

Ultimate Quantum Precision Limit at Colliders: Conditions and Case Studies

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We investigate whether collider experiments can reach the quantum limit of precision, defined by the quantum Fisher information (QFI), using only classical observables such as particle momenta. As a case study, we focus on the $\tau^+\tau^-$ system and the decay channel $\tau \rightarrow \pi\nu$, which offers maximal spin-analyzing power and renders the decay a projective measurement. We develop a general framework to determine when collider measurements can, in principle, saturate the QFI in an entangled biparticle system, and this framework extends naturally to other such systems. Within this framework, QFI saturation occurs if and only if the symmetric logarithmic derivative (SLD) commutes with a complete set of orthonormal separable projectors associated with collider-accessible measurements. This separability condition, reflecting the independence of decay amplitudes, is highly nontrivial. To meet this condition, a key requirement is that the spin density matrix be rank deficient, allowing the SLD sufficient freedom. We show that the classical Fisher information asymptotically saturates the QFI for magnetic dipole moments and CP -violating Higgs interactions in selected phase-space regions, but not for electric dipole moments. These results bridge quantum metrology and collider physics, providing a systematic method to identify quantum-optimal sensitivity in collider experiments.

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Introduction—The quantum Fisher information (QFI) defines the ultimate precision limit for estimating parameters encoded in quantum states, as formalized by the quantum Cramér–Rao bound [1–5]. For a density matrix $\rho(d)$ depending on a single parameter d , the QFI sets a bound on the precision of any unbiased estimator of d : $(\Delta d)^2 \geq 1/[NF_q(d)]$, where N is the number of repeated measurements. The QFI is intrinsically defined at the per-event level, determined solely by the density matrix of a single collider event, and thus quantifies the maximal information content carried by each copy of the state before any measurement is performed. Once a specific measurement is performed, sensitivity is governed by the classical Fisher information (CFI), satisfying $F_c(d) \leq F_q(d)$. The QFI can be saturated if the measurement corresponds to projectors onto the eigenbasis of the symmetric logarithmic derivative (SLD), which typically involves nontrivial quantum observables [6,7].

In collider experiments, direct access to quantum states is limited; measurements are restricted to classical quantities such as particle momenta. However, spin correlations of unstable particles are preserved in the kinematics of their decay products, allowing access to the spin density matrix [8–10]. Techniques such as optimal observables [11,12], matrix element methods [13–15] and machine learning [16,17] have been used to approach the CFI limit in collider measurements. Yet it remains unknown whether the quantum limit—QFI—can be reached in such classical measurement settings.

In this Letter, we develop a general framework to determine when quantum-optimal sensitivity can be achieved in collider experiments using classical observables. As a case study, we apply this framework to the entangled $\tau^+\tau^-$ system [18–24], with the decay $\tau \rightarrow \pi\nu$ providing maximal spin-analyzing power and yielding a rank-one projective measurement.

We show that QFI saturation occurs if and only if the SLD commutes with a complete set of separable (nonentangled) projectors for collider-accessible measurements. This separability condition, rooted in the independence of decay amplitudes, is highly nontrivial. A key requirement is that the spin density matrix be rank deficient, allowing the SLD sufficient flexibility to align with separable eigenstates.

In $\tau^+\tau^-$ system, QFI saturation is achievable for magnetic dipole moments (MDMs) and CP -violating Higgs couplings, but not for electric dipole moments (EDMs).

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This framework applies to other entangled biparticle systems, such as top-quark, baryon, and gauge boson pairs.

Spin density matrix for τ pair—We use a $\tau^+\tau^-$ system as an example to derive the spin density matrix and demonstrate how to calculate the QFI in collider searches.

