

Small- x Factorization in the Target Fragmentation Region

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We consider the differential cross-section for single-inclusive jet production with transverse momentum $P_\perp$ in deep inelastic scattering at small Bjorken x_{Bj} , mediated by a virtual photon with virtuality Q^2 . Unlike most studies at small x , which focus on particle production in the current fragmentation region, we investigate the kinematic regime where the jet is produced in the target fragmentation hemisphere of the Breit frame, and with $P_\perp \ll Q$. For a longitudinally polarized photon, we demonstrate that this cross section is not power suppressed in $P_\perp/Q$ and we derive a factorized expression in terms of extended quark and gluon jet fracture functions. Our formula, valid at next-to-leading order in α_s at small x , is akin to the Altarelli-Martinelli identity for the longitudinal DIS structure function. Numerical estimates show that the extended quark jet fracture function is the most sensitive to saturation effects in large nuclei.

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Transverse momentum dependent (TMD) distribution functions [1], which characterize the distribution of quarks and gluons inside nuclei in terms of their longitudinal and transverse momenta with respect to the nucleus direction of motion, provide fundamental insight on hadron substructure bounded in nuclei [2]. As such, their measurement is one of the main goals of the electron-ion collider (EIC) scientific program [3–5]. A key process to measure the quark TMDs is semi-inclusive hadron production in deep inelastic scattering (DIS) in the regime where the transverse momentum squared $P_\perp^2$ of the hadron (measured in the Breit frame [6–8]) is much smaller than the virtuality Q^2 of the photon probing the target wave function [9–11].

Most studies of semi-inclusive DIS (SIDIS) in the TMD literature have focused on hadrons measured in the current (i.e. virtual photon) fragmentation region of the Breit frame for which a factorization theorem has been established at leading power (LP) in $P_\perp/Q$ [12–14]. On the other hand, the case where the hadron is measured in the target fragmentation region (TFR) has received less attention, although the corresponding cross-section in the Bjorken limit (fixed Bjorken variable $x_{Bj} \lesssim 1$, large Q^2) admits a

factorization theorem [15,16] formulated in terms of extended fracture functions [17–20]. These extended fracture functions describe the distribution of partons carrying longitudinal momentum fraction x in the target probed by the virtual photon under the condition that a spectator parton fragments into a hadron/jet in the TFR with a given longitudinal momentum fraction of the target ξ and transverse momentum $P_\perp$. Fracture functions thus provide additional insight into the internal parton dynamics and, as such, they have recently been measured at moderate x_{Bj} [21,22]. When the hadron measured in the TFR is the recoiling target nucleon in diffractive events, extended fracture functions share the same operator definition [23] and thus contain the same information as diffractive parton distribution functions (PDFs) [16,24]; however, fracture functions are more general objects that have never been computed at small- x .

In this Letter, we investigate for the first time jet production in SIDIS in the TFR and in the Regge limit (small $x_{Bj} \ll 1$, fixed Q^2) [25]. From a broader perspective, our main goal is to bring attention to the TFR as a key probe of saturation dynamics, as the transverse momentum of the spectator parton produced in the TFR reflects the saturation imprint of its parent gluon in a perturbatively calculable manner, positioning the TFR as a valuable region for saturation study at the EIC. This Letter is also motivated by the connection [26] between fracture functions and nucleon energy-energy correlators [27], a novel observable proposed to search for saturation effects at small x [28].

For the sake of simplicity, we consider the longitudinally polarized photon contribution to SIDIS. An analogous analysis can be carried out for the transversely polarized

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case. Surprisingly, unlike hadron/jet measured in the current hemisphere, which give power suppressed contributions to the longitudinal structure function [10,29–32], we demonstrate here (i) that the single-inclusive jet cross-section in SIDIS mediated by a longitudinally polarized photon is LP (i.e., not suppressed by $P_\perp/Q$ for $Q \gg P_\perp$) starting from next-to-leading order (NLO) in the strong coupling α_s , and controlled by jets produced in the TFR, and (ii) that this LP contribution factorizes in terms of extended quark and gluon jet fracture functions for which we give explicit analytic expressions in the saturation regime $P_\perp \sim Q_s$ with Q_s the nucleus saturation scale.

