

Tensor-Network Study of the Roughening Transition in a $(2+1)\text{D}$ $\mathbb{Z}_2$ Lattice Gauge Theory with Matter

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 (Received 26 March 2025; revised 3 May 2025; accepted 27 June 2025; published 17 July 2025)

Within the confined phase of $(2+1)\text{D}$ lattice gauge theories a roughening transition arises between a weakly confined regime with floppy string excitations and a strongly confined regime with stiff string excitations. In this Letter, we use an infinite Density Matrix Renormalization Group (iDMRG) algorithm to quantitatively characterize the properties of confined strings. To this end, we stabilize the state with a string excitation by 't Hooft loop operators. While for zero gauge-matter coupling we can use bare 't Hooft loop operators to do so, for finite gauge-matter coupling we have to transform them to emergent ones, which we achieve with an adiabatic protocol. By analyzing the scaling of both a novel order parameter and the entanglement entropy, our approach allows us to accurately determine the roughening transition, even at finite gauge-matter coupling.

DOI: 10.1103/3lxj-dx76

Introduction—Confinement is a fundamental phenomenon in physics [1] in which particles are bound together and form new composite particles. In condensed matter physics, $(1+1)\text{D}$ spin chains represent an illustrative example for confinement, where mesonic bound states are formed by associating an energy cost with domain-wall separation [2,3], leading to exotic nonequilibrium dynamics [4–9]. Confinement in $(1+1)\text{D}$ spin chains has been experimentally investigated with quantum simulators [10–14]. The situation is fundamentally distinct in higher dimensions, as the string that confines the charges also has transverse degrees of freedom [Fig. 1(a)]. Interestingly, depending on the strength of the confining potential, the width of the flux tube connecting the two charges is qualitatively distinct [15–17], which has recently been demonstrated in the string dynamics for a $(2+1)\text{D}$ lattice gauge theory on a quantum processor [18]. When considering eigenstates with string excitations, the transverse fluctuations of the string are suppressed for a strong linear confinement potential, resulting in a stiff flux tube with finite thickness. In contrast, for weak linear confining potential, the transverse fluctuations are strong, so that the string is floppy and the width of the flux tube diverges logarithmically with the distance between charges [15,19]. A roughening transition

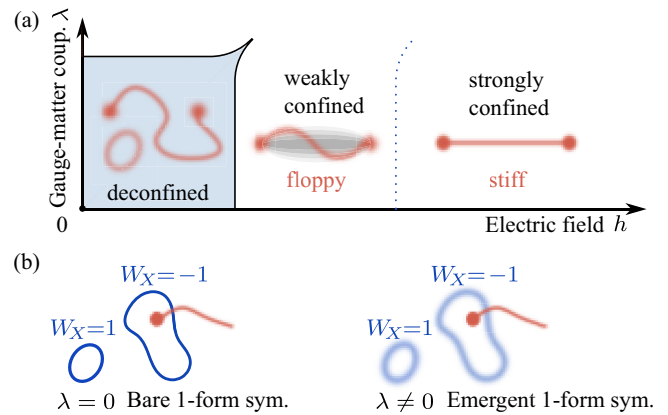


FIG. 1. Confinement of electric charges in a gauge theory. (a) Schematic phase diagram of the $\mathbb{Z}_2$ lattice gauge theory with Ising matter [23]. When the electric field is weak, the electric charges (red dots) are deconfined and the electric string (red lines) between them moves freely without energy cost. With the electric field increasing, there is a phase transition to the confined phase where the electric charges are bound. Within the confined phase a roughening transition arises (dotted line), which we show to bend to the right as the gauge-matter coupling increases. At intermediate electric field, the width of the electric flux tube diverges logarithmically with the distance between two electric charges and the confined electric string is floppy. For strong electric field the width of the electric flux tube is constant and the confined electric string is stiff. (b) When the gauge-matter coupling λ is zero, the 't Hooft loop W_X , measuring the electric charges, is a bare 1-form symmetry and commutes with the Hamiltonian. For finite gauge-matter coupling λ the 't Hooft loop W_X becomes an emergent 1-form symmetry.