The first scenario is a $\tau^+\tau^-$ pair from Higgs decay, $h \rightarrow \tau^+\tau^-$ via a CP -violating phase δ_h [25,26],

$$\mathcal{L}_{\text{CPV}} = -\frac{m_\tau}{v} h \bar{\tau} [\cos \delta_h + i\gamma_5 \sin \delta_h] \tau. \quad (1)$$

The other two scenarios involve $e^+e^- \rightarrow \tau^+\tau^-$ mediated by τ MDM or EDM, which were recently measured by Refs. [27–35]. The corresponding effective interactions with real a_τ and d_τ are

$$\mathcal{L}_{\text{MDM/EDM}} = \frac{1}{2} F_{\mu\nu} \bar{\tau} \sigma^{\mu\nu} \left[-\frac{e}{2m_\tau} a_\tau - i\gamma_5 d_\tau \right] \tau. \quad (2)$$

For each case, we consider the spin density matrix for $\tau^+\tau^-$ production. To avoid the integrated (or fictitious) quantum state [10,36,37], we fix the momentum direction of the τ^+ , denoted by $\hat{k}$, in the center-of-mass frame of the τ pair. For an unpolarized initial state, the spin density matrix can be expressed in the helicity basis [37] as

$$\rho_{\alpha\alpha';\beta\beta'}(\hat{k}) = N_{\text{prod}}^{-1} \sum_{\text{ini}} M(\tau_\alpha^+ \tau_\beta^-) M^*(\tau_{\alpha'}^+ \tau_{\beta'}^-), \quad (3)$$

where α, β label spin states, M is the amplitude, and N_{prod} ensures $\text{Tr}[\rho] = 1$. We aim to extract the small parameter $d = \delta_h, a_\tau, d_\tau$ from τ decays. Thus, ρ can be expanded perturbatively with small d ,

$$\rho = \rho_0 + d\rho_1 + \mathcal{O}(d^2), \quad (4)$$

where ρ_0 is the leading contribution, and ρ_1 encodes the interference contribution.

Quantum metrology and QFI—In quantum metrology, the ultimate precision in estimating a parameter d is set by the QFI. For a state $\rho(d)$, the quantum-optimal measurement is defined by the SLD, $\hat{Q}^{\text{opt}}$, which satisfies [2]

$$\rho_1 = \frac{1}{2} \{ \rho_0, \hat{Q}^{\text{opt}} \}, \quad (5)$$

with the perturbative ρ in Eq. (4) and $\{ \cdot, \cdot \}$ represents the anticommutator. The QFI is,

$$F_q(d) = \text{Tr}[\rho(\hat{Q}^{\text{opt}})^2] \simeq \text{Tr}[\rho_0(\hat{Q}^{\text{opt}})^2]. \quad (6)$$

The quantum Cramér-Rao bound sets a lower limit on the variance of any unbiased estimator based on an observable $\hat{Q}$,

$$\text{Var}(\hat{Q}) F_q(d) \geq (\partial_d \langle \hat{Q} \rangle_\tau)^2, \quad (7)$$

where the mean value $\langle \hat{Q} \rangle_\tau$ and $\text{Var}(\hat{Q})$ are evaluated in the spin Hilbert space of the $\tau^+\tau^-$ pair.

Generalized quantum measurement—Because of its short lifetime ($c\tau \approx 87 \mu\text{m}$), τ spin cannot be measured directly and must be inferred from its decay products. We focus on the two-body decay $\tau \rightarrow \pi\nu$, which offers maximal spin-analyzing power [38], and the final states can be fully reconstructed. At e^+e^- colliders, the neutrino momentum typically has a twofold ambiguity [32]. However, the precise vertex and tracking systems [39,40] enable full neutrino reconstruction using the pion's impact parameter at Belle-II [20]. Machine learning techniques at the LHC also enable full τ decay kinematics [41].