Our strategy to achieve this is simple: we shall extract the LP contribution to the single-inclusive jet cross-section in longitudinally polarized DIS based on the NLO formula *valid for general jet kinematics* at small x . A byproduct of this calculation will be the proof that the LP term in the $P_\perp/Q$ expansion comes indeed from jets produced in the target hemisphere. The benefit of using jets is to avoid nonperturbative inputs like final state fragmentation functions [33,34], although the phase space for producing jets with large enough $P_\perp$ while being in the regime $P_\perp \ll Q$, $x_{\text{Bj}} \ll 1$ is quite narrow at the EIC (the situation will be more favorable at future high energy $e^- - p/A$ colliders [35–38]). A more systematic hadron vs jet analysis should be conducted using suitable jet definitions [39–43] and handling nonperturbative effects.

We now introduce the single-inclusive jet cross section, in the one virtual photon ($\gamma_{T/L}^*$) exchange approximation [29],

$$\frac{d\sigma^{e^- + A \rightarrow e^-' + \text{jet} + X}}{dx_{\text{Bj}} dQ^2 d^2\mathbf{P}_\perp} = \sigma_0^{\text{DIS}} [F_{UU,T} + \varepsilon F_{UU,L}], \quad (1)$$

where $\sigma_0^{\text{DIS}} = 4\pi\alpha_{\text{em}}^2(1-y+y^2/2)/(x_{\text{Bj}}Q^4)$ is the Born level hard factor and $\varepsilon = (1-y)/(1-y+y^2/2)$ is the ratio between the longitudinal and transverse photon fluxes. Q^2 , x_{Bj} and $y = Q^2/(sx_{\text{Bj}})$, respectively, denote the virtuality, the Bjorken variable, and the inelasticity for a fixed $e^- + A$ center-of-mass energy per nucleon $\sqrt{s}$. α_{em} is the fine structure constant. $F_{UU,T}$ and $F_{UU,L}$ are, respectively, the transverse and longitudinal contributions to the single-inclusive jet cross section. In the limit $P_\perp \ll Q$, $F_{UU,T}$ is leading-twist at tree level and the cross section for transversely polarized γ_T^* admits TMD factorization; the NLO calculation of $F_{UU,T}$ for jets at moderate x_{Bj} can be found in [42].

As a proof of principle, we shall compute $F_{UU,L}(x_{\text{Bj}}, Q^2, \mathbf{P}_\perp)$ in Eq. (1) in the small x_{Bj} limit. This is sufficient to prove that $F_{UU,L}$ is LP. The calculation at small x_{Bj} is performed in the color glass condensate (CGC) effective field theory [44–48] and in the dipole frame [49–52] where the virtual photon has 4-momentum $q^\mu = [q^+, -Q^2/(2q^+), \mathbf{0}_\perp]$ and a nucleon from the nucleus A has 4-momentum $P^\mu = (0, P^-, \mathbf{0}_\perp)$ in light-cone coordinates [53]. Another advantage of the small x limit is that the CGC

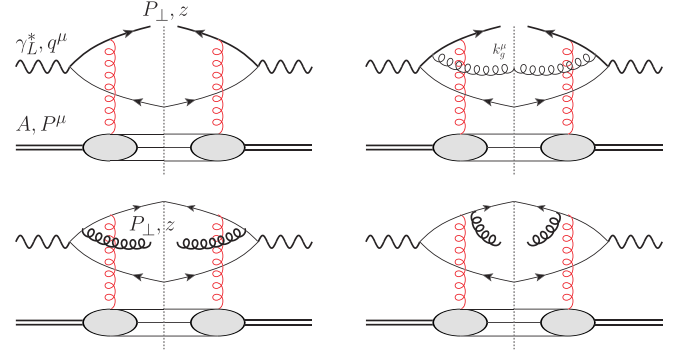


FIG. 1. Feynman graphs for the $\gamma_L^* + A \rightarrow \text{jet} + X$ subprocess at small x_{Bj} . Top: LO (left) and LP NLO diagram (right) for a quark jet measurement. Bottom: examples of LP NLO diagrams for gluon jet production ($q \leftrightarrow \bar{q}$ exchanged and interference graphs are not displayed but contribute at LP).

provides an input for the nonperturbative part for the TMD PDF [54–60]. This also applies to extended jet fracture functions, so our calculation provides their CGC expressions at small x for the first time. We note that Eq. (1) is not differential with the jet’s rapidity; including this dependence, together with the hadron production case, will be addressed in a future paper.