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separates the two cases in the limit of infinitely long strings, which has been predicted for various lattice gauge theories [20–22].

For a pure $\mathbb{Z}_2$ lattice gauge theory with vanishing gauge-matter coupling, the roughening transition can be understood from a duality with the Ising model [24]. A string excitation in gauge theory leads to a domain-wall interface in the Ising model; the roughening transition of the pure $\mathbb{Z}_2$ gauge theory corresponds to the roughening of a domain-wall interface in the ferromagnetic phase of the Ising model, which has been intensively studied by analyzing the width of the interface [25–31]. However, since the roughening transition is a Berezinskii-Kosterlitz-Thouless (BKT) transition [17,32–34], which is notoriously hard to locate numerically, a quantitative way of determining the roughening transition point is still lacking. To study confinement in gauge theories, one detects electric charges and the string connecting them by 't Hooft loops, which can be interpreted as a generalized looplike 1-form symmetry [35–39]; see Fig. 1(b). However, for finite gauge-matter coupling, the 1-form 't Hooft loop symmetry becomes an emergent symmetry [40–46], and is a dressed loop operator that is not directly accessible. For the same reason, the duality to a simple Ising model breaks down, and one needs to develop novel methods to explore the roughening transition at finite gauge-matter couplings, which is a challenging problem and has not been investigated thus far.

In this work, we develop and utilize tensor network methods to study the roughening transition at finite gauge matter coupling. We use entanglement and symmetry to quantitatively characterize the roughening transition, distinct from previous qualitative observations based on the flux tube width or the so-called “kink mass” [47,48]. The effective model of the roughening transition is $(1+1)\text{D}$ and describes the transverse displacement of the 1D string excitation, although the lattice gauge theory is $(2+1)\text{D}$ [17,31]. Because of this dimensional reduction, we consider quasi-1D lattice gauge theories on cylinder geometries and find the infinite density matrix renormalization group (iDMRG) algorithm [49–52] to be well suited to study the roughening transition in $(2+1)\text{D}$ gauge theories. To access the roughening transition at finite gauge-matter coupling, we develop an adiabatic protocol by utilizing a sequence of iDMRG simulations and show that the floppy string is in fact stable against gauge-matter coupling. Our method provides a versatile computational tool for simulating and characterizing confinement, allowing us to locate the roughening transitions of $(2+1)\text{D}$ quantum lattice gauge theories with both zero and finite gauge-matter couplings.

$(2+1)\text{D}$ lattice gauge theory—We consider the Fradkin-Shenker model as a $\mathbb{Z}_2$ lattice gauge theory with the $\mathbb{Z}_2$ Ising matter [23,53,54]. By choosing the unitary gauge, the Ising spin matter degrees of freedom can be discarded and the Fradkin-Shenker model is simplified to

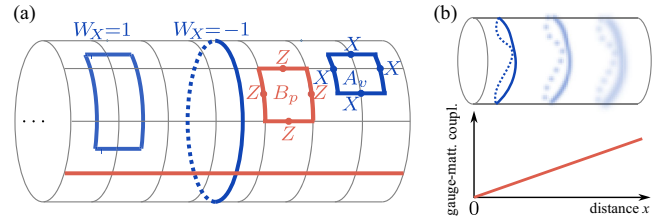


FIG. 2. Simulating an electric string of the $\mathbb{Z}_2$ lattice gauge theory on a cylinder using tensor networks. (a) Vertex operator (A_v), plaquette operator (B_p), bare 1-form 't Hooft loop operator on the dual lattice (blue loop), and electric string on the primal lattice (red line). (b) An infinite system-size DMRG (iDMRG) simulation with slowly increasing gauge-matter coupling λ can be interpreted as a spatially inhomogeneous system in which the gauge-matter coupling slowly increases with distance x . This process transforms the bare 't Hooft loop operator into an emergent 't Hooft loop operator and allows us to probe the roughening transition for finite gauge matter coupling.

