

Double Spacelike Collinear Limits from Multi-Regge Kinematics

Claude Duhr^{*,†}, Aniruddha Venkata[‡], and Chi Zhang (张驰)[‡]
 Bethe Center for Theoretical Physics, Universität Bonn, D-53115, Germany

 (Received 24 July 2025; revised 24 October 2025; accepted 19 November 2025; published 9 December 2025)

We study scattering amplitudes and form factors in planar $\mathcal{N} = 4$ super Yang-Mills theory in the limit where two pairs of gluons become collinear. We find that, when the virtualities of both collinear pairs are spacelike, the collinear factorization of the amplitude involves a generalized splitting amplitude that correlates the two collinear directions, confirming a recent proposal in the literature. Remarkably, we find that our generalized splitting amplitude agrees with the Bern-Dixon-Smirnov subtracted six-point amplitude in multi-Regge kinematics. The latter can be explicitly evaluated using integrability to all orders in the coupling. We also present compelling evidence for the universality of the generalized splitting amplitude by showing that the same function governs the double spacelike collinear limit of scattering amplitudes with up to eight particles and three loops and form factors with up to six particles and two loops.

DOI: 10.1103/3bxb-zfwc

Introduction—It is well-known that scattering amplitudes develop singularities whenever subsets of massless partons become collinear. From a phenomenological perspective, these collinear singularities govern the parton evolution in the initial and final states. In particular, the evolution of parton distribution functions (PDFs) in deep inelastic scattering is described by the celebrated Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equations [1–3], which resum large logarithms resulting from initial-state collinear singularities. The same PDFs appear in perturbative calculations of differential cross sections for the hadronic process producing a colorless final state X [4–6],

$$d\sigma_{h_1 h_2 \rightarrow X} = \sum_{i,j} \int dx_1 dx_2 f_{i/h_1}(x_1) f_{j/h_2}(x_2) d\hat{\sigma}_{ij \rightarrow X} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right), \quad (1)$$

where Q is the hard scale of the process and the sum runs over all parton species. The PDF $f_{i/h}(x)$ is the probability density for finding a parton i inside the hadron h carrying a momentum fraction x . The quantity $d\hat{\sigma}_{ij \rightarrow X}$ is the partonic differential cross section to produce the final state X from a collision of the parton species i and j . The latter quantity

exhibits infrared divergences stemming from initial-state collinear radiation, which are absorbed into the renormalization of the PDFs. The factorization formula in (1) is the basis for almost all predictions of observables at hadron colliders, like the Large Hadron Collider (LHC) at CERN. We stress that (1) has been strictly proven only for final states where all cuts are summed over and sufficient phase space integration has been carried out [4,7]. It is therefore important to understand possible limitations of (1). Since PDFs are closely tied to initial-state collinear singularities, it is essential to understand how scattering amplitudes factorize in the collinear limit, especially at higher orders in perturbation theory.

In this Letter, we study the collinear limit of a scattering amplitude $\mathcal{A}_N$ for N massless colored partons. It is well-known that if a subset of partons with momenta p_i and helicities h_i , $1 \leq i \leq m < N$, become collinear to a light-like direction P , then the amplitude factorizes as

$$\mathcal{A}_N \sim \sum_{c, h_P} \mathbf{Sp}_{h_P}^{(m)} \mathcal{A}_{N-m+1}, \quad (2)$$

where $\sim$ indicates that we neglect terms that are power suppressed in the limit. The right-hand side involves the hard scattering amplitude $\mathcal{A}_{N-m+1}$, where the m collinear partons have been replaced by a single parton with momentum P , helicity h_P , and color c [8]. $\mathbf{Sp}_{h_P}^{(m)}$ is an operator, called a *splitting amplitude*, that acts on the color space of the hard amplitude. It only depends on the momenta $p_1, \dots, p_m$ from the collinear set. Moreover, if the splitting is timelike [meaning that $(p_1 + \dots + p_m)^2 > 0$], then also the color degrees of freedom factorize, and $\mathbf{Sp}_{h_P}^{(m)}$ only depends on the color charges of the collinear particles (cf. [9,10] for a proof of the factorization). For spacelike splittings, however,

^{*}Contact author: cduhr@uni-bonn.de

[†]Contact author: avenkata@uni-bonn.de

[‡]Contact author: czhang@uni-bonn.de

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](#). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

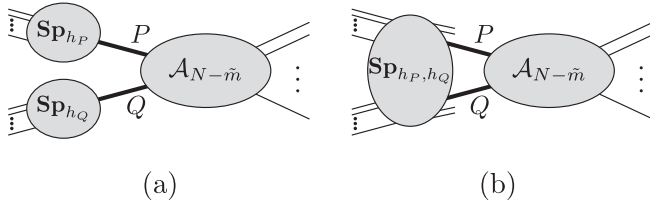


FIG. 1. Timelike (a) versus spacelike (b) collinear factorization where $\tilde{m} = N - m - m' + 2$. The two heavy lines represent the parent partons with momenta and helicities (P, h_P) and (Q, h_Q) , respectively. Nearly parallel lines on the same side of the bubble correspond to the timelike collinear limit, while those on opposite sides correspond to the spacelike collinear limit.

the color degrees of freedom may not completely factorize, and $\mathbf{Sp}_{h_P}^{(m)}$ may depend on the color charges of the particles [11–13]. Splitting amplitudes are known up to $m = 4$ at tree level, $m = 3$ up to one loop, and for $m = 2$ up to three loops [12–17].