The leading order (LO) cross-section is given by the top-left diagram in Fig. 1, where the measured quark with four-momentum k^μ and fractional electric charge e_q has a longitudinal momentum fraction $z = k \cdot P / (P \cdot q) = k^+ / q^+$ and transverse momentum $\mathbf{P}_\perp = \mathbf{k}_\perp$ with respect to the photon direction. In this diagram, multiple scattering of the quark-antiquark color dipole with transverse size $\mathbf{r}_{xy}$ off the shockwave—pictured by a red gluon in Fig. 1 and representing the dense gluon field of the target—is encoded into the dipole S matrix, generically defined in the representation $R = F, A$ of $SU(3)$ (F = fundamental, A = adjoint) as

$$\mathcal{D}_R(x, \mathbf{q}_\perp) \equiv \int_{\mathbf{r}_{xy}} \frac{e^{-iq_\perp \cdot \mathbf{r}_{xy}}}{(2\pi)^2 d_R} \left\langle \text{Tr} \left[V_{R, \mathbf{x}_\perp} V_{R, \mathbf{y}_\perp}^\dagger \right] \right\rangle_x, \quad (2)$$

with $V_{R, \mathbf{x}_\perp}$ a lightlike Wilson line in representation R with transverse coordinate $\mathbf{x}_\perp = \mathbf{B}_\perp + \mathbf{r}_{xy}/2$ and d_R the dimension of R . $V_{F, \mathbf{x}_\perp}$ describes the color precession of the quark while eikally interacting with the target field at $\mathbf{x}_\perp$ (resp. $\mathbf{y}_\perp = \mathbf{B}_\perp - \mathbf{r}_{xy}/2$ for the antiquark). We use the shorthand notation $\int_{\mathbf{r}_{xy}} = \int d^2\mathbf{r}_{xy}$. The dependence upon the impact parameter $\mathbf{B}_\perp$ of the dipole in Eq. (2) is omitted. The x dependence of the CGC average $\langle \dots \rangle_x$ over gluon field configurations is determined by the Balitsky-Kovchegov (BK) and Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK) evolution equation [61–69] or by its linear version corresponding to the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [70,71] in the limit $q_\perp \gg Q_s$, with Q_s the nucleus saturation scale. Ultimately, x is fixed by four-momentum conservation at the scale $x = x_{\text{Bj}}$. Gathering

the dipole S matrix and the hard factor $\mathcal{H}_{\text{LO}}$ for producing the $q\bar{q}$ pair from the virtual photon, we get [56,72]

$$F_{UU,L}^{\text{LO}} = \frac{N_c}{2\pi^4} \sum_{i=q} e_i^2 \int_{\mathbf{B}_\perp, \mathbf{q}_\perp} \mathcal{D}_F(x, \mathbf{q}_\perp) \times \int_0^1 dz \left| \frac{\bar{Q}^2}{(\mathbf{P}_\perp - \mathbf{q}_\perp)^2 + \bar{Q}^2} - \frac{\bar{Q}^2}{P_\perp^2 + \bar{Q}^2} \right|^2, \quad (3)$$

where the sum runs over light quark flavors and $\bar{Q}^2 = z(1-z)Q^2$. At this stage, Eq. (3) contains all kinematic $\mathcal{O}(P_\perp/Q)$ and saturation $\mathcal{O}(Q_s/Q)$ power corrections at small x [73]. Performing the integral over z and expanding the result in the limit $Q \gg P_\perp$ yields $F_{UU,L}^{\text{LO}} = \mathcal{O}(P_\perp^2/Q^2)$ —the explicit expression for the subleading power contribution from $F_{UU,L}^{\text{LO}}$ is shown in Supplemental Material [74] (see also Refs. [89–91]). In this limit, the integral over z in Eq. (3) is dominated by the two endpoints $z=0$ and $z=1$, that is by the so-called aligned jet configurations [42,56,58,72] where the produced $q\bar{q}$ pair is highly asymmetric in longitudinal space with either $z \sim P_\perp^2/Q^2$ or $1-z \sim P_\perp^2/Q^2$. The overall $z^2(1-z)^2$ factor, originating from the $\bar{Q}^4$ dependence in Eq. (3), renders the LO cross section power suppressed.

