

Parent Hamiltonians for Stabilizer Quantum Many-Body Scars

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(Received 20 January 2026; accepted 27 May 2026; published 15 June 2026)

Quantum many-body scars (QMBS) have attracted considerable interest due to their role in weak ergodicity breaking in many-body systems. We present a general construction that embeds stabilizer states as QMBS of local Hamiltonians. The method relies on a notion of factorizability of Pauli strings on a lattice, which is used to convert stabilizer elements into local, few-body operators that annihilate the stabilizer state. This enables the systematic construction of parent Hamiltonians with zero-energy stabilizer QMBS typically near the middle of the spectrum. The method reproduces several known results in a unified framework, including recent examples of volume-law entangled QMBS, such as the “rainbow” QMBS and the entangled antipodal Bell pair state. We also apply the framework to construct examples of stabilizer QMBS with a more complex entanglement structure, such as the cluster state, the toric code state, and a volume-law entangled state we dub the antipodal toric code state. Exact diagonalization confirms our results and reveal the stabilizer states as exact eigenstates of their parent Hamiltonian.

DOI: [10.1103/3vk5-483c](https://doi.org/10.1103/3vk5-483c)

Introduction—Many-body Hamiltonians with generic local interactions have energy eigenstates that are “thermal” [1,2]. Roughly speaking, this means that near the middle of the spectrum the eigenstates are qualitatively similar to random states, e.g., in terms of expectation values of local observables [3,4], their entanglement entropy [5], or their nonstabilizerness (also known as magic) [6]. Quantum many-body scars (QMBS) have attracted significant attention as anomalous *nonthermal* eigenstates in an otherwise thermal spectrum [7–9]. Such eigenstates can inhibit thermalization, which has been observed experimentally [10–16] and may have practical relevance for quantum information processing [17–19]. While early examples of QMBS were identified in constrained models such as the PXP chain [20–23] and projector embedded Hamiltonians [24,25], subsequent work has revealed a growing zoo of models with QMBS [26–34].

In parallel, stabilizer states occupy a central place in quantum information theory as efficiently characterizable quantum states that can be generated with Clifford circuits [35]. These include Bell states, Greenberger-Horne-Zeilinger states [36], cluster states [37], the toric code states [38], and other error-correcting code states [39]. Although stabilizer states can be highly entangled, they are nonthermal since, by definition, they have zero magic in

contrast to thermal states, which have magic that scales with the system size [6].

It is well known that any stabilizer state is an eigenstate of its “stabilizer Hamiltonian,” constructed as a linear combination of stabilizer group elements. However, such stabilizer Hamiltonians often involve multiqubit interactions or nonlocal interactions that are difficult to realize in the laboratory. Whether there is a general construction that embeds the same stabilizer states as eigenstates of more physically plausible Hamiltonians with few-body local interactions remains an interesting open question.

In this Letter, we introduce such a construction, based on a notion of (k, ℓ, b) factorizability of Pauli strings. Whenever a stabilizer element admits a $(2, \ell, b)$ factorization, it yields a corresponding ℓ -local, b -body operator that annihilates the stabilizer state. A linear combination all such contributions produces an ℓ -local, b -body parent Hamiltonian whose stabilizer state lies at zero energy, which is typically near the middle of the spectrum. This mechanism thereby provides a systematic route to construct Hamiltonian models with exact stabilizer QMBS.

Our approach can be used to understand several known examples in a unified framework, including the stabilizer QMBS of Ref. [34] and recent examples of volume-law entangled QMBS [40–43]. Going beyond known results, we construct a number of models with stabilizer QMBS, including the cluster state, toric code states, and models with volume-law entangled stabilizer QMBS. In these examples, the corresponding parent Hamiltonian is typically local and two-body, even when the underlying stabilizers involve three-body operators (for the cluster state) or four-body operators (for the toric code).

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Pauli strings on a lattice—We consider a system of N qubits arranged on a D -dimensional lattice. A *Pauli string* is an N -qubit operator of the form $\hat{P} = \theta \bigotimes_{n=0}^{N-1} \hat{\sigma}_n^{\mu_n}$, where $\theta \in \{\pm 1\}$ and $\hat{\sigma}_n^{\mu_n}$ are the single-qubit Pauli operators acting on qubit n , with $\mu_n \in \{I, X, Y, Z\}$. A Pauli string is b -body if it acts nontrivially on at most b qubits, while a linear combination of Pauli strings (e.g., a Hamiltonian) is called b -body if each term is b -body. We also say that a Pauli string is ℓ -local if all qubits on which it acts nontrivially are contained in a contiguous subset of at most ℓ neighboring qubits on the lattice. Any linear combination of ℓ -local Pauli strings is also ℓ -local.

Stabilizer states—A pure state $|\Psi\rangle$ is *stabilized* by the Pauli string $\hat{P}$ if $\hat{P}|\Psi\rangle = |\Psi\rangle$. If $|\Psi\rangle$ is stabilized by N independent and commuting Pauli strings $\{\hat{P}_n\}_{n=0}^{N-1}$, it is called a *stabilizer state*. These N operators generate the *stabilizer group* of $|\Psi\rangle$, denoted $\mathcal{S} = \langle \{\hat{P}_n\}_{n=0}^{N-1} \rangle$. Given a stabilizer state $|\Psi\rangle$ with stabilizer group $\mathcal{S}$, a *stabilizer Hamiltonian* is any linear combination of stabilizer elements $\hat{H}_{\text{stab}} = \sum_{\hat{P} \in \mathcal{S}} c_{\hat{P}} \hat{P}$, where $c_{\hat{P}} \in \mathbb{R}$. The stabilizer state is an energy eigenstate $\hat{H}_{\text{stab}}|\Psi\rangle = E|\Psi\rangle$, with the energy eigenvalue $E = \sum_{\hat{P} \in \mathcal{S}} c_{\hat{P}}$, and is a ground state if every coupling constant is chosen so that $c_{\hat{P}} \leq 0$. Well-known examples of stabilizer Hamiltonians are the cluster state Hamiltonian [44] and the toric code Hamiltonian [38]. If every element $\hat{P} \in \mathcal{S}$ of a stabilizer group is at least ℓ -local and b -body, then the stabilizer Hamiltonian is also at least ℓ -local and b -body. Since two-local, two-body interactions are the most common in nature, stabilizer Hamiltonians can be difficult to realize in practice if there are no two-local, two-body stabilizer elements (for instance, the toric code stabilizers elements are at least four-body).

Parent Hamiltonian construction—We now introduce a notion of factorizability of Pauli strings on a lattice, which allows us to construct local, few-body parent Hamiltonians for stabilizer states, even if the stabilizer elements are highly nonlocal or multibody.