So far the discussion was restricted to the case of a subset of partons becoming collinear to the same direction P . In the case of multiple timelike collinear directions, the factorization involves a product of splitting amplitudes [cf. Fig. 1(a)], one for each collinear direction. It was recently proposed that, in the case of two spacelike collinear directions P and Q , collinear factorization involves a *generalized splitting amplitude* that depends on the combined set of quantum numbers from both collinear sets [18] [cf. Fig. 1(b)],

$$A_N \sim \sum_{c, h_P, c', h_Q} \mathbf{Sp}_{h_P, h_Q}^{(m_1, m_2)} A_{N-m_1-m_2+2}, \quad (3)$$

where $\mathbf{Sp}_{h_P, h_Q}^{(m_1, m_2)}$ is the generalized splitting amplitude from [18]. Equation (3) implies that the two spacelike collinear clusters do not completely factorize from each other, but there are correlations involving both color and kinematics between the two collinear sets. In particular, (3) may lead to a breakdown of (1) for some observables, because (1) does not account for collinear singularities that correlate the two incoming beams.

Up to now there has been no study of generalized splitting amplitudes. It is easy to see that correlations between two spacelike collinear sets require loop amplitudes with many legs. In a general gauge theory, we do currently not have access to results for amplitudes with more than five legs and two loops. The situation is very different in the planar limit of the maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills (SYM) theory, where analytic results for amplitudes with many loops and legs are available (cf., e.g., [19–25]). In the remainder of this Letter we study scattering amplitudes in planar $\mathcal{N} = 4$ SYM in the limit where two spacelike pairs of massless particles become collinear. This will allow us to test for the very first time the proposal of [18]. Moreover, we will show

that in this case there is a duality between the double spacelike (DSL) collinear limit and another kinematic limit, namely the so-called multi-Regge limit of the six-point amplitude. Remarkably, the latter can be described to all order from integrability [26], and the duality between the two limits implies that we can compute our generalized splitting amplitude to all orders in the coupling.

Amplitudes in planar $\mathcal{N} = 4$ SYM—Amplitudes in planar $\mathcal{N} = 4$ SYM are well studied (see Ref. [27] for a review). Many analytic results for multileg multiloop amplitudes are available, due to the large number of symmetries of the theory. Besides the usual superconformal symmetry, the S matrix of planar $\mathcal{N} = 4$ SYM is invariant under a dual superconformal symmetry [28], which closes with the ordinary superconformal symmetry into a Yangian algebra [29]. It is convenient to consider superamplitudes, where each external state is characterized by a lightlike momentum p_i , an adjoint color index a_i , and a Grassmann variable η_i transforming in the fundamental representation of the R -symmetry group $SU(4)$ [30,31]. In the planar limit only single-trace color structures contribute, so that the superamplitude reads

$$A_N = \sum_{\sigma \in S_N / \mathbb{Z}_N} \text{Tr}(T^{a_{\sigma_1}} \cdots T^{a_{\sigma_N}}) A_N(\Lambda_{\sigma_1}, \dots, \Lambda_{\sigma_N}), \quad (4)$$

where T^a are the generators of the gauge group $SU(N_c)$, A_N are the partial amplitudes that depend on the cyclic ordering of the particles, and $\Lambda_i := (p_i, \eta_i)$, but are independent of the color. As a consequence of dual (super-) conformal symmetry, the partial amplitude can be cast in the form [32]

$$A_N(\Lambda_N) = A_{\text{MHV}}^{\text{tree}}(\Lambda_N) A_N^{\text{BDS}} R_N(\Lambda_N), \quad (5)$$

where $\Lambda_N := (\Lambda_1, \dots, \Lambda_N)$, $A_{\text{MHV}}^{\text{tree}}$ is the maximally helicity violating (MHV) tree superamplitude, and A_N^{BDS} is the Bern-Dixon-Smirnov (BDS) amplitude, defined as the exponential of the one-loop MHV amplitude (divided by the tree level) weighted by the cusp and collinear anomalous dimensions [33]. The BDS amplitude captures infrared singularities to all orders in perturbation theory, so that the *BDS-subtracted amplitude* R_N is finite and dual conformally invariant [32,34]. In particular, its kinematic dependence can only be through dual conformally invariant cross ratios,

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}, \quad x_{ij} = x_i - x_j = p_i + \cdots + p_{j-1}, \quad (6)$$

where the x_i are the dual coordinates; therefore, R_N is nontrivial only for $N \geq 6$. Explicit analytic results for BDS-subtracted amplitudes are available both for MHV and next-to-MHV (NMHV) for $N = 6$ up to seven and six loops [19], for $N = 7$ up to four loops [20,21], and for $N = 8$ up to three and two loops [23,25], respectively.

The symbols of all master integrals for all two-loop amplitudes in $\mathcal{N} = 4$ SYM have been computed in [22,35].