For single τ , its decay defines a spin-space operator via its amplitude, which acts as a generalized measurement [42–45]. For $\tau^- \rightarrow \pi^-\nu_\tau$, the operator is

$$\hat{D}_{\alpha\alpha'}^-(\hat{q}_-) = N_D^{-1} M[\tau_\alpha^- \rightarrow \pi^-(\hat{q}_-)\nu] M^*(\tau_{\alpha'}^- \rightarrow \cdots), \quad (8)$$

where N_D is a normalization factor, and $\hat{q}_-$ is the π^- direction in the τ^- rest frame. The phase space is denoted Ω_- , with angular coordinates θ_- and ϕ_- . The operator $\hat{D}^-(\hat{q}_-)$ is positive semidefinite and satisfies the completeness relation $\int d\Omega_- \hat{D}^-(\hat{q}_-) = 2\pi\mathbb{I}_2$, making it a legitimate measurement operator [45]. Because of maximal spin-analyzing power, $\hat{D}^-$ reduces to a rank-one projector $|\hat{q}_-\rangle\langle\hat{q}_-|$, satisfying $(\hat{D}^-)^2 = \hat{D}^-$. A similar operator $\hat{D}^+(\hat{q}_+)$ exists for τ^+ .

For $\tau^+\tau^-$ system, the decay defines a rank-one operator in the spin Hilbert space,

$$\hat{E}_{\alpha\alpha';\beta\beta'}(\hat{q}_+, \hat{q}_-) = \hat{D}_{\alpha\alpha'}^+(\hat{q}_+) \hat{D}_{\beta\beta'}^-(\hat{q}_-). \quad (9)$$

Each $\hat{E}$ is a direct product of $\hat{D}^\pm(\hat{q}_\pm)$, associated with the pion directions $(\hat{q}_+, \hat{q}_-)$. It satisfies the properties $\hat{E}^2 = \hat{E}$ and the completeness relation $\int d\Omega_+ d\Omega_- \hat{E} = 4\pi^2\mathbb{I}_4$, see Supplemental Material [46] Secs. I and II.

The normalized differential distribution for the process $e^+e^- \rightarrow \tau^+\tau^- \rightarrow \pi^+\pi^-\nu_\tau\bar{\nu}_\tau$, derived in Supplemental Material [46] Sec. I, is

$$\frac{df}{d\Omega_+ d\Omega_-} \equiv \frac{1}{\sigma} \frac{d\sigma}{d\Omega_+ d\Omega_-} = \text{Tr} \left[\frac{\hat{E}(\hat{q}_+, \hat{q}_-)}{4\pi^2} \rho(\hat{k}) \right]. \quad (10)$$

Using Eq. (4), the distribution expands as

$$\frac{df}{d\Omega_+ d\Omega_-} = \Sigma_0 + d \cdot \Sigma_1 + \mathcal{O}(d^2), \quad (11)$$

which captures the leading dependence on d . This directly links τ spin correlations to observed pion kinematics via projective measurements.

Optimal observables for CFI—For classical measurements based on pion directions $\hat{q}_\pm$, an observable $O_d(\hat{q}_+, \hat{q}_-)$ can be constructed to estimate the parameter d . The statistical precision is bounded by the classical Cramér-Rao inequality [49],

$$\text{Var}(O_d)F_c(d) \geq (\partial_d \langle O_d \rangle_\pi)^2, \quad (12)$$

where $\langle O_d \rangle_\pi$ and $\text{Var}(O_d)$ are computed from the distribution in Eq. (10). The CFI for small d is

$$F_c(d) = \int d\Omega_+ d\Omega_- \frac{\Sigma_1^2}{\Sigma_0}, \quad O_d^{\text{opt}}(\hat{q}_+, \hat{q}_-) = \Sigma_1/\Sigma_0, \quad (13)$$

where O_d^{opt} is the classical optimal observable that saturates the classical bound Eq. (12). Many collider analyses aim to reach this classical limit by constructing such observables or using machine learning techniques [50].

The relation between CFI and QFI—Each classical observable $O(\hat{q}_+, \hat{q}_-)$ defined on pion directions corresponds to a unique quantum operator $\hat{Q}_O$ in the τ spin Hilbert space, via

$$\hat{Q}_O \equiv \int d\Omega_+ d\Omega_- \hat{E}(\hat{q}_+, \hat{q}_-) O(\hat{q}_+, \hat{q}_-). \quad (14)$$

This construction ensures that the quantum and classical expectation values are identical using Eq. (10),

$$\langle \hat{Q}_O \rangle_\tau = \text{Tr}[\rho(\hat{k}) \hat{Q}_O] = \langle O(\hat{q}_+, \hat{q}_-) \rangle_\pi. \quad (15)$$

The variance of the classical observable also satisfies [3]

$$\text{Var}(O) \geq \text{Var}(\hat{Q}_O), \quad (16)$$

reflecting the information loss in mapping quantum to classical observables.