Definition 1—A Pauli string $\hat{P}$ is (k, ℓ, b) factorizable if it can be factorized as a product of k nontrivial Pauli strings that are ℓ -local and b -body, i.e.,

$$\hat{P} = \hat{P}^{(1)} \hat{P}^{(2)} \dots \hat{P}^{(k)}, \quad (1)$$

where each $\hat{P}^{(i)} \neq \mathbb{I}^{\otimes N}$ is ℓ -local and b -body.

In this Letter, we focus exclusively on $k = 2$. In general, a Pauli string can be factorized in many different ways. For example, on a $D = 1$ -dimensional chain, the Pauli string $\hat{P} = \hat{\sigma}_n^Z \hat{\sigma}_{n+1}^Z$ is $(2, 1, 1)$ factorizable, since it can be factorized as $\hat{P} = \hat{P}^{(1)} \hat{P}^{(2)}$, where $\hat{P}^{(1)} = \hat{\sigma}_n^Z$ and $\hat{P}^{(2)} = \hat{\sigma}_{n+1}^Z$ are both one-local and one-body. However, it is also $(2, 2, 2)$ factorizable as $\hat{P} = \hat{P}^{(1)} \hat{P}^{(2)}$, where $\hat{P}^{(1)} = -\hat{\sigma}_n^X \hat{\sigma}_{n+1}^X$ and $\hat{P}^{(2)} = \hat{\sigma}_n^Y \hat{\sigma}_{n+1}^Y$ are both two-local and

two-body. Based on this factorizability property, we present the following parent Hamiltonian construction.

Result 1—Given a stabilizer state $|\Psi\rangle$ with the stabilizer group $\mathcal{S}$, let $\{\hat{P}_\alpha\}_\alpha \subset \mathcal{S}$ be the subset of $(2, \ell, b)$ -factorizable stabilizer group elements $\hat{P}_\alpha = \hat{P}_\alpha^{(1)} \hat{P}_\alpha^{(2)}$, which we enumerate with the index α . Then $|\Psi\rangle$ is a zero-energy eigenstate of the ℓ -local, b -body parent Hamiltonian:

$$\hat{H} = \sum_\alpha J_\alpha (\hat{P}_\alpha^{(1)} - \hat{P}_\alpha^{(2)}), \quad (2)$$

where $J_\alpha \in \mathbb{R}$ are arbitrary coupling constants.

The proof is straightforward: Since $\hat{P}_\alpha = \hat{P}_\alpha^{(1)} \hat{P}_\alpha^{(2)}$ is a $(2, \ell, b)$ -factorizable stabilizer element and $|\Psi\rangle$ is a stabilizer state, we have $\hat{P}_\alpha^{(1)} \hat{P}_\alpha^{(2)} |\Psi\rangle = |\Psi\rangle$, where $\hat{P}_\alpha^{(1)}$ and $\hat{P}_\alpha^{(2)}$ are both ℓ -local and b -body. Multiplying both sides by $\hat{P}_\alpha^{(1)}$ and using $\hat{P}_\alpha^{(1)} \hat{P}_\alpha^{(1)} = \mathbb{I}^{\otimes N}$ (since any Pauli string squares to the identity operator) gives $\hat{P}_\alpha^{(2)} |\Psi\rangle = \hat{P}_\alpha^{(1)} |\Psi\rangle$, from which it follows that $(\hat{P}_\alpha^{(1)} - \hat{P}_\alpha^{(2)}) |\Psi\rangle = 0$ for all α . The state $|\Psi\rangle$ is, therefore, annihilated by any linear combination $\hat{H} = \sum_\alpha J_\alpha (\hat{P}_\alpha^{(1)} - \hat{P}_\alpha^{(2)})$ of such terms. ■

Since the parent Hamiltonian Eq. (2) is a linear combination of nontrivial Pauli strings, it is traceless and generally has both positive and negative eigenvalues. The zero-energy stabilizer eigenstate is typically near the middle of the spectrum, i.e., a QMBS. We emphasize that the Hamiltonian is ℓ -local and b -body, even if the stabilizer elements are more nonlocal or multibody.

A large class of parent Hamiltonians for simple stabilizer states, such as product states or products of Bell pairs, can be easily constructed using Result 1, as shown in Supplemental Material [45]. Here, we provide some examples for more complex stabilizer states.

Example 1: Cluster state ($D = 1$)—Consider the cluster state $|\Psi_\theta^{\text{cluster}}\rangle$ on a $D = 1$ -dimensional chain of N qubits, defined by the stabilizer group $\mathcal{S}_\theta^{\text{cluster}} = \langle \{\theta_n \hat{\sigma}_{n-1}^X \hat{\sigma}_n^Z \hat{\sigma}_{n+1}^X\}_{n=0}^{N-1} \rangle$. Here, $\theta_n \in \{\pm 1\}$ and we assume periodic boundary conditions $n + N \equiv n$. Each of the N independent stabilizer generators $\theta_n \hat{\sigma}_{n-1}^X \hat{\sigma}_n^Z \hat{\sigma}_{n+1}^X$ [illustrated in the inset in Fig. 1(a)] is three-local and three-body but is $(2, 2, 2)$ factorizable in two ways:

$$\theta_n \hat{\sigma}_{n-1}^X \hat{\sigma}_n^Z \hat{\sigma}_{n+1}^X = \begin{cases} \theta_n (\hat{\sigma}_{n-1}^X) (\hat{\sigma}_n^Z \hat{\sigma}_{n+1}^X), \\ \theta_n (\hat{\sigma}_{n-1}^X \hat{\sigma}_n^Z) (\hat{\sigma}_{n+1}^X). \end{cases} \quad (3)$$

Also, the product of nearest-neighbor generators is $(2, 2, 2)$ factorizable:

$$(\theta_n \hat{\sigma}_{n-1}^X \hat{\sigma}_n^Z \hat{\sigma}_{n+1}^X) (\theta_{n+1} \hat{\sigma}_n^X \hat{\sigma}_{n+1}^Z \hat{\sigma}_{n+2}^X) = \theta_n \theta_{n+1} (\hat{\sigma}_{n-1}^X \hat{\sigma}_n^Y) (\hat{\sigma}_{n+1}^Y \hat{\sigma}_{n+2}^X), \quad (4)$$

as is the product of next-nearest-neighbor generators:

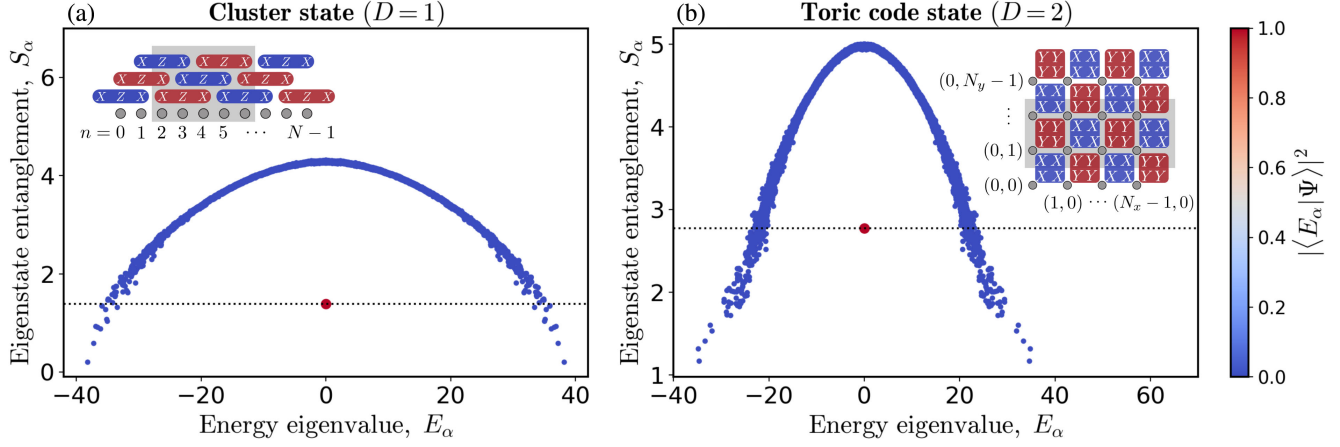


FIG. 1. (a) Inset: the generators of the cluster state on the $D = 1$ chain. We compute the half-chain entanglement entropy (i.e., the von Neumann entropy for the subsystem highlighted in gray) of each eigenstate of the parent Hamiltonian in Eq. (6). The cluster state (red marker) is a zero-energy QMBS (the color of each marker denotes the overlap of the corresponding eigenstate with the cluster state). The dashed black line shows the entanglement entropy $S^{\text{cluster}} = 2 \ln 2$ of the cluster state. [Parameters: $N = 14$, $\vec{\theta} = (-1, -1, \dots, -1)$ and ω_n, ω'_n, J_n , and J'_n are, for each n , generated uniformly at random from the range $[0.7, 1.3]$.] (b) Inset: the vertex (blue) and plaquette (red) operators on the square lattice. We compute the half-system entanglement entropy (i.e., the von Neumann entropy of the subsystem highlighted in gray) of each eigenstate of the parent Hamiltonian in Eq. (6). The toric code state (red marker) is a zero-energy QMBS. Its entanglement entropy $S^{\text{toric}} = N_x \ln 2$ scales with the size of the boundary of the subsystem rather than its volume. [Parameters: $N_x = N_y = 4$, $\vec{\theta} = (-1, -1, \dots, -1)$. The Hamiltonian parameters are given in the caption of Fig. 4 in End Matter.].

$$\begin{aligned} & (\theta_{n-1} \hat{\sigma}_{n-2}^X \hat{\sigma}_{n-1}^Z \hat{\sigma}_n^X) (\theta_{n+1} \hat{\sigma}_n^X \hat{\sigma}_{n+1}^Z \hat{\sigma}_{n+2}^X) \\ &= \theta_{n-1} \theta_{n+1} (\hat{\sigma}_{n-2}^X \hat{\sigma}_{n-1}^Z) (\hat{\sigma}_{n+1}^Z \hat{\sigma}_{n+2}^X). \end{aligned} \quad (5)$$

By Result 1, the $(2, 2, 2)$ factorizations in Eqs. (3)–(5) lead to a two-local and two-body parent Hamiltonian for the $D = 1$ cluster state of the form

$$\begin{aligned} \hat{H}^{\text{cluster}} = & \sum_n \omega_n (\hat{\sigma}_{n-1}^X - \theta_n \hat{\sigma}_n^Z \hat{\sigma}_{n+1}^X) \\ & + \sum_n \omega'_n (\hat{\sigma}_{n+1}^X - \theta_n \hat{\sigma}_{n-1}^X \hat{\sigma}_n^Z) \\ & + \sum_n J_n (\hat{\sigma}_{n-1}^X \hat{\sigma}_n^Y - \theta_n \theta_{n+1} \hat{\sigma}_{n+1}^Y \hat{\sigma}_{n+2}^X) \\ & + \sum_n J'_n (\hat{\sigma}_{n-2}^X \hat{\sigma}_{n-1}^Z - \theta_{n-1} \theta_{n+1} \hat{\sigma}_{n+1}^Z \hat{\sigma}_{n+2}^X), \end{aligned} \quad (6)$$

where ω_n, ω'_n, J_n , and J'_n are arbitrary real coupling constants. A numerical analysis verifies that $\hat{H}_{\vec{\theta}}^{\text{cluster}}$ is nonintegrable (see End Matter, Sec. I). However, the cluster state is a nonthermal eigenstate (i.e., a QMBS), since it has zero magic and also a nonthermal value $S^{\text{cluster}} = 2 \ln 2$ of half-chain entanglement entropy [50], in contrast to its surrounding thermal states which obey a volume-law scaling $S \propto N$ of entanglement entropy [see Fig. 1(a)]. We note that the freedom to choose the coupling constants and the stabilizer state phases θ_n gives a large degree of flexibility to construct a parent Hamiltonian $\hat{H}_{\vec{\theta}}^{\text{cluster}}$ with a desired property, for instance, r -site translation invariance, or with a particular subset of interaction types turned on or off.

Longer-range two-body interactions could be added to the Hamiltonian by also finding the $(2, \ell, 2)$ -factorizable stabilizer elements with $\ell > 2$.