Let us briefly comment on the behavior of A_N in the limit where a pair of particles i and j become collinear. First of all, it is easy to see that the leading power contribution only occurs when two collinear particles are adjacent in the color ordering. Moreover, since R_N is unity or an R invariant [36] for $N \leq 5$, it follows that the two-particle splitting amplitude is entirely determined by the BDS ansatz [33,37]. For example, in the collinear limit $p_1 \parallel p_2$, an MHV component amplitude factorizes as

$$A_5^{\text{MHV}}(p_1^{h_1}, p_2^{h_2}, \dots) \sim \text{Sp}_+^{(2)}(p_1^{h_1}, p_2^{h_2}; \xi) A_4(P^{h_P}, \dots), \quad (7)$$

where the subscript on Sp now denotes its MHV degree (which can be $\pm$, requiring $h_1 + h_2 - h_P = \pm 1$), and ξ is the fraction of the parent momentum P carried by p_1 , i.e., $p_1 = \xi P$ in the strict collinear limit. Since the splitting amplitude is completely determined by $N = 5$, the BDS-subtracted amplitude R_6 must tend to its tree value in all two-particle collinear limits. Using the fact that amplitudes in this theory are dual to Wilson loops [38], it is possible to compute a systematic expansion of the BDS-subtracted amplitudes around collinear limits. The quantities that control this near-collinear expansion can be computed to all orders using tools from integrability [39].

Generalized splitting amplitude—We now discuss the main result of this Letter, namely the generalized splitting amplitude for two adjacent spacelike collinear pairs, say $p_1 \parallel p_2$ and $p_3 \parallel p_4$, in planar $\mathcal{N} = 4$ SYM to all orders in the 't Hooft coupling constant a . Note that the generalized splitting amplitudes, which require $P \cdot Q := (p_1 + p_2) \cdot (p_3 + p_4) \neq 0$, appear starting from $N = 6$.

Before presenting the result, we first clarify what is meant by “spacelike.” Amplitudes in planar $\mathcal{N} = 4$ SYM are typically computed in the Euclidean region where all invariants x_{ij}^2 are negative and the amplitude is real. We refer to the region where all invariants are timelike, i.e., positive, as the *timelike Euclidean region*. It is identical to the Euclidean region up an overall phase (and the BDS-subtracted amplitude is independent of this phase). In the following, the term “spacelike” refers to analytic continuation in some two-particle invariants $x_{i,i+2}^2$ from the timelike Euclidean region to the region where these invariants are negative.

We are now in a position to state our claim about the generalized splitting amplitude, which consists of two points: (i) The splitting amplitudes factorize in *all but two* distinct double spacelike collinear regions, which, in addition to satisfying $s_{12} = x_{13}^2 < 0$ and $s_{34} = x_{35}^2 < 0$, require

$$(I): P \cdot Q > 0, s_{23} = x_{24}^2 > 0, \xi_1 > 1, \xi_2 > 1, \quad (8)$$

$$(II): P \cdot Q < 0, s_{23} = x_{24}^2 < 0, \xi_1 > 1, \xi_2 > 1, \quad (9)$$

respectively, and ξ_1 and ξ_2 are the momentum fractions of p_1 and p_4 in the collinear limit. (ii) Although the (component) amplitudes in these two regions do not factorize, they are compensated by *universal* corrections,

$$A_N \sim \sum_{h,h'} \text{Sp}_h^{(2)}(-p_1, p_2; \xi_1 + i\epsilon) \text{Sp}_{h'}^{(2)}(\pm p_3, \mp p_4; \xi_2 + i\epsilon) \times \Delta_{I/II}^{h,h'}(\tau, z) A_{N-2}(P^{h_P}, Q^{h_Q}, \dots), \quad (10)$$

where two choices of signs correspond to two regions, with the $+$ or $-$ sign of the momentum indicating that it is outgoing or incoming, respectively. Here, $\text{Sp}_h^{(2)}$ is the two-particle splitting amplitude defined in Eq. (7), which is determined by the BDS amplitude. The new ingredient $\Delta_{I/II}^{h,h'}$ receives contributions from the BDS-subtracted amplitudes and is completely determined by the multi-Regge limit of the six-point BDS-subtracted amplitude R_6 , as we will demonstrate below.

Six-point case: To analyze the amplitudes in the collinear regions (8) and (9), we need to first analytically continue the amplitudes to the kinematic regions defined in [40,41] (where they are called *Mandelstam regions*) containing these collinear regions, and then take x_{13}^2 and x_{35}^2 to 0 at the same rate (see Supplemental Material [42]). For simplicity, we present only the analysis for region (8) here, as other regions follow similarly.

