonconstant $E_h(\mathbf{k})$ (depending on the fluctuation of quantum metric [71,72]) in this case is nearly uniformly distributed in moiré Brillouin zone (MBZ) (as shown in SM [59]), we expect that the effect of $H_{\text{single-hole}}$ becomes significant only in large systems with sufficiently many accessible $\mathbf{k}$ points [73]. This makes the observation of state selection very challenging in small systems within the capability of numerical simulation.

The mixed nature of Pfaffian and anti-Pfaffian states can be further verified by studying the chirality of their graviton excitations [74–77], which has been recently generalized to moiré systems [78–80]. Using graviton operators respecting the crystalline symmetry of the moiré lattice, we found that for both even and odd number of electrons, the corresponding graviton spectra carry nearly equal weights of positive and negative chiralities, indicating a superposition of Pfaffian and anti-Pfaffian states (see more details in SM [59]). Such a phenomenon has also been observed in the first LL, where exact particle-hole symmetry enforces identical graviton spectra for both chiralities [81]. In the present case, particle-hole symmetry is only weakly broken, consistent with the nearly uniform single-hole energy. To suppress one of the two chiralities, an alternative approach is to include three-body interactions (arising from band mixing effects), which explicitly break particle-hole symmetry. Then, one of Pfaffian or anti-Pfaffian states may be selected out, depending on the sign of such interactions. A systematic exploration of such a phenomenon in non-Abelian moiré FCIs warrants future verification.

Composite Fermi liquids—A third competing phase in these systems is comprised by gapless CFLs. It originally occurs in the lowest LL at $1/2$ or $1/4$ band filling [82]. Within our system, this phenomenon arises at weak coupling strength $\gamma \rightarrow 0$, where the full system decouples into two independent chiral TBGs, each resembling the characteristic behaviors of the original cTBG, which has been verified that its flat band captures the lowest LL physics [9]. This then offers a convincing explanation for the emergence of gapless CFLs. We confirm this by the low-lying energy spectrum, the electron occupation of the CFL ground states, and the static structure factor, which aligns with the recent reports of CFLs in tMoTe₂ [56,57]. In particular, the structure factor becomes singular around $|\mathbf{q}| = 2k_F$, due to the backscattering process of composite fermions at the Fermi surface of $|\mathbf{k}| = k_F$ (see SM [59], which includes Ref. [83]).

Discussion—In this Letter we have strongly corroborated recent claims that non-Abelian MR states may be realized in moiré systems. At the same time our detailed entanglement spectroscopy together with the structure factor highlights the fact that ground-state degeneracies and momentum quantum numbers alone are insufficient for establishing the presence of this exotic phase of matter. In particular our analysis demonstrates that competing CDW order can be the underlying reason for states that at first glance appear to be MR states.

We then uncover genuine MR states in the intermediate coupling strength regime, where these states isolated from excited states consistently emerge across various system sizes and are accompanied by the correct quasihole excitations. Moreover, the Pfaffian and anti-Pfaffian nature of these states is verified by their geometric excitations, in

which the nearly identical spectral weights suggest a mixed Pfaffian and anti-Pfaffian character, due to the weak breaking of particle-hole symmetry.

By tuning the coupling strength between two sheets of TBGs, we further identify the possibility for a direct phase transition between the charge density wave and the Moore-Read states, where the entanglement spectrum exhibits a consistent behavior with the energy gap closing and reopening.

Our Letter serves as a prototypical example of the interplay between Moore-Read states and competing states in moiré systems, offering compelling avenues for future research. Firstly, while non-Abelian topological states have long been theorized, their (likely) realization has been limited to systems with strong magnetic field and low temperatures, such as LL systems [20]. Our findings extend the possibility of observing these states in a more favorable setting in moiré materials without the need for external fields, which has recently been utilized for realizing Abelian topological states at zero magnetic field and relatively high temperatures. From a theoretical standpoint, although the MR states represent the simplest manifestation of non-Abelian topological states, our Letter can be readily adapted to more complicated scenarios, such as the Read-Rezayi states at fractional fillings $2/5$ and $3/5$ [39].

Secondly, while evidence of charge density wave is observed both from entanglement spectrum and structure factor, the prominent peaks outside the MBZ suggest the potential existence of novel phases of matter. Finally, the transition between the CFL and MR states warrants further investigation.

Acknowledgments—We acknowledge useful discussions and related collaborations with Ahmed Abouelkomsan, Liang Fu, Aidan Reddy and Donna Sheng. H. L. and E. J. B. were supported by the Swedish Research Council (VR, Grant No. 2018-00313), the Wallenberg Academy Fellows program of the Knut and Alice Wallenberg Foundation (2018.0460), and the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine. Z. L. was supported by the National Natural Science Foundation of China (Grants No. 12350403 and No. 12374149).

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End Matter

Transition between CDWs and Moore-Read states— Having provided evidence for both the CDW and the MR states, we now revisit and provide more details on the phase transition between them by tuning γ as was already shown in Fig. 1. Within the range $\gamma \in (0, 10]$, the two layers of TBGs remain coupled and the targeted flat band remains isolated. We identify the phase transition between the CDW order and the MR states through two criteria. One is the many-body energy spectrum gap Δ_E between the degenerated ground states and excited states. Here, we choose odd number of electrons to clearly show the phase transition. (In even-electron case, the gap is less conclusive as only two states evolve to CDW orders.) It then reveals a phase transition around

$\gamma \approx 4.5$. Our second tool is the PES gap at the CDW and MR countings. Here, the PES gap is defined as $\Delta_{\text{PES}} = \xi_{m+1} - \xi_m$, where ξ_m represents the m th PES level in ascending order, with m the counting for either the CDW or the MR states. We find Δ_{PES} shows a consistent behavior as Δ_E . Before the phase transition, the PES gap for the CDW is roughly 0, while it is significant for the MR states (the largest $\Delta_{\text{PES}}^{\text{MR}} \approx 0.6$), which serves as a supplementary confirmation of MR states. Upon crossing the phase transition point, we observe a rapid decrease of $\Delta_{\text{PES}}^{\text{MR}}$ to 0, while $\Delta_{\text{PES}}^{\text{CDW}}$ undergoes a pronounced jump, attaining a finite value. This serves as compelling evidence for the phase transition from MR states to the CDW.