Applying this to the classical optimal observable O_d^{opt} , one obtains a hierarchy of sensitivities,

$$F_q(d) \geq \frac{(\partial_d \langle \hat{Q}_{O_d^{\text{opt}}} \rangle_\tau)^2}{\text{Var}(\hat{Q}_{O_d^{\text{opt}}})} \geq \frac{(\partial_d \langle O_d^{\text{opt}} \rangle_\pi)^2}{\text{Var}(O_d^{\text{opt}})} = F_c(d). \quad (17)$$

This illustrates the general principle $F_q \geq F_c$: classical optimal observables are always less precise than the quantum-optimal operators.

Conditions to reach QFI—In quantum metrology, QFI saturation is achieved by performing projective measurements in the eigenbasis of the SLD,

$$\hat{Q}^{\text{opt}} = \sum_i \lambda_i \hat{\Pi}_i, \quad (18)$$

where $\hat{\Pi}_i = |\psi_i\rangle\langle\psi_i|$ project onto the orthonormal eigenstates of $\hat{Q}^{\text{opt}}$.

However, in τ pair decays, the collider measurement is limited to the pion momenta $q_\pm$, which determine the spin-space projector

$$\hat{E}(-\hat{q}_+, \hat{q}_-) = |\hat{q}_+\rangle \otimes |\hat{q}_-\rangle \langle \hat{q}_+| \otimes \langle \hat{q}_-| \equiv |E\rangle \langle E|, \quad (19)$$

where $|\hat{q}_\pm\rangle$ are the spin states based on the pion directions in each τ rest frame. These operators $\hat{E}$ are rank-one projectors onto separable states, reflecting the independent τ decay amplitudes in Eq. (9) (see the exclusion of decay dependence on local variables [51,52]). Collider measurements only access such separable projectors. Given the first state $|E_1\rangle = |\hat{q}_+, \hat{q}_-\rangle \equiv |\hat{q}_+\rangle \otimes |\hat{q}_-\rangle$, the remaining orthonormal states $|E_j\rangle$ ($j = 2, 3, 4$) are forced to be separable

$$|-\hat{q}_+, \hat{q}_-\rangle, \quad |\hat{q}_+, -\hat{q}_-\rangle, \quad |-\hat{q}_+, -\hat{q}_-\rangle, \quad (20)$$

where $|\hat{q}_+, \hat{q}_-\rangle$ has components

$$\{c_+c_-, c_+s_-e^{i\phi_-}, s_+c_-e^{i\phi_+}, s_+s_-e^{i(\phi_++\phi_-)}\}, \quad (21)$$

where $c_\pm = \cos(\theta_\pm/2)$ and $s_\pm = \sin(\theta_\pm/2)$. The minus sign indicates that $|-\hat{q}_+\rangle$ is orthogonal to $|\hat{q}_+\rangle$ in the τ^+ spin space.

This structure imposes a key constraint: if any SLD eigenstate is entangled, the QFI cannot be saturated via collider measurements. Thus, the necessary and sufficient condition for QFI saturation is that the SLD shares its eigenbasis with the accessible separable projectors, which can be expressed as

$$[\hat{Q}^{\text{opt}}, \hat{E}_j] = 0 \Leftrightarrow \hat{Q}^{\text{opt}}|E_j\rangle = \lambda_j|E_j\rangle, \quad (22)$$

for $j = 1, 2, 3, 4$. This means that the SLD must be simultaneously diagonalizable with all four experimental projectors, which gives four linear constraints to determine whether QFI saturation is possible.