the toric code in a field [55,56],

$$H = -J_E \sum_v A_v - J_M \sum_p B_p - h \sum_e X_e - \lambda \sum_e Z_e, \quad (1)$$

where $A_v = \prod_{e \in v} X_e$ and $B_p = \prod_{e \in p} Z_e$ are the vertex and plaquette operators of the toric code model [57], and X_e and Z_e are Pauli matrices on the edges e of the lattice; see Fig. 2(a). Here, $J_E > 0$ and $J_M > 0$ control the gaps of electric and magnetic excitations, respectively, h is the electric field, and λ the gauge-matter coupling (the matter field disappeared because of the unitary gauge). A schematic phase diagram of the model is shown in Fig. 1(a). When both h and λ are small, electric charges are deconfined. When h is large and λ is small, electric charges are confined.

For vanishing gauge-matter coupling $\lambda = 0$, the model has a bare 1-form 't Hooft loop symmetry $W_X = \prod_{e \in \hat{C}} X_e$, where $\hat{C}$ is a loop along the dual lattice, i.e., $[H, W_X] = 0$; see Fig. 2(a). The 1-form 't Hooft loop symmetry measures the $\mathbb{Z}_2$ electric charges surrounded by the loop $\hat{C}$. For finite gauge-matter coupling $\lambda \neq 0$, the bare 1-form 't Hooft loop symmetry is no longer a symmetry of H and the electric charges are measured by an emergent 1-form 't Hooft loop symmetry [40] challenging their numerical detection; see Fig. 1(b).

Characterizing the roughening transition for vanishing gauge matter coupling—We will first illustrate our numerical approach and introduce the observables for detecting the roughening transition for the pure lattice gauge theory ($\lambda = 0$). To obtain the eigenstate of H in Eq. (1) with an electric flux string, we calculate the ground state of the Hamiltonian: $\tilde{H} = H + J_W \sum_x W_X^{[y]}(x)$ using iDMRG [49–52], where $W_X^{[y]}(x)$ is a noncontractible 't Hooft loop operator wrapping in the y direction whose horizontal

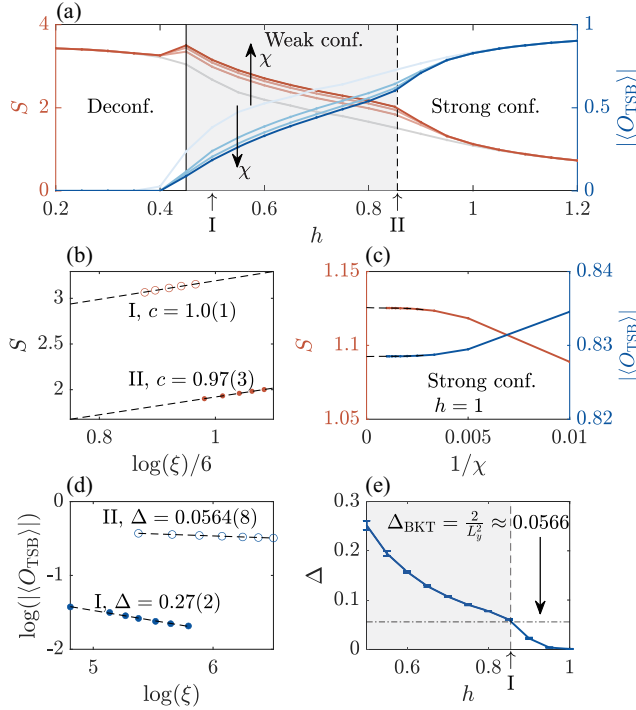


FIG. 3. Roughening transition for vanishing gauge-matter coupling ($\lambda = 0$). (a) Entanglement entropy S and translational symmetry breaking order parameter $|\langle O_{\text{TSB}} \rangle|$ obtained from iDMRG simulations with bond dimensions $\chi = 100, 400, 700, 1000$ and $L_y = 6$. (b) The central charge c extracted from the entanglement entropy as a function of the correlation length ξ at $h = 0.5$ (“I”) and at the estimated BKT transition point at $h_{\text{BKT}} \approx 0.855$ (“II”). (c) S and $|\langle O_{\text{TSB}} \rangle|$ as a function of χ at $h = 1$. The dashed lines represent extrapolations to $\chi \rightarrow \infty$. (d) Order parameter $|\langle O_{\text{TSB}} \rangle|$ as a function of ξ . (e) The extracted scaling dimension Δ of $|\langle O_{\text{TSB}} \rangle|$ as a function of h . The dash-dot line indicates the scaling dimension Δ_{BKT} of $|\langle O_{\text{TSB}} \rangle|$ at the roughening (BKT) transition point.