For general N , there may be multiple regions containing the collinear region of interest. However, there is only one physical region containing region (8) for $N = 6$, corresponding to the scattering $p_1 p_4 \rightarrow p_2 p_3 p_5 p_6$:

$$x_{24}^2 > 0, \quad x_{15}^2 > 0, \quad \text{and all other } x_{ij}^2 < 0. \quad (11)$$

There are three independent conformally invariant cross ratios for $N = 6$:

$$u_1 = u_{1346}, \quad u_2 = u_{2451}, \quad u_3 = u_{3562}. \quad (12)$$

The DSL collinear limit then corresponds to analytically continuing $u_2 \rightarrow u_2 e^{2\pi i}$ and then taking the limit $(u_1, u_2, u_3) \rightarrow (0, 1, 0)$, with the following ratios held fixed:

$$\frac{u_1}{1 - u_2} = \frac{z\bar{z}}{(1 - z)(1 - \bar{z})}, \quad \frac{u_3}{1 - u_2} = \frac{1}{(1 - z)(1 - \bar{z})}. \quad (13)$$

Remarkably, at the level of the cross ratios, the DSL collinear limit is *identical* to the multi-Regge limit for the six-point BDS-subtracted amplitude, where the produced particles are strongly ordered in rapidity. It was first observed in [40] that the BDS-subtracted amplitude R_6 in the Euclidean region becomes trivial in multi-Regge

kinematics (MRK), but the result is nontrivial if continued to the Riemann sheet defined by (11) due to the presence of a Regge cut. Since both the analytic continuation and the limit agree for the DSL collinear limit and MRK at the level of cross ratios, we can immediately identify the generalized splitting amplitude $\Delta_I^{h,h'}(\tau := \sqrt{u_1 u_3}, z)$ in Eq. (10) with various six-point BDS-subtracted amplitudes in MRK.

More precisely, we find that $\Delta_I^{++} = \Delta_I^{--}$ can be obtained from the MHV BDS-subtracted amplitude R_6^{MHV} in MRK [40,41,43–45], and $\Delta_I^{+-} = \Delta_I^{+}|_{z \leftrightarrow \bar{z}}$ from the NMHV BDS-subtracted amplitude R_6^{NMHV} in MRK [46,47] (for more details, see Supplemental Material [42]). For example, from an MHV configuration in $2 \rightarrow 4$ kinematics, we have (cf. [48])

$$\begin{aligned} \Delta_I^{++}(\tau, z) &= e^{i\delta_6} \mathcal{M}(R_6^{\text{MHV}}) = \cos \pi \omega_{\text{ab}} \\ &+ \frac{ia}{2} \sum_{n=-\infty}^{+\infty} (-1)^n \left(\frac{z}{\bar{z}} \right)^{n/2} \\ &\times \mathcal{P} \int_{-\infty}^{+\infty} \frac{|z|^{2i\nu} \Phi_{\nu n}^{\text{reg}} d\nu}{\nu^2 + n^2/4} (-\tau)^{-\omega_{\nu n}}, \quad (14) \end{aligned}$$

where $\mathcal{P}$ represents the principal value integral, and $\mathcal{M}(R)$ indicates that the function R is evaluated in MRK. Here $\omega_{\text{ab}} = \frac{1}{8} \gamma_K(a) \log |z|^2$ is the Regge pole contribution, and $\gamma_K(a)$ is the cusp anomalous dimension. The remaining contribution takes the form of a Fourier-Mellin transform and involves the Balitsky-Fadin-Kuraev-Lipatov (BFKL) eigenvalue $\omega_{\nu n}$ and the (regularized) impact factor $\Phi_{\nu n}^{\text{reg}}$. Similarly, $\Delta_{\text{II}}(\tau, z)$ for the region (9) is determined by $e^{-i\delta_6} \mathcal{M}(R_6^{\text{MHV}})$ in $3 \rightarrow 3$ kinematics [26,43,49], which has an expression similar to Eq. (14).

Our analysis reveals a duality between the DSL collinear limit and MRK in planar $\mathcal{N} = 4$ SYM. The fact that the BDS-subtracted amplitude vanishes in Euclidean MRK corresponds to the fact that the amplitude factorizes into two-particle splitting amplitudes in the double timelike collinear limit. Correspondingly, the nonfactorizing contributions in the DSL collinear limit can be attributed to the same Glauber gluon exchange [6,50] described by the Fourier-Mellin integral in Eq. (14). We note that both kinematic regimes explore aspects of forward scattering. Nevertheless, the two regimes are distinct, as no rapidity ordering is imposed in the collinear limit, making the duality between the collinear limit and MRK nontrivial.

As a consequence of the conformal equivalence of the two limits, we can directly use knowledge about R_6 in MRK and apply it to the DSL collinear limit. In particular, we see that $\Delta_{\text{I/II}}$ is completely determined by the cusp anomalous dimension γ_K , the BFKL eigenvalue $\omega_{\nu n}$, and the impact factor $\Phi_{\nu n}^{\text{reg}}$, where the latter two quantities have been extracted to high orders from fixed-order computations [43,44,51,52]. More importantly, since all three quantities are known to all orders from integrability [26,53]

(at least conjecturally), we conclude that also $\Delta_{\text{I/II}}$ is fixed to all orders in the 't Hooft coupling. It is known how to systematically evaluate the Fourier-Mellin transform in Eq. (14) order by order in terms of Brown's single-valued harmonic polylogarithms [54–56], and so we conclude that the same statement must hold for $\Delta_{\text{I/II}}$.

Universality of generalized splitting amplitudes: At this point, we would like to check that $\Delta_{\text{I/II}}$ is universal. It is independent of the hard scattering and is the same function for all higher-point amplitudes and form factors. We now give compelling evidence that it is.