Physically, a longitudinally polarized photon cannot directly couple to a (sea) quark at LO from the target because of helicity conservation [92,93]. The same argument applies for the longitudinal structure function $F_L(x, Q^2)$, which also vanishes at LO in the naive parton model by virtue of the Callan-Gross relation [92]. That said, we note in anticipation of a future discussion that the integral over $\mathbf{P}_\perp$ of $F_{UU,L}^{\text{LO}}$ in Eq. (3) nevertheless contributes to F_L at LP: one gets $F_L^{\text{LO}} = \alpha_s(\Sigma_i e_i^2)/(3\pi) xg(x, Q^2)$ [94–97] (cf. Supplemental Material [74]), where the gluon PDF is defined, for $x \ll 1$, as [56,59,98,99]

$$xg(x, Q^2) \equiv \int^Q d^2\mathbf{q}_\perp \int_{\mathbf{B}_\perp} \frac{N_c}{2\pi^2\alpha_s} q_\perp^2 \mathcal{D}_F(x, \mathbf{q}_\perp). \quad (4)$$

We now consider the NLO correction to Eq. (3). The complete NLO correction at small x_{Bj} to the SIDIS jet cross section for a longitudinally polarized virtual photon has been computed in [100,101] (see also Refs. [102–104]). We shall focus on the top right real diagram in Fig. 1, labeled $\text{R1} \times \text{R1}^*$, where the quark-jet $P_\perp$ is measured (the case where the antiquark jet is tagged is identical). This diagram contributes to $F_{UU,L}$ as [cf. Eq. (3.33) in [101]],

$$F_{UU,L}^{\text{R1} \times \text{R1}^*} = \frac{\alpha_s N_c}{16\pi^8} \sum_{i=q} e_i^2 \int_{\mathbf{x}_\perp, \mathbf{x}'_\perp, \mathbf{y}_\perp, \mathbf{z}_\perp} e^{-i\mathbf{P}_\perp \cdot \mathbf{r}_{xx'}} \int_0^1 dz \times \int_0^{1-z} \frac{dz_g}{z_g} (1-z-z_g)^2 Q^4 K_0(QX_R) K_0(QX'_R) \times \frac{\mathbf{r}_{zx} \cdot \mathbf{r}_{zx'}}{r_{zx}^2 r_{zx'}^2} [z^2 + (z+z_g)^2] \Xi(\mathbf{x}_\perp, \mathbf{y}_\perp, \mathbf{z}_\perp, \mathbf{x}'_\perp), \quad (5)$$

with $\mathbf{r}_{ab} \equiv \mathbf{a}_\perp - \mathbf{b}_\perp$. Here $z_g = k_g \cdot P/(P \cdot q)$ is the longitudinal momentum fraction of the gluon with 4-momentum k_g^μ and $\mathbf{z}_\perp$ its transverse coordinate when it eikonal crosses the shockwave. The integral over z_g is divergent as $z_g \rightarrow 0$, yet as we shall see, the high virtuality limit of Eq. (5) is insensitive to this rapidity divergence [105]. Further, K_0 is the zeroth order modified Bessel function entering into the $q\bar{q}g$ light-cone wave function with the effective $q\bar{q}g$ size given by $X_R^2 = z(1-z-z_g)r_{xy}^2 + zz_g r_{zx}^2 + (1-z-z_g)z_g r_{zy}^2$ and likewise for X'_R with the replacement $\mathbf{r}_{xy} \rightarrow \mathbf{r}'_{xy}$ and $\mathbf{r}_{zx} \rightarrow \mathbf{r}'_{zx}$. Last, the CGC correlator Ξ associated with the eikonal interaction of the $q\bar{q}g$ system with the target gluon field reads

$$\Xi = \frac{N_c}{2} \langle D_{xx'} - D_{xz} D_{zy} - D_{yz} D_{zx'} + 1 \rangle_x - \frac{1}{2N_c} \langle D_{xx'} - D_{xy} - D_{yx'} + 1 \rangle_x, \quad (6)$$

with the shorthand notation $D_{xy} = \text{Tr}(V_{F,\mathbf{x}_\perp} V_{F,\mathbf{y}_\perp}^\dagger)/N_c$.

We would like to demonstrate that Eq. (5) has a leading twist component in the limit $Q^2 \gg P_\perp^2$. As previously mentioned, this limit is dominated by aligned jet configurations, with end points $z \sim 0$ and $1-z-z_g \sim 0$ corresponding to soft quark and soft antiquark, respectively. Yet for this diagram and photon polarization, the end point $1-z-z_g \sim 0$ is power suppressed due the factor $(1-z-z_g)^2$ in Eq. (5). Let us then focus on the end point $z \sim 0$ with the scaling $z \sim P_\perp^2/Q^2 \ll 1$. In this regime, one can set $z=0$ in the third line of Eq. (5). The additional power of z_g^2 cancels the light-cone divergence in $1/z_g$ of the z_g integral such that one can safely integrate over z_g between 0 and 1, with typically $z_g \sim 1/2$.