Example 2: Toric code state ($D = 2$)—As another example, we consider one of the most well-known many-body stabilizers, the toric code on an $N_x \times N_y$ square lattice with periodic boundary conditions (here, N_x and N_y are both assumed to be even numbers). The stabilizer group of the toric code is generated by a set of “vertex” and “plaquette” operators, with the vertex operators defined as

$$\hat{O}_{x,y}^X = \theta_{x,y} \hat{\sigma}_{x,y}^X \hat{\sigma}_{x+1,y}^X \hat{\sigma}_{x,y+1}^X \hat{\sigma}_{x+1,y+1}^X, \quad \theta_{x,y} \in \{\pm 1\}, \quad (7)$$

for coordinates $(x, y) \in \mathbb{Z}_{N_x} \times \mathbb{Z}_{N_y}$ such that $x + y$ is even, and the plaquette operators defined as:

$$\hat{O}_{x,y}^Y = \theta_{x,y} \hat{\sigma}_{x,y}^Y \hat{\sigma}_{x+1,y}^Y \hat{\sigma}_{x,y+1}^Y \hat{\sigma}_{x+1,y+1}^Y, \quad \theta_{x,y} \in \{\pm 1\}, \quad (8)$$

at coordinates (x, y) such that $x + y$ is odd [see the inset in Fig. 1(b)]. Although there are $N = N_x N_y$ commuting vertex and plaquette operators [one for each site of the lattice], only $N - 2$ of them are independent. For a complete set of N independent, commuting generators, one can also append the Wilson loop operators $\hat{W}_1 = \theta_{W_1} \prod_{n \in C_1} \hat{\sigma}_n^Z$ and $\hat{W}_2 = \theta_{W_2} \prod_{n \in C_2} \hat{\sigma}_n^Z$, where C_1 is a noncontractible “horizontal” loop around the torus, C_2 is a noncontractible “vertical” loop, and $\theta_{W_1}, \theta_{W_2} \in \{\pm 1\}$.

Although the vertex and plaquette operators are four-local and four-body, they are $(2, 2, 2)$ factorizable in

multiple ways. For example, the vertex operator can be factorized as

$$\hat{\mathcal{O}}_{x,y}^X = \theta_{x,y}(\hat{\sigma}_{x,y}^X \hat{\sigma}_{x+1,y}^X)(\hat{\sigma}_{x,y+1}^X \hat{\sigma}_{x+1,y+1}^X) \quad (9)$$

or as

$$\hat{\mathcal{O}}_{x,y}^X = \theta_{x,y}(\hat{\sigma}_{x,y}^X \hat{\sigma}_{x,y+1}^X)(\hat{\sigma}_{x+1,y}^X \hat{\sigma}_{x+1,y+1}^X), \quad (10)$$

which, by Result 1, means that the toric code state is annihilated by the two-local operators $\hat{\sigma}_{x,y}^X \hat{\sigma}_{x+1,y}^X - \theta_{x,y} \hat{\sigma}_{x,y+1}^X \hat{\sigma}_{x+1,y+1}^X$ and $\hat{\sigma}_{x,y}^Y \hat{\sigma}_{x+1,y}^Y - \theta_{x,y} \hat{\sigma}_{x,y+1}^Y \hat{\sigma}_{x+1,y+1}^Y$. Similarly, the plaquette operator $\hat{\mathcal{O}}_{x,y}^Y$ yields two-local terms $\hat{\sigma}_{x,y}^Y \hat{\sigma}_{x+1,y}^Y - \theta_{x,y} \hat{\sigma}_{x,y+1}^Y \hat{\sigma}_{x+1,y+1}^Y$ and $\hat{\sigma}_{x,y}^X \hat{\sigma}_{x+1,y}^X - \theta_{x,y} \hat{\sigma}_{x,y+1}^X \hat{\sigma}_{x+1,y+1}^X$ that annihilate the toric code state for $x + y$ odd. Putting all of these annihilating terms together in a linear combination gives a quantum XY Hamiltonian on the square lattice with correlated interaction strengths on neighboring parallel bonds. The parent Hamiltonian in its most general form is given in End Matter, Sec. II, but if, for simplicity, we choose $\theta_{x,y} = -1$ for all (x, y) and demand translation invariance in both x and y directions, then we have the parent Hamiltonian:

$$\hat{H}^{\text{toric}} = \sum_{x,y} [J^X \hat{\sigma}_{x,y}^X \hat{\sigma}_{x+1,y}^X + \tilde{J}^X \hat{\sigma}_{x,y}^X \hat{\sigma}_{x,y+1}^X + J^Y \hat{\sigma}_{x,y}^Y \hat{\sigma}_{x+1,y}^Y + \tilde{J}^Y \hat{\sigma}_{x,y}^Y \hat{\sigma}_{x,y+1}^Y], \quad (11)$$

where J^X , J^Y , $\tilde{J}^X$, and $\tilde{J}^Y$ are real coupling parameters.

The quantum XY Hamiltonian is nonintegrable (as verified by an analysis of the level spacing statistics in End Matter, Sec. II) but has the toric code state as a zero-energy QMBS. The entanglement entropy of the toric code state [corresponding to the bipartition highlighted in gray in the inset in Fig. 1(b)] is $S^{\text{toric}} = N_x \ln 2$, which scales with the size of the boundary of the subsystem rather than the volume-law scaling $S \propto N_x N_y$ of the surrounding thermal eigenstates [see Fig. 1(b)]. Since the Wilson loop operators play no role in constructing the parent Hamiltonian, there are, in fact, four degenerate toric code QMBS at zero energy, corresponding to the four combinations of the Wilson loop operator phases $\theta_{W_1} \in \{\pm 1\}$ and $\theta_{W_2} \in \{\pm 1\}$. We note that this construction may shed light on recent results in Ref. [51], which discovered QMBS in a $D = 2$ -dimensional quantum XY model with a U(1) symmetry. Our parent Hamiltonian, however, does not require U(1) symmetry and explicitly identifies the QMBS as the toric code states. A similar construction was also used in Ref. [34], which discovered stabilizer QMBS in a particular lattice gauge theory closely related to the toric code.

Volume-law entangled stabilizer QMBS—Given a bipartition of an N -qubit system into subsystems A and B and a stabilizer state $|\Psi\rangle$, its stabilizer group can be decomposed

as $\mathcal{S} = \mathcal{S}_A \times \mathcal{S}_B \times \mathcal{S}_{AB}$, where $\mathcal{S}_A$ and $\mathcal{S}_B$ are the subgroups made up of stabilizer elements acting nontrivially only on subsystems A and B , respectively, and the subgroup $\mathcal{S}_{AB}$ contains all elements not in $\mathcal{S}_A \times \mathcal{S}_B$. With this decomposition, the bipartite entanglement entropy of the stabilizer state is $S = \frac{1}{2} \ln |\mathcal{S}_{AB}|$, where $|\mathcal{S}_{AB}|$ is the number of elements in the subgroup $\mathcal{S}_{AB}$ [50]. This suggests an approach to constructing ℓ -local and b -body parent Hamiltonians for volume-law entangled stabilizer QMBS: One should try to find stabilizers with $|\mathcal{S}_{AB}|$ growing exponentially with the volume of subsystem A (or B) and containing an extensive number of $(2, \ell, b)$ -factorizable elements in the subgroup $\mathcal{S}_{AB}$. This was, in fact, already done for several recent examples of volume-law entangled QMBS, such as the “rainbow” Bell pair state [40] and the “antipodal” Bell pair state [41,43], although without employing the stabilizer formalism and without noting that the corresponding QMBS are stabilizer states. In Supplemental Material [45], we discuss these recent results in the context of our framework. However, to show how our method can be used to construct parent Hamiltonians for volume-law entangled stabilizer QMBS, we present an example, which we dub the antipodal toric code (ATC) state.