For the higher-point amplitudes, the main difference to the six-point case is that there may be several physical regions compatible with either DSL collinear region (8) or (9). For instance, physical regions compatible with both collinear regions can have either $x_{61}^2 > 0$ or $x_{61}^2 < 0$ for $N = 7$. Through analytic continuation to various compatible regions (cf. [57]), we have performed explicit computations of the DSL collinear limits for the symbols of seven- and eight-point MHV and NMHV amplitudes up to three loops [20,25,38,58]. Remarkably, we find that all these amplitudes exhibit the same pattern as in (10), where τ and z are defined as in the six-point case with p_6 replaced by p_N .

The four-point MHV $\text{Tr}\phi^2$ form factor is the first to exhibit nontrivial DSL collinear limits. This form factor has been computed through three-loop order [59,60], while the five- and six-point MHV form factors have been computed through two-loop order [61]. We have verified that all these form factors exhibit the same DSL collinear limit behavior [with the obvious identification $x_{13}^2 = s_{12} := (p_1 + p_2)^2$, $x_{24}^2 = s_{23}$, etc.] as the amplitudes. In particular, the four-point form factor fails to factorize not only in regions I and II described in (8) and (9), but also in regions I' and II', which are related to regions I and II via $\xi_i \rightarrow 1 - \xi_i$, due to an additional symmetry. Nevertheless, the form factor exhibits the same pattern as in Eq. (10) in all aforementioned regions with the correct identification of τ and z .

Based on our explicit computations, we conjecture that the generalized splitting amplitude is universal, and that the function $\Delta_{\text{I/II}}$ governs the behavior of any quantity in planar $\mathcal{N} = 4$ SYM in the limit where two spacelike pairs of particles become collinear.

Conclusions—In this Letter we have obtained for the first time results for high-loop scattering amplitudes in the collinear limit where two spacelike pairs of massless particles become simultaneously collinear, and we have obtained an analytic expression for a generalized splitting amplitude, first introduced in [18].

We have focused our Letter on the planar $\mathcal{N} = 4$ SYM theory. We have uncovered a duality between the DSL collinear limit and MRK for the six-point BDS-subtracted amplitude. MRK amplitudes in planar $\mathcal{N} = 4$ SYM have been studied extensively, both for $N = 6$ [26,40,41,43–46,49,51,52,62–65] and beyond [48,57,66–80]. Our duality allows us to immediately apply known results for the

six-point BDS-subtracted amplitude in MRK to obtain the generalized splitting amplitude $\Delta_{I/II}$. The universality of the DSL collinear limit can be used to turn the MRK results for R_6 into constraints for the a future bootstrap program for higher-point amplitudes and form factors. It would be interesting to understand if there are also kinematic regimes for $N > 6$ that are dual to the BDS-subtracted amplitudes R_N in MRK, or if the DSL collinear limit considered here is connected to the self-crossing limits for polygonal Wilson loops considered in [81–83], which have an overlap with MRK.

Our findings confirm the expectations of [18] about the structure of the DSL collinear limit. Unlike other sources of collinear factorization breaking considered in the literature [11,12], this correlation between different spacelike collinear directions contributes already at leading color. While our analysis was restricted to the planar $\mathcal{N} = 4$ SYM theory, this is sufficient to establish the existence of correlations between multiple spacelike collinear directions at leading color.

It is interesting to ask what possible phenomenological implications of our findings could be, in particular for the factorization of observables in hadronic collisions into PDFs multiplied by partonic cross sections. Since the factorization breaking enters through the imaginary part at one loop, it can contribute to cross-section calculations for the first time at N^4 LO through the square of the one-loop amplitude with two initial-state jets collinear to the beam directions. It would be interesting to see if this nonfactorizable contribution leaves an imprint in some (differential) observables after phase integration.

Acknowledgments—We are grateful to Zhenjie Li for his instruction regarding the higher-point form factors and to Lance Dixon for his comments on the manuscript. This research was supported by the European Research Council (ERC) under the European Union’s research and innovation programme Grant Agreement No. 101043686 (ERC Consolidator Grant LoCoMotive).

Views and opinions expressed are, however, those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

Data availability—No data were created or generated in this study.

-
- [1] V. N. Gribov and L. N. Lipatov, Deep inelastic e-p scattering in perturbation theory, *Sov. J. Nucl. Phys.* **15**, 438 (1972).
 - [2] Y. L. Dokshitzer, Calculation of the structure functions for deep inelastic scattering and e^+e^- annihilation by perturbation theory in quantum chromodynamics., *Sov. Phys. JETP* **46**, 641 (1977).