coordinate is x ; see Fig. 2(a). Since both $W_X^{[y]}(x)$ and A_v commute with H , the ground state of $\tilde{H}$ does not depend on J_W and J_E as long as they are sufficiently large.

As a first observable, we will compute the entanglement entropy S of the eigenstates obtained from iDMRG to detect the different behaviors of the strings at weak and strong confinement; see Fig. 3(a). We consider cylinders that are infinitely long in x direction and have a finite circumference $L_y = 6$ in y direction; results for larger systems are shown in Supplemental Material [58]. Analyzing the entanglement entropy S of the state obtained with different bond dimensions χ , we find that both for very small and very large electric field h , corresponding to the deconfined phase and the strongly confined regime, respectively, the entanglement S converges with bond dimension χ [see Fig. 3(c) for strong confinement]. In contrast, for intermediate electric field strength h , S increases with bond dimension, indicating a gapless intermediate phase. In particular, for a critical phase described by a conformal

field theory, we expect that the entanglement scales as $S = c \log(\xi)/6$, with c being the central charge and ξ the correlation length extracted from the state with a bond dimension χ . Performing finite entanglement scaling by extracting the entanglement entropy S and the correlation correlation length ξ of the states from the iDMRG simulations with various bond dimensions χ [64,65], we obtain $c \approx 1$, as shown in Fig. 3(b), which is the expected behavior for a BKT phase. The entanglement entropy thus clearly distinguishes between the strong and weak confinement. A related but qualitative observation from finite DMRG has been studied in a lattice $U(1)$ gauge theory [66]. Since at zero gauge-matter coupling, the $\mathbb{Z}_2$ gauge theory is dual to the Ising model, we demonstrate that we can also use the entanglement entropy of the Ising model to determine the weakly and strongly confined regimes [58].

To quantitatively locate the position of the BKT transition and characterize the BKT phase, we construct an order parameter based on the symmetry-breaking aspect of the roughening transition [16]. On a cylinder with a circumference L_y [see Fig. 2(a)], the Hamiltonian H in Eq. (1) has $\mathbb{Z}_{L_y}$ translational symmetry. In the strongly confined regime, there are L_y degenerate eigenstates with electric strings [the horizontal red line in Fig. 2(a)] located at $y = 0, 1, \dots, L_y - 1$; so the translation symmetry along the y direction is broken spontaneously. However, in the weakly confined regime a continuous translational symmetry emerges [16], giving rise to a BKT phase [67]. We can define such an order parameter $|\langle O_{\text{TSB}} \rangle|$ to detect the translational symmetry breaking, by requiring that O_{TSB} satisfies $\mathcal{T}^{[y]} O_{\text{TSB}} \mathcal{T}^{[y]\dagger} = \exp(2i\pi/L_y) O_{\text{TSB}}$ with $\mathcal{T}^{[y]}$ being the translation operator along the y direction. We choose

$$O_{\text{TSB}} = \sum_{e \in \hat{C}_y} \exp \left[i \frac{2\pi y(e)}{L_y} \right] \frac{1 - X_e}{2}, \quad (2)$$

where $y(e) = 0, 1, \dots, L_y - 1$ labels the y coordinate of the edge e and $\hat{C}_y$ is a vertical noncontractible loop on the dual lattice. The order parameter detects the spatial distribution of the electric string in the y direction by considering a nonzero Fourier mode of the distribution, and captures translational symmetry breaking. Since the roughening transition is a BKT transition accompanied by spontaneous $\mathbb{Z}_{L_y}$ symmetry breaking, the $\mathbb{Z}_{L_y}$ -deformed sine-Gordon model [69,70] is a well-suited effective field theory for this transition. From the $\mathbb{Z}_{L_y}$ -deformed sine-Gordon model, we derive the scaling dimension of O_{TSB} ; see Supplemental Material [58] and Table I. From the scaling dimensions, we then obtain the Luttinger parameter R (compactified radius of the boson field) in the BKT phase and locate the BKT transition point. For example, since the electric field