Example 3: ATC state—Consider the previous toric code example, but on a thin torus with $N_y = 2$ and N_x an odd number. To obtain a $D = 1$ -dimensional chain, we can map the planar coordinates (x, y) to a linear coordinate n , via the transformation $n = x + yN_x$, as shown on the left side in Fig. 2. When the $D = 1$ -dimensional chain is wrapped into a circle, one can see that the vertex and plaquette operators are distributed antipodally; i.e., they are highly nonlocal Pauli strings (see the right side in Fig. 2). For the bipartition highlighted in gray in Fig. 2, an inspection of the stabilizer group shows that $|\mathcal{S}_{AB}| = 2^{N-2}$, so that the ATC state has entanglement entropy $S = \frac{1}{2}(N - 2) \ln 2$, which is a

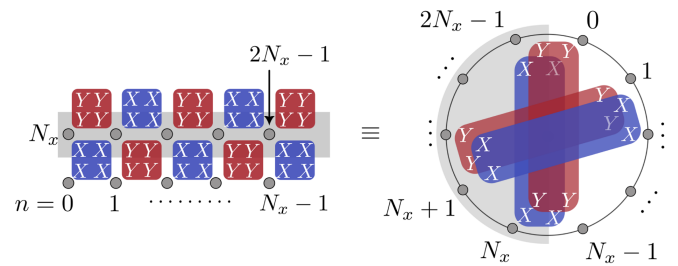


FIG. 2. Left: the vertex and plaquette operators for the toric code on a $N_x \times 2$ square lattice, with its coordinates (x, y) mapped onto a linear chain with coordinates $n = 0, 1, \dots, 2N_x - 1$. Right: an equivalent representation of the operators the circle, showing that they are distributed nonlocally (antipodally). For clarity, only a few vertex and plaquette operators are shown. These vertex and plaquette operators are generators for the stabilizer group of the ATC state.

volume-law scaling (see End Matter, Sec. I for details). Let us now construct a parent Hamiltonian for the ATC state.

For simplicity, we choose the phases $\theta_n = +1$ for each vertex [Eq. (7)] and plaquette [Eq. (8)] stabilizer element. With this choice, the product of ℓ nearest-neighbor vertex operators is a stabilizer element of the form $\hat{\sigma}_n^X \hat{\sigma}_{n+\ell}^X \hat{\sigma}_{n+N_x}^X \hat{\sigma}_{n+N_x+\ell}^X$ ($n \in \{0, 1, \dots, N_x - 1\}$), while the product of ℓ nearest-neighbor plaquette operators is a stabilizer element of the form $\hat{\sigma}_n^Y \hat{\sigma}_{n+\ell}^Y \hat{\sigma}_{n+N_x}^Y \hat{\sigma}_{n+N_x+\ell}^Y$. Their product gives another stabilizer element $\hat{\sigma}_n^Z \hat{\sigma}_{n+\ell}^Z \hat{\sigma}_{n+N_x}^Z \hat{\sigma}_{n+N_x+\ell}^Z$. Each of these stabilizer elements is $(2, \ell, 2)$ factorizable, so that Result 1 can be applied to give the ℓ -local, two-body annihilating terms $\hat{\sigma}_n^X \hat{\sigma}_{n+\ell}^X - \hat{\sigma}_{n+N_x}^X \hat{\sigma}_{n+N_x+\ell}^X$, $\hat{\sigma}_n^Y \hat{\sigma}_{n+\ell}^Y - \hat{\sigma}_{n+N_x}^Y \hat{\sigma}_{n+N_x+\ell}^Y$, and $\hat{\sigma}_n^Z \hat{\sigma}_{n+\ell}^Z - \hat{\sigma}_{n+N_x}^Z \hat{\sigma}_{n+N_x+\ell}^Z$, respectively. If, for simplicity, we insist that the parent Hamiltonian has alternating couplings, then it has the form of an alternating Heisenberg XYZ chain with range ℓ interactions:

$$\hat{H}_\ell^{\text{ATC}} = \sum_{n=0}^{N-1} (-1)^n (J_\ell^X \hat{\sigma}_n^X \hat{\sigma}_{n+\ell}^X + J_\ell^Y \hat{\sigma}_n^Y \hat{\sigma}_{n+\ell}^Y + J_\ell^Z \hat{\sigma}_n^Z \hat{\sigma}_{n+\ell}^Z). \quad (12)$$

The ATC state is a zero-energy eigenstate of this Hamiltonian for any interaction range ℓ , so it is also a zero-energy eigenstate of the sum $\hat{H}^{\text{ATC}} = \sum_{\ell=1}^{N/2} \hat{H}_\ell^{\text{ATC}}$. The interaction strengths J_ℓ^μ can be chosen to decay with the distance ℓ with any desired function. For instance, it includes the all-to-all interacting model of Ref. [52] as a special case. Although the ATC state has volume-law entanglement, it is nonthermal by construction, since it is a stabilizer state. The parent Hamiltonian can be further generalized by noting that, for our thin torus, a Wilson loop operator $\hat{W}_2 = \theta_{W_2} \hat{\sigma}_n^Z \hat{\sigma}_{n+N_x}^Z$ is $(2, 1, 1)$ factorizable and leads to additional Hamiltonian terms that annihilate the ATC state (see End Matter, Sec. III).

Discussions—Stabilizer states have an efficient classical description in terms of a generating set of Pauli strings, in contrast to thermal states, which are generally believed to lack any efficient classical representation. Nevertheless, stabilizer states can support rich entanglement structures, including multipartite entanglement, long-range entanglement, topological order, and volume-law scaling of entanglement entropy. In this Letter, we introduce a method for constructing physically plausible ℓ -local, b -body Hamiltonians with stabilizer states as QMBS. We have shown that this method enables the construction of many new Hamiltonians with stabilizer states as zero-energy QMBS and also explains several previously known results in a unified framework. More examples are provided in Supplemental Material [45], including parent Hamiltonians for product states, products of Bell pairs, and volume-law entangled states which we dub the “rainbow cluster state”

and “antipodal cluster state.” We also use our framework to show that the PXP model has the stabilizer QMBS recently discovered in Ref. [42]. We note that the present construction relies on (k, ℓ, b) factorizability with $k = 2$ and does not straightforwardly generalize to $k > 2$.