- [3] G. Altarelli and G. Parisi, Asymptotic freedom in parton language, *Nucl. Phys.* **B126**, 298 (1977).
- [4] J. C. Collins, D. E. Soper, and G. F. Sterman, Soft gluons and factorization, *Nucl. Phys.* **B308**, 833 (1988).
- [5] J. C. Collins, D. E. Soper, and G. F. Sterman, Factorization of hard processes in QCD, *Adv. Ser. Dir. High Energy Phys.* **5**, 1 (1989).
- [6] J. Collins, *Foundations of Perturbative QCD*, Vol. 32. (Cambridge University Press, Cambridge, England, 2011).
- [7] S. B. Libby and G. F. Sterman, Jet and lepton pair production in high-energy lepton-hadron and hadron-hadron scattering, *Phys. Rev. D* **18**, 3252 (1978).
- [8] The flavor of the parton is fixed uniquely from the flavors of the m collinear partons.
- [9] D. A. Kosower, All order collinear behavior in gauge theories, *Nucl. Phys.* **B552**, 319 (1999).
- [10] I. Feige and M. D. Schwartz, Hard-soft-collinear factorization to all orders, *Phys. Rev. D* **90**, 105020 (2014).
- [11] S. Catani, D. de Florian, and G. Rodrigo, Space-like (versus time-like) collinear limits in QCD: Is factorization violated?, *J. High Energy Phys.* **07** (2012) 026.
- [12] J. Henn, R. Ma, Y. Xu, K. Yan, Y. Zhang, and H. X. Zhu, Two-loop spacelike splitting amplitude for $N = 4$ Super-Yang-Mills theory, *Phys. Rev. D* **112**, 076003 (2025).
- [13] X. Guan, F. Herzog, Y. Ma, B. Mistlberger, and A. Suresh, Splitting amplitudes at N^3 LO in QCD, *J. High Energy Phys.* **01** (2025) 090.
- [14] S. Catani, D. de Florian, and G. Rodrigo, The triple collinear limit of one loop QCD amplitudes, *Phys. Lett. B* **586**, 323 (2004).
- [15] C. Duhr, T. Gehrmann, and M. Jaquier, Two-loop splitting amplitudes and the single-real contribution to inclusive Higgs production at N^3 LO, *J. High Energy Phys.* **02** (2015) 077.
- [16] S. Badger, F. Buciumi, and T. Peraro, One-loop triple collinear splitting amplitudes in QCD, *J. High Energy Phys.* **09** (2015) 188.
- [17] M. Czakon and S. Sapeta, Complete collection of one-loop triple-collinear splitting operators for dimensionally-regulated QCD, *J. High Energy Phys.* **07** (2022) 052.
- [18] L. Cieri, P. K. Dhani, and G. Rodrigo, Catani’s generalization of collinear factorization breaking, [arXiv:2402.14749](https://arxiv.org/abs/2402.14749).
- [19] S. Caron-Huot, L. J. Dixon, F. Dulat, M. von Hippel, A. J. McLeod, and G. Papathanasiou, Six-gluon amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory at six and seven loops, *J. High Energy Phys.* **08** (2019) 016.
- [20] L. J. Dixon, J. Drummond, T. Harrington, A. J. McLeod, G. Papathanasiou, and M. Spradlin, Heptagons from the Steinmann cluster bootstrap, *J. High Energy Phys.* **02** (2017) 137.
- [21] J. Drummond, J. Foster, O. Gürdoğan, and G. Papathanasiou, Cluster adjacency and the four-loop NMHV heptagon, *J. High Energy Phys.* **03** (2019) 087.
- [22] S. Caron-Huot, Superconformal symmetry and two-loop amplitudes in planar $N = 4$ super Yang-Mills, *J. High Energy Phys.* **12** (2011) 066.
- [23] S. He, Z. Li, and C. Zhang, Two-loop octagons, algebraic letters and $\bar{Q}$ equations, *Phys. Rev. D* **101**, 061701 (2020).
- [24] S. He, Z. Li, and C. Zhang, The symbol and alphabet of two-loop NMHV amplitudes from $\bar{Q}$ equations, *J. High Energy Phys.* **03** (2021) 278.