Furthermore, the hierarchy of scales $Q \gg P_\perp$ must be reflected in a hierarchy of transverse distances among the $q\bar{q}g$ system while interacting with the shockwave. From the phase in Eq. (5), we have $r_{xx'} \sim 1/P_\perp$ (and likewise for r_{xy} and r'_{xy} since $\mathbf{r}_{xx'} = \mathbf{r}_{xy} - \mathbf{r}'_{xy}$) such that the $q\bar{q}$ pair is widely separated in the amplitude and complex conjugate amplitude. Moreover, the $\bar{q}g$ pair must have a small size as the condition $X_R \lesssim 1/Q$ from the K_0 Bessel function implies $r_{zy} \sim 1/Q$. We can therefore approximate $X_R^2 \approx z r_{xy}^2 + z_g(1-z_g) r_{zy}^2$ (and likewise for X'_R) where it is now easy to verify that both terms in X_R are of the same order. Similarly, the Weizsäcker-Williams gluon emission kernel $\mathbf{r}_{zx}^i/r_{zx}^2$ in Eq. (5) is approximated by $-\mathbf{r}_{xy}^i/r_{xy}^2$. Finally, using $\mathbf{z}_\perp \sim \mathbf{y}_\perp$, the CGC correlator Eq. (6) simplifies into a sum of dipoles, with an overall C_F Casimir factor,

$$\Xi \approx C_F \langle D_{xx'} - D_{xy} - D_{yx'} + 1 \rangle_x, \quad (7)$$

$$= C_F \int_{\mathbf{q}_\perp} (e^{i\mathbf{q}_\perp \cdot \mathbf{r}_{xy}} - 1)(e^{-i\mathbf{q}_\perp \cdot \mathbf{r}'_{xy}} - 1) \mathcal{D}_F(x, \mathbf{q}_\perp). \quad (8)$$

Under these approximations, one can analytically perform the transverse coordinate integrals in Eq. (5), and extract the $Q \gg P_\perp$ limit of the final z integration (cf. Supplemental Material [74]), such that

$$F_{UU,L}^{\text{NLO,LP-}q} = \frac{\alpha_s C_F}{2\pi} \sum_{i=q} e_i^2 \times \frac{N_c}{4\pi^4} \int_{B_\perp, q_\perp} \mathcal{D}_F(x, q_\perp) \times \left[1 - \frac{P_\perp \cdot (P_\perp - q_\perp)}{P_\perp^2 - (P_\perp - q_\perp)^2} \ln \left(\frac{P_\perp^2}{(P_\perp - q_\perp)^2} \right) \right], \quad (9)$$

which is the complete NLO result at LP and small x for the quark-jet contribution to $F_{UU,L}$ (using similar arguments, one shows that other diagrams not displayed in Fig. 1 are not LP). This expression neglects power corrections in $P_\perp/Q$, but it resums to all orders saturation power corrections in $Q_s/P_\perp$. It is then applicable in the saturation regime $P_\perp \sim Q_s$ so long as $P_\perp, Q_s \ll Q$.

Most importantly, the NLO correction to $F_{UU,L}$ is leading twist; unlike Eq. (3), it is not suppressed by $P_\perp/Q$. The mathematical derivation gives physical insight on why it is so: the leading twist component comes from the regime $z \sim P_\perp^2/Q^2 \ll 1$, meaning that in the Breit frame where the jet rapidity reads $\eta = \ln(zQ/P_\perp) \sim \ln(P_\perp/Q) < 0$, the jet is produced in the target fragmentation hemisphere (cf. Fig. 2, left) and not in the current (or γ^*) fragmentation one where the leading hadron is measured in hadronic SIDIS and where the corresponding TMD factorization theorem holds [14,30,32,106].