An advantage of our construction is that the well-developed stabilizer formalism enables efficient computation of various properties of stabilizer states, such as multipartite entanglement [50]. Also, stabilizer QMBS can be classified in terms of their complexity, given by the depth of the Clifford circuit that is needed to create them starting from a product stabilizer state (see Supplemental Material [45]).

Our framework provides wide scope for future work, for example, exploring the quasiparticle excitations on top of stabilizer QMBS, and generalizations to systems of qutrits (or higher-dimensional local systems), to parent Hamiltonians with multiple stabilizer QMBS or to stabilizer QMBS at nonzero energies. A further generalization to Floquet circuit models would enable experimental observation on a digital quantum simulator, similar to Ref. [16].

Note added—Recently, related and complementary results were reported in Ref. [53].

Acknowledgments—I thank L. Coady for invaluable support and encouragement. This publication has emanated from research conducted with the financial support of Taighde Éireann—Research Ireland under Grant No. 22/PATH-S/10812.

Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

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- [45] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/3vk5-483c>, supports the results of the main text by providing some additional technical details and more examples of our parent Hamiltonian construction, which includes Refs. [46–49].

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End Matter

Further details for Example 1 (the $D = 1$ cluster state)—In Fig. 3 below, we plot the energy level spacing statistics for the parent Hamiltonian of the cluster state, given in Eq. (6). Restricting to energy eigenvalues E_α in the middle half of the energy spectrum, we see that level spacings $s_\alpha = E_\alpha - E_{\alpha-1}$ between consecutive energy levels (normalized by the average level spacing $\bar{s}$) follows the same distribution as would be expected for a Hamiltonian drawn from the Gaussian unitary ensemble (GUE). This indicates that the Hamiltonian is nonintegrable and that the Hamiltonian is not time-reversal symmetric [1].

The half-chain entanglement entropy of the cluster state is straightforwardly computed by the methods of Ref. [50]. Roughly speaking, one can see in the inset in Fig. 1(a) that there are four generators that cross the bipartition (and cannot be constructed as a product of generators acting trivially on each subsystem), so that $|S_{AB}| = 2^4$. The entanglement entropy of the cluster state is, therefore, $S = \frac{1}{2} \ln |S_{AB}| = 2 \ln 2$.

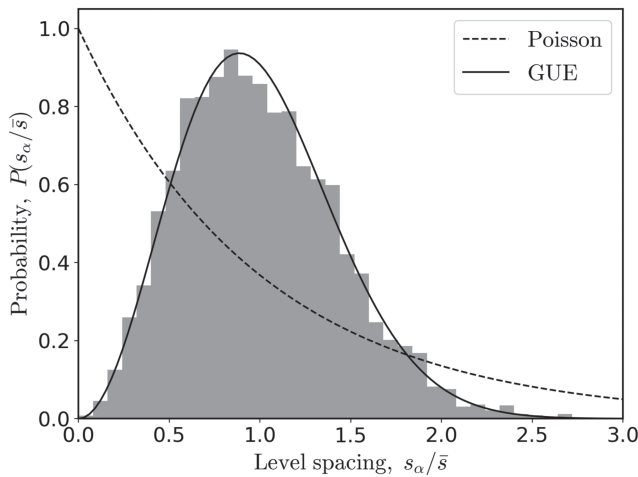


FIG. 3. The level spacing statistics for the parent Hamiltonian of the cluster state, given in Eq. (6). All parameters are the same as for Fig. 1(a).

Further details for Example 2 (the toric code state)—From the (2, 2, 2) factorizations of the vertex operator given in Eqs. (9) and (10), we know that the toric code state is annihilated by the two-local, two-body terms $\hat{\sigma}_{x,y}^X \hat{\sigma}_{x+1,y}^X - \theta_{x,y} \hat{\sigma}_{x,y+1}^X \hat{\sigma}_{x+1,y+1}^X$ and $\hat{\sigma}_{x,y}^X \hat{\sigma}_{x,y+1}^X - \theta_{x,y} \hat{\sigma}_{x+1,y}^X \hat{\sigma}_{x+1,y+1}^X$ (for $x+y$ even). Similarly, the plaquette operator $\hat{O}_{x,y}^Y = \theta_{x,y} \hat{\sigma}_{x,y}^Y \hat{\sigma}_{x+1,y}^Y \hat{\sigma}_{x,y+1}^Y \hat{\sigma}_{x+1,y+1}^Y$ can be (2, 2, 2) factorized as $\theta_{x,y} (\hat{\sigma}_{x,y}^Y \hat{\sigma}_{x,y+1}^Y) (\hat{\sigma}_{x+1,y}^Y \hat{\sigma}_{x+1,y+1}^Y)$ or as $\theta_{x,y} (\hat{\sigma}_{x,y}^Y \hat{\sigma}_{x,y+1}^Y) (\hat{\sigma}_{x+1,y}^Y \hat{\sigma}_{x+1,y+1}^Y)$, yielding the two-local, two-body annihilating terms $\hat{\sigma}_{x,y}^Y \hat{\sigma}_{x+1,y}^Y - \theta_{x,y} \hat{\sigma}_{x,y+1}^Y \hat{\sigma}_{x+1,y+1}^Y$ and $\hat{\sigma}_{x,y}^Y \hat{\sigma}_{x,y+1}^Y - \theta_{x,y} \hat{\sigma}_{x+1,y}^Y \hat{\sigma}_{x+1,y+1}^Y$ (for $x+y$ odd). The toric code state is annihilated by any linear combination of these terms:

$$\begin{aligned} \hat{H}^{\text{toric}} = & \sum_{\substack{x,y \\ (x+y)\text{even}}} J_{x,y}^X (\hat{\sigma}_{x,y}^X \hat{\sigma}_{x+1,y}^X - \theta_{x,y} \hat{\sigma}_{x,y+1}^X \hat{\sigma}_{x+1,y+1}^X) \\ & + \sum_{\substack{x,y \\ (x+y)\text{even}}} \tilde{J}_{x,y}^X (\hat{\sigma}_{x,y}^X \hat{\sigma}_{x,y+1}^X - \theta_{x,y} \hat{\sigma}_{x+1,y}^X \hat{\sigma}_{x+1,y+1}^X) \\ & + \sum_{\substack{x,y \\ (x+y)\text{odd}}} J_{x,y}^Y (\hat{\sigma}_{x,y}^Y \hat{\sigma}_{x+1,y}^Y - \theta_{x,y} \hat{\sigma}_{x,y+1}^Y \hat{\sigma}_{x+1,y+1}^Y) \\ & + \sum_{\substack{x,y \\ (x+y)\text{odd}}} \tilde{J}_{x,y}^Y (\hat{\sigma}_{x,y}^Y \hat{\sigma}_{x,y+1}^Y - \theta_{x,y} \hat{\sigma}_{x+1,y}^Y \hat{\sigma}_{x+1,y+1}^Y). \quad (\text{B1}) \end{aligned}$$