- [25] Z. Li and C. Zhang, The three-loop MHV octagon from $\bar{Q}$ equations, *J. High Energy Phys.* **12** (2021) 113.
- [26] B. Basso, S. Caron-Huot, and A. Sever, Adjoint BFKL at finite coupling: A short-cut from the collinear limit, *J. High Energy Phys.* **01** (2015) 027.
- [27] N. Arkani-Hamed, L. J. Dixon, A. J. McLeod, M. Spradlin, J. Trnka, and A. Volovich, Solving scattering in $N = 4$ Super-Yang-Mills theory, [arXiv:2207.10636](https://arxiv.org/abs/2207.10636).
- [28] J. M. Drummond, J. Henn, V. A. Smirnov, and E. Sokatchev, Magic identities for conformal four-point integrals, *J. High Energy Phys.* **01** (2007) 064.
- [29] J. M. Drummond, J. M. Henn, and J. Plefka, Yangian symmetry of scattering amplitudes in $N = 4$ super Yang-Mills theory, *J. High Energy Phys.* **05** (2009) 046.
- [30] V. P. Nair, A current algebra for some gauge theory amplitudes, *Phys. Lett. B* **214**, 215 (1988).
- [31] N. Arkani-Hamed, F. Cachazo, and J. Kaplan, What is the simplest quantum field theory?, *J. High Energy Phys.* **09** (2010) 016.
- [32] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Dual superconformal symmetry of scattering amplitudes in $N = 4$ super-Yang-Mills theory, *Nucl. Phys.* **B828**, 317 (2010).
- [33] Z. Bern, L. J. Dixon, and V. A. Smirnov, Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond, *Phys. Rev. D* **72**, 085001 (2005).
- [34] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu, and A. Volovich, The two-loop six-gluon MHV amplitude in maximally supersymmetric Yang-Mills theory, *Phys. Rev. D* **78**, 045007 (2008).
- [35] A. Spiering, M. Wilhelm, and C. Zhang, All planar two-loop amplitudes in maximally supersymmetric Yang-Mills theory, *Phys. Rev. Lett.* **134**, 071602 (2025).
- [36] L. J. Mason and D. Skinner, Dual superconformal invariance, momentum twistors and grassmannians, *J. High Energy Phys.* **11** (2009) 045.
- [37] C. Anastasiou, Z. Bern, L. J. Dixon, and D. A. Kosower, Planar amplitudes in maximally supersymmetric Yang-Mills theory, *Phys. Rev. Lett.* **91**, 251602 (2003).
- [38] S. Caron-Huot, Notes on the scattering amplitude / Wilson loop duality, *J. High Energy Phys.* **07** (2011) 058.
- [39] B. Basso, A. Sever, and P. Vieira, Spacetime and flux tube S-matrices at finite coupling for $N = 4$ supersymmetric Yang-Mills theory, *Phys. Rev. Lett.* **111**, 091602 (2013).
- [40] J. Bartels, L. N. Lipatov, and A. Sabio Vera, BFKL Pomeron, Reggeized gluons and Bern-Dixon-Smirnov amplitudes, *Phys. Rev. D* **80**, 045002 (2009).
- [41] J. Bartels, L. N. Lipatov, and A. Sabio Vera, $N = 4$ supersymmetric Yang Mills scattering amplitudes at high energies: The Regge cut contribution, *Eur. Phys. J. C* **65**, 587 (2010).
- [42] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/3bxb-zfwc> for additional details, which contains the parametrization of the DSL collinear limit in terms of momentum twistors and a derivation for various $\Delta_{I/II}$.
- [43] L. N. Lipatov and A. Prygarin, BFKL approach and six-particle MHV amplitude in $N = 4$ super Yang-Mills, *Phys. Rev. D* **83**, 125001 (2011).
- [44] V. S. Fadin and L. N. Lipatov, BFKL equation for the adjoint representation of the gauge group in the next-to-leading approximation at $N = 4$ SUSY, *Phys. Lett. B* **706**, 470 (2012).
- [45] S. Caron-Huot, When does the gluon reggeize?, *J. High Energy Phys.* **05** (2015) 093.
- [46] L. Lipatov, A. Prygarin, and H. J. Schnitzer, The multi-Regge limit of NMHV Amplitudes in $N = 4$ SYM Theory, *J. High Energy Phys.* **01** (2013) 068.
- [47] L. J. Dixon and M. von Hippel, Bootstrapping an NMHV amplitude through three loops, *J. High Energy Phys.* **10** (2014) 065.
- [48] V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou, and B. Verbeek, The seven-gluon amplitude in multi-Regge kinematics beyond leading logarithmic accuracy, *J. High Energy Phys.* **06** (2018) 116.
- [49] J. Bartels, L. N. Lipatov, and A. Prygarin, MHV amplitude for $3 \rightarrow 3$ gluon scattering in Regge limit, *Phys. Lett. B* **705**, 507 (2011).
- [50] A. Sen, Asymptotic behavior of the fermion and gluon exchange amplitudes in massive quantum electrodynamics in the Regge limit, *Phys. Rev. D* **27**, 2997 (1983).
- [51] L. J. Dixon, C. Duhr, and J. Pennington, Single-valued harmonic polylogarithms and the multi-Regge limit, *J. High Energy Phys.* **10** (2012) 074.
- [52] L. J. Dixon, J. M. Drummond, C. Duhr, and J. Pennington, The four-loop remainder function and multi-Regge behavior at NNLLA in planar $N = 4$ super-Yang-Mills theory, *J. High Energy Phys.* **06** (2014) 116.
- [53] N. Beisert, B. Eden, and M. Staudacher, Transcendentality and crossing, *J. Stat. Mech.* (2007) P01021.
- [54] E. Remiddi and J. A. M. Vermaseren, Harmonic polylogarithms, *Int. J. Mod. Phys. A* **15**, 725 (2000).
- [55] F. C. Brown, Single-valued multiple polylogarithms in one variable, *C. R. Acad. Sci. Paris, Ser. I* **338**, 527 (2004).
- [56] F. C. Brown, Single-valued hyperlogarithms and unipotent differential equations, 2004, <https://www.ihes.fr/~brown/RHpaper5.pdf>.
- [57] V. Del Duca, C. Duhr, F. Dulat, and B. Penante, All two-loop MHV remainder functions in multi-Regge kinematics, *J. High Energy Phys.* **01** (2019) 162.
- [58] J. M. Drummond, G. Papathanasiou, and M. Spradlin, A symbol of uniqueness: The cluster bootstrap for the 3-loop MHV heptagon, *J. High Energy Phys.* **03** (2015) 072.
- [59] L. J. Dixon, O. Gürdoğan, Y.-T. Liu, A. J. McLeod, and M. Wilhelm, Antipodal self-duality for a four-particle form factor, *Phys. Rev. Lett.* **130**, 111601 (2023).
- [60] L. J. Dixon and S. Xin, A two-loop four-point form factor at function level, *J. High Energy Phys.* **01** (2025) 012.
- [61] Z. Li, Two-loop MHV form factors from the periodic Wilson loop, *J. High Energy Phys.* **05** (2025) 209.
- [62] L. N. Lipatov and A. Prygarin, Mandelstam cuts and light-like Wilson loops in $N = 4$ SUSY, *Phys. Rev. D* **83**, 045020 (2011).
- [63] J. Pennington, The six-point remainder function to all loop orders in the multi-Regge limit, *J. High Energy Phys.* **01** (2013) 059.
- [64] J. Broedel and M. Sprenger, Six-point remainder function in multi-Regge-kinematics: An efficient approach in momentum space, *J. High Energy Phys.* **05** (2016) 055.