TABLE I. Scaling dimensions Δ in the gapless BKT phase and Δ_{BKT} at the roughening transition, where R is the Luttinger parameter, or equivalently the radius of a compactified free boson conformal field theory, and L_y is the cylinder circumference. The electric field $\sum_e X_e$ drives the roughening transition, O_{TSB} is the order parameter detecting the translational symmetry breaking in the strongly confined phase.

Operator	$X_e - \langle X_e \rangle$	O_{TSB}
Δ	L_y^2/R^2	$1/R^2$
Δ_{BKT}	2	$2/L_y^2$

drives the BKT transition, the corresponding field in the sine-Gordon model becomes marginally irrelevant with a scaling dimension 2 at the roughening transition. Based on this, we obtain the scaling dimensions of O_{TSB} at the BKT transition as $2/L_y^2$.

The results for $|\langle O_{\text{TSB}} \rangle|$ obtained from iDMRG simulations with various bond dimensions are shown in Fig. 3(a). In the deconfined phase $|\langle O_{\text{TSB}} \rangle| = 0$ while $|\langle O_{\text{TSB}} \rangle|$ converges to a nonzero value in the strongly confined regime. see inset of Fig. 3(c). For weak confinement, $|\langle O_{\text{TSB}} \rangle|$ slowly decreases with bond dimension χ . A finite correlation scaling of $|\langle O_{\text{TSB}} \rangle|$ using the correlation length ξ induced by the finite bond dimensions shows that $|\langle O_{\text{TSB}} \rangle|$ decays algebraically to zero with ξ in the weakly confining regime; see Fig. 3(d). The exponent of the algebraic decay corresponds to the scaling dimension Δ of $|\langle O_{\text{TSB}} \rangle|$, whose relation with the Luttinger parameter is displayed in Table I. We plot the extracted values of Δ as a function of h in Fig. 3(e) [71]. Since the scaling dimension of $|\langle O_{\text{TSB}} \rangle|$ is $2/L_y^2$ at the BKT transition (see Table I), we locate the BKT transition at $h_{\text{BKT}} \approx 0.855$ for $L_y = 6$ in Figs. 3(d) and 3(e) and $h_{\text{BKT}} \approx 0.865$ for $L_y = 8$ [58]. This is consistent with the expectation that with L_y increasing, h_{BKT} becomes slightly larger, because the smaller cylinder circumference suppresses the transverse fluctuation of the electric string. Note that our results do not quantitatively agree with previous estimates of $h_{\text{BKT}} \approx 0.59$ in Ref. [17] and $h_{\text{BKT}} \approx 0.72$ in Ref. [31] using the effective solid-on-solid model describing the confined string. A possible reason for the discrepancy is that the effective solid-on-solid model only accounts for the transverse fluctuation of the electric string, and the longitudinal fluctuation of the electric string and the fluctuation of the electric loops are ignored, yet their effects can be significant at weak confinement. Since we directly consider the original model, the effects of these fluctuations are accounted for. For pure gauge theories, the Wilson loop operator is also commonly used to detect the transition between confined and deconfined phases [24,54]. In Supplemental Material [58] we show, that the decay of Wilson loop operator correlations also distinguishes strong and weak confinement.