Difficulties and rank deficit as the solution—The conditions in Eq. (22) require all eigenstates of $\hat{Q}^{\text{opt}}$, generically written as $a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle$, to be separable. This condition translates to a vanishing Schmidt rank, i.e., $ad - bc = 0$, which is a stringent constraint for a biparticle system produced at colliders.

In processes like $pp \rightarrow t\bar{t}$ [53], contributions from both quark- and gluon-initiated diagrams typically produce a full-rank mixed spin density matrix [10,55,56], a feature further reinforced by convolution with parton distribution functions (PDFs). In such cases, the SLD is uniquely fixed, and its eigenstates are generally entangled. Full reconstruction of $t\bar{t}$ kinematics might resolve the convolution of PDFs, but in general, achieving QFI saturation remains difficult.

TABLE I. Four optimal projective measurements for CPV Higgs decay and the $\pi^\pm$ directions are perpendicular to tau directions.

CPV h	θ_+	θ_-	ϕ_+	ϕ_-
$ E_1\rangle$	$\pi/2$	$\pi/2$	ϕ_1	ϕ_2
$ E_2\rangle$	$\pi/2$	$\pi/2$	$\phi_1 + \pi$	ϕ_2
$ E_3\rangle$	$\pi/2$	$\pi/2$	ϕ_1	$\phi_2 + \pi$
$ E_4\rangle$	$\pi/2$	$\pi/2$	$\phi_1 + \pi$	$\phi_2 + \pi$

However, in our perturbative setup, the rank deficiency of the spin density matrix ρ_0 alters the situation. Specifically, the SLD equation Eq. (5) remains unchanged under the addition of any operator in the null space of ρ_0 , see its general form in Supplemental Material [46] Sec. III. These components do not affect QFI, leaving $\hat{Q}^{\text{opt}}$ nonunique and allowing a family of solutions.

CPV Higgs decay—We consider the first scenario $h \rightarrow \tau^+ \tau^-$, which serves as a simple and illustrative, yet counter-intuitive example. The perturbative expansion of the spin density matrix gives

$$\rho_0 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (23)$$

with ranks 1 and 2, respectively. Both ρ_0 and ρ_1 are nonzero only in the central 2×2 block spanned by the spin states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$. It is thus natural to seek a solution for the SLD Q_h within this subspace. A straightforward choice for Q_h and the QFI are given as

$$Q_h = 2\rho_1, \quad F_q(\delta_h) = 4. \quad (24)$$

It is clear that two eigenstates of Q_h are entangled, so it cannot be implemented through separable projective measurements.

Remarkably, by exploiting the rank deficiency of ρ_0 , one can construct a modified SLD operator that does satisfy the separability requirements. The full solution is

$$Q_h^{\text{opt}} = \frac{2}{\sin \varphi} \begin{pmatrix} -\cos \varphi & 0 & 0 & e^{-i\Phi} \\ 0 & -\cos \varphi & e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} & -\cos \varphi & 0 \\ e^{i\Phi} & 0 & 0 & -\cos \varphi \end{pmatrix}, \quad (25)$$

with $\varphi \equiv \phi_1 - \phi_2$, $\Phi \equiv \phi_1 + \phi_2$, and $\phi_{1,2}$ being two free phase parameters. Q_h^{opt} clearly differs from Q_h , because no

 TABLE II. Four projective measurements for MDM, with pion directions along the $\hat{k}$ axis.

MDM	θ_+	θ_-	ϕ_+	ϕ_-
$ E_1\rangle = \downarrow\downarrow\rangle$	π	π	$\dots$	$\dots$
$ E_2\rangle = \downarrow\uparrow\rangle$	π	0	$\dots$	$\dots$
$ E_3\rangle = \uparrow\downarrow\rangle$	0	π	$\dots$	$\dots$
$ E_4\rangle = \uparrow\uparrow\rangle$	0	0	$\dots$	$\dots$

choice of $\phi_{1,2}$ can reduce it back to Q_h . The eigenstates $|E_i\rangle$ of Q_h^{opt} form a complete set of orthonormal projectors, corresponding to four pion configurations with fixed angular directions in the τ rest frames, summarized in Table I. In each case, the $\pi^\pm$ are emitted in the transverse plane, perpendicular to $\hat{k}$.