This is the quantum XY Hamiltonian on a square lattice, with the restriction that couplings across parallel neighbouring bonds are correlated (if $\theta_{x,y} = -1$) or anticorrelated (if $\theta_{x,y} = 1$). If we assume that $\theta_{x,y} = -1$ for all x and y and that the coupling constants are spatially homogeneous ($J_{x,y}^{X/Y} = J^{X/Y}$ and $\tilde{J}_{x,y}^{X/Y} = \tilde{J}^{X/Y}$), then the Hamiltonian simplifies to Eq. (11), which is the usual translation invariant quantum XY Hamiltonian on a square lattice.

It is known that the $D = 2$ -dimensional quantum XY Hamiltonian is nonintegrable. Here, we verify this numerically by computing the level spacing statistics of the Hamiltonian in Eq. (B1). Since $\hat{H}^{\text{toric}}$ commutes with the X -parity operator $\hat{\Pi}^X = \otimes_{x,y} \hat{\sigma}_{x,y}^X$ and also the Z -parity

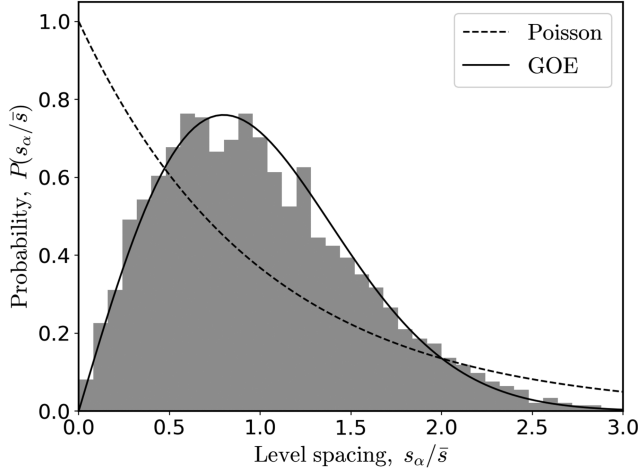


FIG. 4. The level spacing statistics for the parent Hamiltonian of the toric code state, given in Eq. (B1), in its $\hat{\Pi}^X = \hat{\Pi}^Z = +1$ parity sector. Here, the square lattice has the size $N_x = N_y = 4$, the toric code state is chosen with $\theta_{x,y} = -1$ for all x and y , and the coupling constants $J_{x,y}^{X/Y}$ and $\tilde{J}_{x,y}^{X/Y}$ are chosen uniformly at random from the range $[0.7, 1.3]$. The same parameters are also used for Fig. 1(b), except that it also shows data in all parity sectors $\hat{\Pi}^X = \pm 1$ and $\hat{\Pi}^Z = \pm 1$.

operator $\hat{\Pi}^Z = \otimes_{x,y} \hat{\sigma}_{x,y}^Z$, to obtain sensible level spacing statistics we must restrict to sectors of fixed X and Z parity. In Fig. 4 below, we plot the energy level spacing statistics in the $\hat{\Pi}^X = \hat{\Pi}^Z = +1$ parity sector (the sector that includes the toric code states). We see that the level spacing statistics follow the same distribution as would be expected for a Hamiltonian from the Gaussian orthogonal ensemble (GOE). This indicates that the Hamiltonian is nonintegrable and time-reversal symmetric, as expected.

Again, the half-system entanglement entropy of the toric code state can be straightforwardly computed by the methods of Ref. [50]. Intuitively, one can see in the inset in Fig. 1(b) that there are $2N_x$ generators that are cut by the bipartition between the two subsystems (and cannot be constructed as a product of generators acting trivially on each subsystem), so that $|\mathcal{S}_{AB}| = 2^{2N_x}$. The entanglement entropy of the cluster state is, therefore, $S = \frac{1}{2} \ln |\mathcal{S}_{AB}| = N_x \ln 2$.

Further details for Example 3 (antipodal toric code state)—In the main text, we showed that the ATC state (with stabilizer phases $\theta_n = 1$ for all vertex and plaquette operators) is annihilated by the ℓ -local, two-body terms $\hat{\sigma}_n^X \hat{\sigma}_{n+\ell}^X - \hat{\sigma}_{n+N_x}^X \hat{\sigma}_{n+N_x+\ell}^X$, $\hat{\sigma}_n^Y \hat{\sigma}_{n+\ell}^Y - \hat{\sigma}_{n+N_x}^Y \hat{\sigma}_{n+N_x+\ell}^Y$, and $\hat{\sigma}_n^Z \hat{\sigma}_{n+\ell}^Z - \hat{\sigma}_{n+N_x}^Z \hat{\sigma}_{n+N_x+\ell}^Z$, where $n \in \{0, 1, \dots, N_x - 1\}$. The ATC state is, therefore, annihilated by the parent Hamiltonian obtained by a linear combination of all such terms:

$$\begin{aligned} \hat{H}_\ell^{\text{ATC}} = & \sum_{n=0}^{N_x-1} J_{\ell,n}^X (\hat{\sigma}_n^X \hat{\sigma}_{n+\ell}^X - \hat{\sigma}_{n+N_x}^X \hat{\sigma}_{n+N_x+\ell}^X) \\ & + \sum_{n=0}^{N_x-1} J_{\ell,n}^Y (\hat{\sigma}_n^Y \hat{\sigma}_{n+\ell}^Y - \hat{\sigma}_{n+N_x}^Y \hat{\sigma}_{n+N_x+\ell}^Y) \\ & + \sum_{n=0}^{N_x-1} J_{\ell,n}^Z (\hat{\sigma}_n^Z \hat{\sigma}_{n+\ell}^Z - \hat{\sigma}_{n+N_x}^Z \hat{\sigma}_{n+N_x+\ell}^Z). \end{aligned} \quad (\text{C1})$$

Since we have assumed that N_x is an odd number, choosing the coupling constants as $J_{\ell,n}^\mu = (-1)^n J_\ell^\mu$ gives the Hamiltonian for the Heisenberg XYZ chain with ℓ -range alternating couplings given in Eq. (12).