Roughening transition for finite gauge-matter coupling—When the gauge-matter coupling λ is nonzero, the bare 1-form 't Hooft loop operator is no longer a symmetry of the Hamiltonian H . Instead there exists an emergent 1-form 't Hooft loop symmetry which depends on the details of the model. It is difficult to find the explicit expression of an emergent 1-form symmetry. We thus develop an adiabatic protocol to approach the case of the emergent 1-form 't Hooft loop operator using iDMRG. For this, we first obtain the eigenstate with the electric string at zero gauge-matter coupling and then slowly ramp up the gauge-matter coupling while targeting the eigenstate with the electric string. We mimic the adiabatic evolution by sequential iDMRG simulations [72,73]; a technique which has been used to simulate charge pumping through adiabatic flux insertion in topological states [74]. The basic idea is that we consider a path given by a sequence of $\{\lambda_0, \dots, \lambda_i, \dots\}$ with $\lambda_0 = 0$ and a small interval $\epsilon = \lambda_{i+1} - \lambda_i$. For λ_0 , we calculate the eigenstate with the confined string using the Hamiltonian $\tilde{H} = H + J_W \sum_x W_X^{[y]}(x)$. Then, for λ_i with $i > 0$, we run iDMRG by initializing the iDMRG environment using the converged iDMRG environment at λ_{i-1} and the Hamiltonian $H(J_E, J_M, h, \lambda = \lambda_i)$. The technical details of the sequential iDMRG simulations are shown in Supplemental Material [58]. Since in iDMRG the system size effectively grows with the number of sweeps, the gauge-matter coupling is slowly increasing in x direction. When pulling through a 't Hooft loop operator on such a system, it changes from the bare 1-form symmetry to an emergent 1-form symmetry; see Fig. 2(b). By imposing the electric string using the bare 't Hooft loop operator at $\lambda = 0$, we obtain the eigenstate with the electric string at $\lambda \neq 0$. We can track the eigenstate of the effective iDMRG Hamiltonian with the maximum overlap of the eigenstate from the previous sweep, in analogy with the DMRG-X method [75,76], which further improves stability of the approach.

To investigate the robustness of the floppy string at finite gauge-matter coupling, we calculate the eigenstate with an electric string for $J_E = 2, J_M = 1, \lambda = 0.2$ and tune h using the sequential iDMRG simulations outlined above. The entanglement entropy S and the translational symmetry breaking order parameter $|\langle O_{\text{TSB}} \rangle|$ confirm the existence of the floppy string for intermediate electric field h ; see Fig. 4(a). The central charge c extracted from the states of the sequential iDMRG simulations is close to 1 in the weakly confined regime; see Fig. 4(b), confirming further that the weakly confined regime is robust against perturbations which explicitly break the exact 1-form 't Hooft loop symmetry and that it is still a critical BKT phase. In Fig. 4(c), at $h = 1$, both the entanglement entropy and the order parameter saturate to finite values as $\chi \rightarrow \infty$, indicating that the electric flux tube is strongly confined. In Figs. 4(d) and 4(e), we extract the scaling dimension of $|\langle O_{\text{TSB}} \rangle|$ as a function of h , yielding $h_{\text{BKT}}(\lambda = 0.2) \approx 0.875$, which is