In Eq. (9), the expression after the $\times$ symbol is interpreted as the LO quark extended jet fracture function,

$$x\mathcal{F}_q(x, P_\perp) = \frac{N_c}{4\pi^4} \int_{B_\perp, q_\perp} \mathcal{D}_F(x, q_\perp) \times \left[1 - \frac{P_\perp \cdot (P_\perp - q_\perp)}{P_\perp^2 - (P_\perp - q_\perp)^2} \ln \left(\frac{P_\perp^2}{(P_\perp - q_\perp)^2} \right) \right]. \quad (10)$$

Strictly speaking, this is the ξ -integrated version of the extended fracture function [15], but the dependence on $\xi = k^-/P^-$ can easily be restored, see Eq. (22) in Supplemental Material [74]. Because the jet produced in the TFR arises from perturbative parton splitting and does not come from the intact recoiling target in a diffractive event with a rapidity gap, the $P_\perp$ dependence of the extended fracture functions differs from the t dependence of diffractive PDFs computed in [23,107–113]. Indeed, at very low values

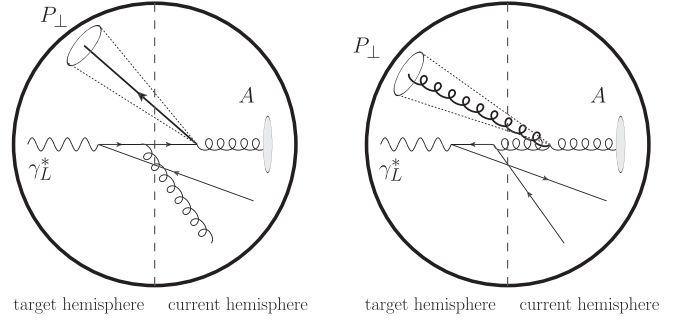


FIG. 2. Breit frame views of the LP jet produced in $\gamma_L^* - A$ collision at small x in the quark (left) and gluon (right) channels.

of $P_\perp \lesssim 1/R_A$ (where R_A is the transverse size of the target) and when the hadron produced is the same species as the target, the diffractive SIDIS contribution is indistinguishable and must be taken into account. For values of $P_\perp \gtrsim 1/R_A$, the probability that the nucleus remains intact is negligible, and the mechanism of extended fracture functions, presented in this Letter, is the dominant contribution. Equation (10) turns out to be identical, at this α_s order, to the standard TMD sea quark distribution function probed in small x SIDIS mediated by a transversely polarized photon in the current fragmentation region [54–58,114], including in the saturation regime $P_\perp \sim Q_s$. This is nontrivial as the longitudinal photon does not directly couple to sea quarks in the target, as illustrated in Fig. 2, left.

In addition to the diagrams where the fermionic jet is tagged, there are LP NLO contributions to the SIDIS jet cross section where the gluon jet with transverse momentum $P_\perp = k_{g\perp}$ and $z = z_g \lesssim P_\perp^2/Q^2$ is measured [115] (examples of diagrams are shown in the second row of Fig. 1). They are computed in detail in the Supplemental Material [74] following the method outlined above. Remarkably, the LP component of these diagrams now depends on the gluon-gluon dipole $\mathcal{D}_A(x, q_\perp)$. Indeed, in the limit $Q \gg P_\perp$ with $P_\perp$ now that of the gluon, the $q\bar{q}$ pair becomes the smallest dipole with transverse size $r_{xy} \sim 1/Q$ much smaller than the transverse distance $r_{zx} \sim 1/P_\perp$ between the $q\bar{q}$ pair and the gluon. Like in semi-inclusive diffractive DIS [116,117], an effective gluon-gluon dipole is built in the amplitude out of the NLO gluon and this very small $q\bar{q}$ dipole. The gluon-jet contribution eventually reads

$$F_{UU,L}^{\text{NLO,LP-g}} = \frac{\alpha_s}{3\pi} \sum_{i=q} e_i^2 \times x\mathcal{F}_g(x, P_\perp), \quad (11)$$

with the gluon extended jet fracture function $x\mathcal{F}_g$,

$$x\mathcal{F}_g(x, P_\perp) = \frac{N_c^2 - 1}{4\pi^4} \int_{B_\perp, q_\perp} \mathcal{D}_A(x, q_\perp) \left\{ \ln \left(\frac{1}{x} \right) \frac{q_\perp^2}{P_\perp^2} - 1 - \frac{(P_\perp - q_\perp)^2 - q_\perp^2}{2P_\perp^2} \ln \left(\frac{(P_\perp - q_\perp)^2}{P_\perp^2} \right) + \left(1 - \frac{2(P_\perp \cdot (P_\perp - q_\perp))^2}{P_\perp^2 (P_\perp - q_\perp)^2} \right) \frac{(P_\perp - q_\perp)^2}{P_\perp^2 - (P_\perp - q_\perp)^2} \ln \left(\frac{(P_\perp - q_\perp)^2}{P_\perp^2} \right) \right\}. \quad (12)$$