MDM and EDM of τ —For the process $e^+ e^- \rightarrow \tau^+ \tau^-$ at low-energy lepton colliders with unpolarized beams, the calculations of density matrices $\rho_{0,1}$ and QFI for MDM and EDM scenarios are detailed in Supplemental Material [46] Sec. IV.

For the MDM case, the SLD satisfying Eq. (22) is diagonal in the helicity basis. The corresponding projective measurements are the four orthonormal product states $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$. These are experimentally accessible by selecting pion momentum configurations aligned or antialigned with the τ momentum $\hat{k}$ in the rest frame. Notably, these directions do not have the twofold ambiguity for neutrinos. The four measurement settings are listed in Table II.

In contrast, for the EDM scenario, we find no SLD that satisfies Eq. (22). To systematically explore possible SLDs in both MDM and EDM cases, we proposed and utilized the amplitude-based method, which is equivalent to the matrix-based method, see Supplemental Material [46] Sec. III.

Approaching QFI in collider measurements—While projective measurements using the operators $\hat{E}_{1,2,3,4}$ can,

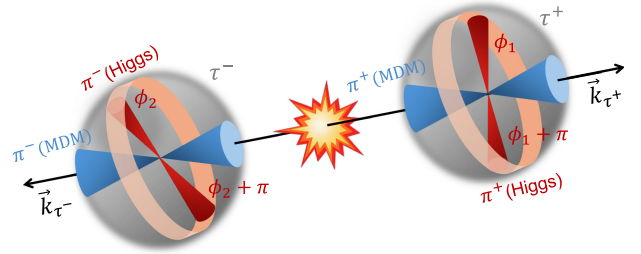


FIG. 1. Illustration of the optimal conical phase space for the MDM and Higgs cases. For the MDM case, the phase space includes four fixed $\pi^\pm$ direction configurations (Table II), each expanded into a cone of opening angle $\delta\theta_\pi$, denoted as Cone. For the Higgs case, the optimal directions vary with the free azimuthal parameters $\phi_{1,2}$, effectively forming a belt around the equator widened by $\delta\theta_\pi$, denoted as Belt.

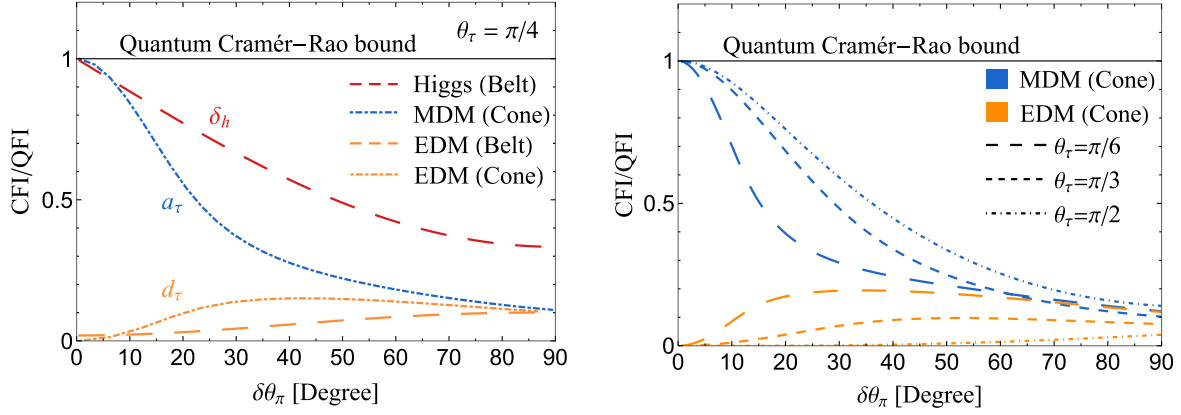


FIG. 2. CFI as a function of cone size $\delta\theta_\pi$ in restricted phase space, normalized to the corresponding QFI for MDM and Higgs cases. The EDM case is shown for comparison using the same restricted phase space. The τ^+ direction is fixed at polar angle $\theta_\tau = \pi/4$ for MDM and EDM cases in the left panel. In the right panel, θ_τ is fixed as $\pi/6$, $\pi/3$, and $\pi/2$, respectively.