There are many more local, few-body terms that annihilate the ATC state, enabling significant generalization of the parent Hamiltonian. For example, as mentioned in the main text, for our thin torus ($N_y = 2$) the Wilson loop operator $\hat{W}_2 = \theta_{W_2} \hat{\sigma}_0^Z \hat{\sigma}_{N_x}^Z$ is a $(2, 1, 1)$ -factorizable stabilizer element. The product of this operator with the stabilizer element $\hat{\sigma}_0^Z \hat{\sigma}_1^Z \hat{\sigma}_{N_x}^Z \hat{\sigma}_{N_x+1}^Z$ gives another $(2, 1, 1)$ -factorizable stabilizer element $\theta_{W_2} \hat{\sigma}_1^Z \hat{\sigma}_{N_x+1}^Z$. Proceeding iteratively across the chain in this way, we see that $\theta_{W_2} \hat{\sigma}_n^Z \hat{\sigma}_{n+N_x}^Z$ is a $(2, 1, 1)$ -factorizable stabilizer element for the ATC state, for each $n \in \{0, 1, \dots, N_x - 1\}$. This leads to an additional annihilating parent Hamiltonian:

$$\hat{H}^Z = \sum_{n=0}^{N_x-1} h_n^Z (\hat{\sigma}_n^Z - \theta_{W_2} \hat{\sigma}_{n+N_x}^Z) \quad (\text{C2})$$

for the ATC state.

Moreover, the product of any of the stabilizer elements $\theta_{W_2} \hat{\sigma}_n^Z \hat{\sigma}_{n+N_x}^Z$ (obtained above via the Wilson loop operator) with its overlapping plaquette operators $\hat{\sigma}_{n-1}^Y \hat{\sigma}_n^Y \hat{\sigma}_{n+N_x-1}^Y \hat{\sigma}_{n+N_x}^Y$ and $\hat{\sigma}_n^Y \hat{\sigma}_{n+1}^Y \hat{\sigma}_{n+N_x}^Y \hat{\sigma}_{n+N_x+1}^Y$ gives the additional $(2, 2, 2)$ -factorizable stabilizer elements $-\theta_{W_2} \hat{\sigma}_{n-1}^X \hat{\sigma}_n^X \hat{\sigma}_{n+N_x-1}^X \hat{\sigma}_{n+N_x}^X$ and $-\theta_{W_2} \hat{\sigma}_n^X \hat{\sigma}_{n+1}^X \hat{\sigma}_{n+N_x}^X \hat{\sigma}_{n+N_x+1}^X$, respectively. Further multiplication by $\ell - 1$ neighboring vertex (or plaquette) operators gives the $(2, \ell, 2)$ -factorizable stabilizer elements $-\theta_{W_2} \hat{\sigma}_n^Y \hat{\sigma}_{n+\ell}^Y \hat{\sigma}_{n+N_x}^Y \hat{\sigma}_{n+N_x+\ell}^Y$ and $-\theta_{W_2} \hat{\sigma}_n^X \hat{\sigma}_{n+\ell}^X \hat{\sigma}_{n+N_x}^X \hat{\sigma}_{n+N_x+\ell}^X$. These lead to the ℓ -local, two-body annihilating terms $\hat{\sigma}_n^Y \hat{\sigma}_{n+1}^Y + \theta_{W_2} \hat{\sigma}_{n+N_x}^Y \hat{\sigma}_{n+N_x+\ell}^Y$ and $\hat{\sigma}_n^X \hat{\sigma}_{n+\ell}^X + \theta_{W_2} \hat{\sigma}_{n+N_x}^X \hat{\sigma}_{n+N_x+\ell}^X$ and the annihilating parent Hamiltonians:

$$\begin{aligned} \hat{H}^{XY} = & \sum_{n=0}^{N_x-1} J_n^{XY} (\hat{\sigma}_n^X \hat{\sigma}_{n+\ell}^Y + \theta_{W_2} \hat{\sigma}_{n+N_x}^X \hat{\sigma}_{n+N_x+\ell}^Y), \\ \hat{H}^{YX} = & \sum_{n=0}^{N_x-1} J_n^{YX} (\hat{\sigma}_n^Y \hat{\sigma}_{n+1}^X + \theta_{W_2} \hat{\sigma}_{n+N_x}^Y \hat{\sigma}_{n+N_x+\ell}^X). \end{aligned} \quad (\text{C3})$$

The half-system entanglement entropy of the ATC state can be computed by the methods of Ref. [50]. One can see in Fig. 2 that there are $N = 2N_x$ vertex and plaquette operators that are cut by the bipartition between the two subsystems. However, as discussed in the main text, for the toric code only $N - 2$ of the vertex and plaquette operators are independent. A complete set of stabilizer generators can be obtained by appending the Wilson loop operators $\hat{W}_1 = \theta_{W_1} \otimes_{n=0}^{N_x-1} \hat{\sigma}_n^Z$ (a string of $\hat{\sigma}^Z$ operators in a non-contractible “horizontal” loop around the torus) and $\hat{W}_2 = \theta_{W_2} \hat{\sigma}_0^Z \hat{\sigma}_{N_x}^Z$ (a string of $\hat{\sigma}^Z$ operators in a noncontractible “vertical” loop around the torus). The operator $\hat{W}_1$ acts

nontrivially only on a subsystem A , which consists of the qubits labeled $n \in \{0, 1, \dots, N_x - 1\}$. Since $\hat{\Pi}^Z = \otimes_{n=0}^{2N_x-1} \hat{\sigma}_n^Z$ is a stabilizer element, the operator $\hat{\Pi}^Z \hat{W}_1 = \otimes_{n=N_x}^{2N_x-1} \hat{\sigma}_n^Z$ is also a stabilizer element, which acts nontrivially only on subsystem B , consisting of the qubits labeled $n \in \{N_x, N_x + 1, \dots, 2N_x - 1\}$. The subgroup $\mathcal{S}_A \times \mathcal{S}_B = \{\hat{\mathbb{I}}^{\otimes N}, \hat{W}_2, \hat{\Pi}_2, \hat{\Pi}^Z \hat{W}_1\}$ contains four elements, while all other stabilizer elements are in the group $\mathcal{S}_{AB}$, which, therefore, has the size $|\mathcal{S}_{AB}| = 2^{N-2}$. It follows, by the results in Ref. [50], that the entanglement entropy of the ATC state is $S = \frac{1}{2} \ln |\mathcal{S}_{AB}| = \frac{1}{2} (N - 2) \ln 2$.