As a cross-check of Eq. (12) [118], we consider the dilute limit $P_\perp \gg q_\perp \sim Q_s$ yielding, up to $Q_s^4/P_\perp^4$ corrections,

$$x\mathcal{F}_g(x, \mathbf{P}_\perp) \approx \frac{\alpha_s}{2\pi^2} \frac{1}{P_\perp^2} \int_x^1 d\xi P_{gg}(\xi) xg(x, P_\perp^2), \quad (13)$$

with $P_{gg}(\xi) = 2N_c(1 - \xi(1 - \xi))^2/[\xi(1 - \xi)_+]$ the $g \rightarrow gg$ splitting function. Thus, the gluon extended fracture function in Eq. (12) can be interpreted [119,120] as the convolution between the (adjoint) dipole gluon TMD $\mathcal{D}_A$ sourcing the parent gluon from the target wave-function and a TMD $g \rightarrow gg$ splitting function [121–124], where one daughter gluon gives the measured jet while the other interacts with the virtual photon via a $q\bar{q}$ pair, as shown in Fig. 2, right.

Gathering the quark and gluon jet contributions, our final result for the single-inclusive jet cross section in longitudinally polarized DIS at LP and small x is

$$F_{UU,L}^{\text{LP}}(x, \mathbf{P}_\perp) = \frac{\alpha_s C_F}{2\pi} \sum_{i=q,\bar{q}} e_i^2 x \mathcal{F}_i(x, \mathbf{P}_\perp) + \frac{\alpha_s}{3\pi} \sum_{i=q} e_i^2 x \mathcal{F}_g(x, \mathbf{P}_\perp). \quad (14)$$

Equation (14), together with Eqs. (10)–(12), are the main results of this Letter. Equation (14) is analogous to the Altarelli-Martinelli identity [125,126] for $F_L(x, Q^2)$ at *one loop* in collinear factorization, whose small x limit is [127,128]

$$F_L \approx \frac{\alpha_s C_F}{2\pi} \sum_{i=q,\bar{q}} e_i^2 x q_i(x, Q^2) + \frac{\alpha_s}{3\pi} \sum_{i=q} e_i^2 x g(x, Q^2). \quad (15)$$

Yet, Eq. (14) differs from Eq. (15) in two crucial aspects: (i) Eq. (14) is valid at the TMD level ($\mathbf{P}_\perp$ unintegrated), (ii) Eq. (14) is a two loop result in the collinear factorization α_s power counting since $x\mathcal{F}_{q/g}$ in Eqs. (10)–(12) implicitly contains one power of α_s through the dipole operator. This new result would be worth confirming via a two-loop collinear calculation, following [115].

Integrating Eq. (14) for $P_\perp \leq Q$ gives the one loop correction to F_L at small x and leading twist. Albeit $\mathcal{O}(\alpha_s^2)$, the result has the same structure as Eq. (15) with the quark PDF $xq_i(x, Q^2)$ at small x defined as the integral over $P_\perp$ up to Q of $x\mathcal{F}_q$ [42,120,129,130]. In the gluon channel, $xg(x, Q^2)$ is now defined by the $\mathbf{P}_\perp$ integral of Eq. (12). This integral is controlled by the perturbative $1/P_\perp^2$ tail obtained in Eq. (13) and gives what one expects: the $x \rightarrow 0$ limit of one DGLAP step of the LO gluon PDF defined in Eq. (4). In the end, Eq. (14) shows that the Altarelli-Martinelli relation in NLO collinear factorization is valid at two loops for $x \ll 1$.

We finally conclude with a numerical study of these extended jet fracture functions at small x . Our goal is to illustrate the $P_\perp$ dependence of $x\mathcal{F}_{q/g}$ and to assess the