in principle, saturate the QFI for the MDM and Higgs cases, the corresponding pion directions occupy zero-measure phase space, yielding no observable events. To overcome this, we define quasi-QFI measurements by expanding each optimal direction into a cone of angular size $\delta\theta_\pi$, recovering the ideal limit as $\delta\theta_\pi \rightarrow 0$. We compute the CFI within these cones using Eq. (13). The optimal conical phase space is illustrated in Fig. 1.

Figure 2 shows the calculated CFI for the MDM with different choice of θ_τ , the polar angle of τ , while the Higgs case is θ_τ independent. Both converge to the QFI as $\delta\theta_\pi \rightarrow 0$. This is analytically confirmed in Supplemental Material [46] Sec. V, which proves that the phase-space-restricted CFI asymptotically reaches the QFI. These results highlight that conventional CFI methods can achieve the quantum precision limit in collider settings. For the EDM case, no separable projective measurements can saturate the QFI, but its CFI and QFI are compared in the same way for completeness.

Except $\tau \rightarrow \pi\nu$ decay, other decay channels have reduced spin-analyzing power and correspond to nonprojective positive operator-valued measures (POVM). In these cases, the resulting CFI is strictly lower, and saturation of the QFI is not possible within our current framework, as shown in Supplemental Material [46] Sec. VA. We restrict to the projective case in this Letter and defer general POVM effects to future studies.

Discussions and conclusions—We developed a general framework to determine when collider experiments, limited to classical observables, can achieve the quantum precision limit set by the QFI. Using the $\tau^+\tau^-$ system as a benchmark, we showed that collider-accessible measurements correspond to projective operators onto separable spin states, due to the independent two decay amplitudes. QFI saturation requires that the SLD commute with all such separable projectors—a nontrivial condition made possible by the rank deficiency of the spin density matrix, which grants the SLD sufficient flexibility.

Applying this framework to three benchmark processes, we found that QFI saturation is achievable for the MDM and CP -violating Higgs cases, but only within restricted regions of phase space. For the EDM case, no such separable SLD exists. We also demonstrate that the conventional CFI method, when confined to suitable phase-space regions, can asymptotically approach the QFI. CP violation does not preclude QFI saturation, as seen in the Higgs example. While our main discussion considers τ pairs with any fixed $\hat{k}$, statistics from different $\hat{k}$ can be combined by averaging $F_q(\hat{k})$ with weight $\sigma^{-1}d\sigma/d\hat{k}$, which leaves our conclusions unchanged.

While our analysis focuses on the $\tau^+\tau^-$ system, the framework naturally extends to a broad class of biparticle systems in which spin information is reconstructed from decay kinematics. Examples include top-quark pairs (e.g., produced by e^+e^- collisions), baryon pairs (e.g., Δ baryons, hyperons, and heavy-flavor baryons), and gauge boson pairs (e.g., WW , ZZ), either produced directly or via heavy meson decays. Extensions to generalized POVMs—relevant for decays with reduced spin-analyzing power—will be explored in future work.

It is worth emphasizing that QFI saturation is achievable only in specific regions of collider phase space; full-phase-space measurements cannot saturate the QFI everywhere but can nevertheless yield greater overall precision because of their larger statistical power.

In this sense, the per-event QFI is a property of the underlying entangled production state and provides a measurement-independent benchmark above the CFI obtainable from any specific differential cross section. While this benchmark does not itself improve the sensitivity of an already-recorded data set with a fixed measurement strategy, it allows one to diagnose how close a given analysis is to the per-event quantum limit, to assess whether further methodological refinements are warranted given the required experimental resources, and to identify

decay modes and regions of phase space that are intrinsically most informative about the parameter of interest, thereby guiding future studies that incorporate realistic detector effects, backgrounds, and analysis choices.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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