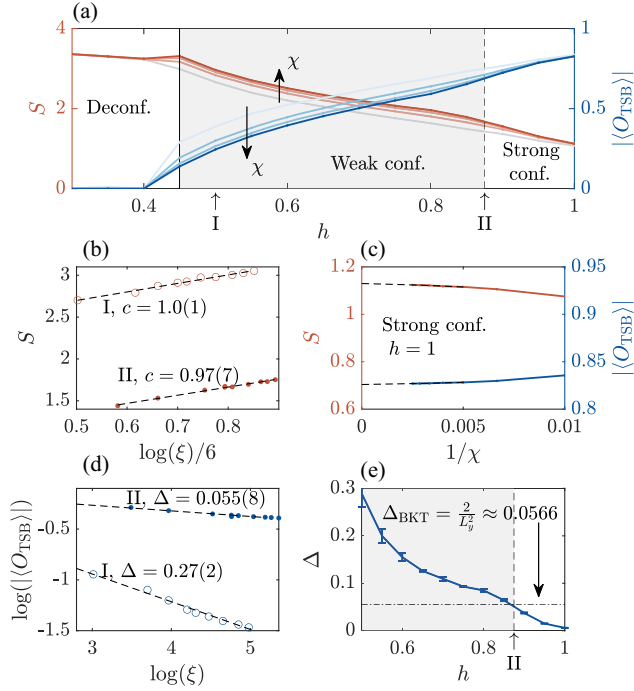


FIG. 4. Roughening transition for finite gauge-matter coupling. (a) Entanglement entropy S and translational symmetry breaking order parameter $|\langle O_{\text{TSB}} \rangle|$ obtained from iDMRG simulations with $\chi = 100, 200, \dots, 400$. (b) The central c extracted from the entanglement entropy as a function of the correlation length ξ at $h = 0.5$ (“I”) and at the estimated BKT transition point $h_{\text{BKT}} \approx 0.875$ (“II”). (c) S and $|\langle O_{\text{TSB}} \rangle|$ as a function of χ at $h = 1$. The dashed lines represent extrapolations to $\chi \rightarrow \infty$. (d) Order parameter $|\langle O_{\text{TSB}} \rangle|$ as function of ξ . We set $L_y = 6, \lambda = 0.2, J_M = 1, J_E = 2$. (e) The extracted scaling dimension Δ of $|\langle O_{\text{TSB}} \rangle|$ as a function of h . The dash-dot line indicates the scaling dimension Δ_{BKT} of $|\langle O_{\text{TSB}} \rangle|$ at the roughening (BKT) transition point.

slightly larger than $h_{\text{BKT}}(\lambda = 0) \approx 0.855$ at zero gauge-matter coupling. For $L_y = 8$, we find $h_{\text{BKT}}(\lambda = 0.2) \approx 0.88$; see Supplemental Material [58].

Conclusion and outlook—We developed and utilized tensor network based algorithms to characterize the roughening transition between a weakly and a strongly confined string excitation. We showed that sequential iDMRG simulations can be used to investigate the confinement at nonzero gauge-matter coupling, from which we find that the weakly confined regime is stable against the nonzero gauge-matter coupling. This enabled us to explore the dependence of the roughening transition point on the gauge-matter coupling. For gauge theories in continuous space it has been argued that the confined flux tube is always rough at any finite field h [16]. This suggests that in our lattice model the roughening transition point h_{BKT} increases with the gauge-matter coupling λ , as the width of the emergent 1-form symmetry grows with λ , effectively reducing the lattice spacing. At a critical value of the gauge-matter coupling, the emergent 1-form ’t Hooft loop

symmetry ceases to exist [42,46]. Near but below this threshold, the width of the 1-form symmetry is expected to become large pushing the roughening transition point to very large fields.

Our method can be applied to a large class of lattice gauge theories, as long as the relation between the emergent ’t Hooft loop operator and the exact ’t Hooft loop is known across the parameter space. For example, we investigate the $\mathbb{Z}_3$ gauge theory in Supplemental Material [58]. In future work, it will be exciting to study the weakly confined string connecting non-Abelian anyons [77–79]. Moreover, we emphasize that in a lattice gauge theory the roughening transition of the confined string depends on how the string is put on the lattice. For instance, when considering a zigzag string along the diagonal direction of a pure $\mathbb{Z}_2$ lattice gauge theory on a square lattice, there is no roughening string transition as Manhattan distance preserving moves do not cost energy [18,80]. By performing iDMRG on a helical lattice, one can interpolate between a zigzag string and a straight string which would be interesting to explore as well [47,48]. Furthermore, it has been predicted that the width of the flux tube at the bulk topological phase transition toward deconfinement may be modified [81,82]. Thus, a challenging problem is to extend our approach to study the electric string in the vicinity of the bulk topological transition, for instance with projected entangled pair states [83,84].

Acknowledgments—We thank Marcello Dalmonte, Torsten Zache, and Rui-Zhen Huang for insightful discussions. Calculations were performed using the TeNPy Library [73]. We acknowledge support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy–EXC-2111-390814868, TRR 360-492547816, and DFG Grants No. KN1254/1-2, No. KN1254/2-1, and No. FOR 5522 (Project ID 499180199), the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (Grant Agreement No. 851161), the European Union (Grant Agreement No. 101169765), as well as the Munich Quantum Valley, which is supported by the Bavarian state government with funds from the Hightech Agenda Bayern Plus.

Data availability—Data and codes are available upon reasonable request on Zenodo [85].

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