- [65] J. Bartels, L. N. Lipatov, and A. Prygarin, Collinear and Regge behavior of $2 \rightarrow 4$ MHV amplitude in $N = 4$ super Yang-Mills theory, [arXiv:1104.4709](#).
- [66] J. Bartels, A. Kormilitzin, L. N. Lipatov, and A. Prygarin, BFKL approach and $2 \rightarrow 5$ maximally helicity violating amplitude in $\mathcal{N} = 4$ super-Yang-Mills theory, *Phys. Rev. D* **86**, 065026 (2012).
- [67] A. Prygarin, M. Spradlin, C. Vergu, and A. Volovich, All two-loop MHV amplitudes in multi-Regge kinematics from applied symbology, *Phys. Rev. D* **85**, 085019 (2012).
- [68] J. Bartels, V. Schomerus, and M. Sprenger, Multi-Regge limit of the n -gluon bubble Ansatz, *J. High Energy Phys.* **11** (2012) 145.
- [69] J. Bartels, A. Kormilitzin, and L. Lipatov, Analytic structure of the $n = 7$ scattering amplitude in $\mathcal{N} = 4$ SYM theory in the multi-Regge kinematics: Conformal Regge pole contribution, *Phys. Rev. D* **89**, 065002 (2014).
- [70] J. Bartels, A. Kormilitzin, and L. N. Lipatov, Analytic structure of the $n = 7$ scattering amplitude in $\mathcal{N} = 4$ theory in multi-Regge kinematics: Conformal Regge cut contribution, *Phys. Rev. D* **91**, 045005 (2015).
- [71] J. Bartels, V. Schomerus, and M. Sprenger, Heptagon amplitude in the multi-Regge regime, *J. High Energy Phys.* **10** (2014) 067.
- [72] J. Bartels, V. Schomerus, and M. Sprenger, The Bethe roots of Regge cuts in strongly coupled $\mathcal{N} = 4$ SYM theory, *J. High Energy Phys.* **07** (2015) 098.
- [73] T. Bargheer, G. Papathanasiou, and V. Schomerus, The two-loop symbol of all multi-Regge regions, *J. High Energy Phys.* **05** (2016) 012.
- [74] J. Broedel, M. Sprenger, and A. Torres Orjuela, Towards single-valued polylogarithms in two variables for the seven-point remainder function in multi-Regge kinematics, *Nucl. Phys.* **B915**, 394 (2017).
- [75] T. Bargheer, Systematics of the multi-Regge three-loop symbol, *J. High Energy Phys.* **11** (2017) 077.
- [76] V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou, and B. Verbeek, Multi-Regge kinematics and the moduli space of Riemann spheres with marked points, *J. High Energy Phys.* **08** (2016) 152.
- [77] R. Marzucca and B. Verbeek, The multi-Regge limit of the eight-particle amplitude beyond leading logarithmic accuracy, *J. High Energy Phys.* **07** (2019) 039.
- [78] V. Del Duca, S. Druc, J. M. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou, and B. Verbeek, All-order amplitudes at any multiplicity in the multi-Regge limit, *Phys. Rev. Lett.* **124**, 161602 (2020).
- [79] T. Bargheer, V. Chestnov, and V. Schomerus, The multi-Regge limit from the Wilson loop OPE, *J. High Energy Phys.* **05** (2020) 002.
- [80] J. Bartels, Analytic structure of the 8-point scattering amplitude in multi-Regge kinematics in $N = 4$ SYM: Conformal Regge pole and Regge cut contributions, [arXiv: 2005.08818](#).
- [81] H. Dorn and S. Wuttke, Wilson loop remainder function for null polygons in the limit of self-crossing, *J. High Energy Phys.* **05** (2011) 114.
- [82] H. Dorn and S. Wuttke, Hexagon remainder function in the limit of self-crossing up to three loops, *J. High Energy Phys.* **04** (2012) 023.
- [83] L. J. Dixon and I. Esterlis, All orders results for self-crossing Wilson loops mimicking double parton scattering, *J. High Energy Phys.* **07** (2016) 116; **08** (2016) 131(E).