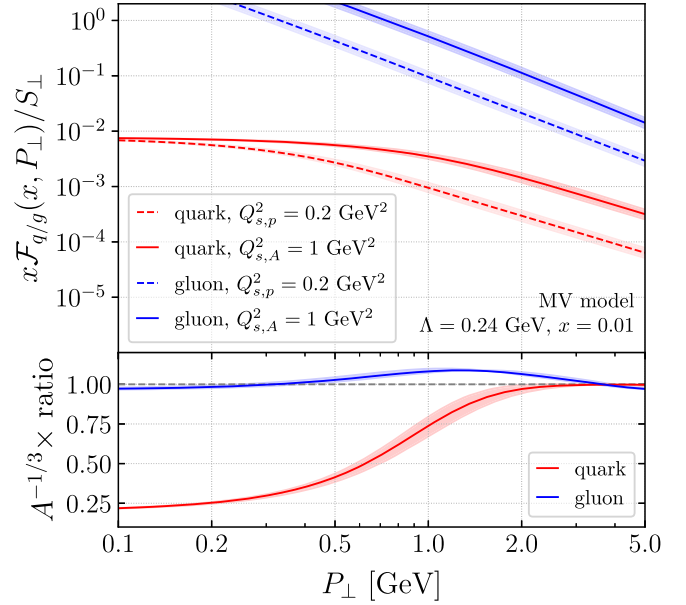


FIG. 3. Small- x quark and gluon extended fracture functions at LO for a proton and a nucleus with $A^{1/3} = 5$ (normalized by the transverse area $S_\perp$ of the target). The lower panel shows the nucleus/proton ratio times $A^{-1/3}$.

importance of gluon saturation. To compute the dipole S -matrix in momentum space, we use the McLerran-Venugopalan model [131,132] for a nucleus with mass number A , $\langle D_{xy} \rangle = \exp[-\frac{1}{4}A^{1/3}Q_{s,p}^2 r_{xy}^2 \ln[(1/r_{xy}\Lambda) + e]]$ with parameters $Q_{s,p}^2 = 0.2 \text{ GeV}^2$ for the bare proton saturation scale at $x = 0.01$, $\Lambda = 0.24 \text{ GeV}$ [133,134]. To get the adjoint dipole in Eq. (2), we simply rescale $Q_{s,p}^2 \rightarrow N_c Q_{s,p}^2 / C_F$. These choices are meant to illustrate the general behavior of $x\mathcal{F}_{q/g}$ with motivated values for the model parameters; we postpone phenomenological studies to future work.

In Fig. 3, top, the red and blue curves, respectively, correspond to the $P_\perp$ dependence of the quark and gluon extended fracture functions, at LO in the small x limit, as given by Eqs. (10)–(12). They are represented for two values of the saturation scale corresponding to a proton (dashed curves) and a large nucleus with $A^{1/3} = 5$ (plain curves). The bottom plot displays the ratio of the quark and gluon extended fracture functions between a large nucleus and a proton, normalized by $A^{-1/3}$, and the bands correspond to a variation of Λ by a factor of 2.

The quark extended jet fracture function has two regimes: at large $P_\perp$, one recognizes the perturbative tail $\propto A^{1/3}Q_{s,p}^2/P_\perp^2$, such that the ratio A/p in the bottom plot converges toward 1 for $P_\perp \gg Q_s$. For $P_\perp \ll Q_s$, the growth of the quark extended fracture function saturates [54–56] and reaches the universal (independent of A) unitary limit.

On the other hand, the gluon extended fracture function Eq. (12) shows very little sensitivity to saturation effect,

since the $1/P_{\perp}^2$ perturbative tail extends down to values of $P_{\perp}$ smaller than Q_s . The small impact of saturation is also manifest in the ratio plot, which is approximatively constant except around $P_{\perp} \sim Q_s$. This is a consequence of the first term in the curly bracket of Eq. (12), which behaves like $1/P_{\perp}^2$ even for $P_{\perp} \ll Q_s$. This term is caused by final state gluon emissions in the dipole frame (bottom-right diagram in Fig. 1), which are insensitive to the scattering of the $q\bar{q}$ dipole off the shockwave. This $1/P_{\perp}^2$ scaling at small $P_{\perp}$ will be modified by quantum corrections; the extended fracture functions presented here are LO and omit quantum evolution, which is essential to accurately describe their $P_{\perp}$ dependence. Although a rigorous proof requires a two-loop CGC calculation, we conjecture that the x dependence of the fracture functions is inherited from the BK evolution of the dipole while the resolution scale Q dependence should follow [17,135] the DGLAP equation [136–138].

To summarize, we have unveiled a new contribution to small- x factorization in SIDIS coming from jets produced in the TFR. We have derived a simple analytic expression for $F_{UU,L}$ at small x and leading twist in terms of extended fracture functions. These results will open new ways to explore saturation and hadron structure phenomenology at the EIC in the TFR [139,140], complementary to diffractive DIS. As part of this program, we intend to conduct an in-depth phenomenological study of small- x processes in the TFR for both jet and hadron production, including quantum evolutions of fracture functions.

Figures 1 and 2 have been created with JaxoDraw [141].

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Data availability—The data that support the findings of this article are openly available [142].

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