

Recurrence Time for Finite Quantum Systems

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We study the time it takes for all states of a finite quantum system to return simultaneously to their original configuration. In particular, we define the recurrence time for a quantum system to be the time at which all time-evolved states are close to their initial configuration, and at least one state has deviated significantly during this interval. Considering finite-dimensional quantum systems evolving unitarily, we find bounds on this notion of recurrence time, for continuous time and discrete time, by using Dirichlet's approximation theorem. We show how the problem of finding a bound on recurrence time can be related to approximating the difference of real numbers by rationals. We present a mathematical result on the latter, which we then use to obtain tighter bounds on recurrence time.

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Introduction—How long does it take for a quantum system to come back to its original state? The Poincaré recurrence theorem proves that any classical bounded system will eventually return arbitrarily close to its initial state, and an analogous result applies to quantum systems evolving via a time-independent Hamiltonian with a discrete spectrum [1–3]. Interestingly, it was noted by Wallace [4] that finite dimensional quantum systems obey a stronger notion of recurrence, in which all states of the system return arbitrarily close to their initial states *at the same time*. This notion of recurrence is referred to as uniform recurrence.

Studying recurrence phenomenon in physical systems is of major philosophical interest, as it is in contrast to the irreversible nature of physical systems as predicted by the second law of thermodynamics. More practically, recurrence phenomenon have applications in quantum complexity [5], quantum chaos [6], quantum speed limits [7], equilibration [8], and cosmology [9,10].

In this Letter, we bound the time for the uniform recurrence of a quantum system, considering unitary evolution in both continuous time and discrete time (e.g., for a quantum walk or quantum cellular automata [11,12]). Bounds on the recurrence time for individual states have previously been obtained for continuous time evolution [7,13–15]. Deriving bounds for uniform recurrence is a natural generalization of these results. Moreover, as this recurrence depends only on the Hamiltonian, and not

on any particular state, it may be of use when the initial state is unknown (e.g., when considering quantum computation on or storage of unknown data). Furthermore, uniform recurrence has the advantage that it will occur for any nontrivial finite-dimensional Hamiltonian, whereas some particular states (e.g., energy eigenstates or states close to them) do not experience any meaningful recurrences.

There have been many interesting works on different notions of recurrence. For discrete time evolution, expected recurrence times involving repeated measurements of the time evolved state have also been investigated [16,17]. While we focus on approximate recurrence in this work, [6] studies exact recurrence for Floquet systems.

Our approach incorporates both pure and mixed states, and uses the trace distance to quantify the strength of recurrence. To bound the recurrence time, we utilize the simultaneous version of Dirichlet's approximation theorem for approximation of real numbers by rationals [[18], p. 27]. The theorem states that given any d real numbers $\alpha_1, \alpha_2, \dots, \alpha_d$ and any integer $N > 1$, there exists integers $q, l_1, l_2, \dots, l_d$ such that $1 \leq q \leq N^d$ and

$$|\alpha_i - l_i/q| \leq \frac{1}{qN} \text{ for all } i. \quad (1)$$

Moreover, we include some developments of this approach where we only care about approximating differences between real numbers, which may be of mathematical interest in other situations.

Quantum recurrence time—Let us first discuss what we mean by recurrence time for an individual state. Suppose we have some initial state and we let it evolve for some time. We consider some small number ϵ . Now, if for some time t_r , the time evolved state is at most ϵ distance away from the initial state, we might want to call t_r a recurrence

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time. However, it might be the case that the state never got farther than ϵ from its initial state. Thus, we will require that the state has to have evolved more than ϵ away from the initial state at some time before t_r in order to count as a nontrivial recurrence.

In order to define this more formally, consider a Hilbert space $\mathcal{H}$, with $\mathfrak{D}(\mathcal{H})$ the space of all valid density operators on $\mathcal{H}$. We will use the trace distance as a measure of distance between two states, which quantifies the probability of determining which state you have via the optimal measurement. Given some $\rho, \sigma \in \mathfrak{D}(\mathcal{H})$, the trace distance between them is defined as $T(\rho, \sigma) \equiv \text{tr}|\rho - \sigma|/2$.

Definition 1 (State recurrence)—Suppose for some initial state $\rho(0) \in \mathfrak{D}(\mathcal{H})$, we have a family of time-evolved states $\rho(t) \in \mathfrak{D}(\mathcal{H})$, with $t \in \tau$, where $\tau = \mathbb{R}^+$ (continuous time) or $\tau = \mathbb{Z}^+$ (discrete time). We say that the state $\rho(0) \in \mathfrak{D}(\mathcal{H})$ is ϵ -recurrent (with $0 \leq \epsilon < 1$) at time $t \in \tau$ if

$$T(\rho(t_r), \rho(0)) \leq \epsilon, \quad (2)$$

and there exists some $t' < t$ (with $t' \in \tau$) such that

$$T(\rho(t'), \rho(0)) > \epsilon. \quad (3)$$

The second condition is important for the recurrence to be nontrivial. (Otherwise, we consider the recurrence to be trivial.) This definition is close to the notion of recurrence considered in [1,14]. However, we are interested in a stronger, state-independent notion of recurrence.

Definition 2 (System recurrence)—Suppose for some Hilbert space $\mathcal{H}$, for every initial state $\rho(0) \in \mathfrak{D}(\mathcal{H})$, we have a family of time-evolved states $\rho(t) \in \mathfrak{D}(\mathcal{H})$, with $t \in \tau$, where either $\tau = \mathbb{R}^+$ (continuous time) or $\tau = \mathbb{Z}^+$ (discrete time). We say that the system is ϵ -recurrent (with $0 \leq \epsilon < 1$) at time $t_r \in \tau$ if (i) for all $\rho(0) \in \mathfrak{D}(\mathcal{H})$,

$$T(\rho(t_r), \rho(0)) \leq \epsilon, \quad (4)$$

and (ii) there exists some state $\sigma(0) \in \mathfrak{D}(\mathcal{H})$ and some time $t' \leq t_r$ (with $t' \in \tau$) such that

$$T(\sigma(t'), \sigma(0)) > \epsilon. \quad (5)$$

In other words, for a system to be recurrent at some time, all states must recur trivially or nontrivially with at least one state recurring nontrivially [19]. This is close to the notion of uniform recurrence in [4].

We are primarily interested in continuous time Hamiltonian evolution and discrete-time unitary evolution. In these cases, time-evolution preserves the purity of the initial state. Thus, for every initial pure state $|\psi(0)\rangle$, we can write the time-evolved states as $|\psi(t)\rangle$. Then, the following lemma follows from the joint convexity of the trace distance (See End Matter for proof).

Lemma 1—Suppose we have a quantum system (associated with some Hilbert space $\mathcal{H}$) where time evolution is

unitary. The quantum system is ϵ -recurrent at time t_r iff (i) for all pure states $|\psi(0)\rangle \in \mathcal{H}$,

$$T(|\psi(t_r)\rangle, |\psi(0)\rangle) \leq \epsilon, \quad (6)$$

and (ii) there exists some pure state $|\varphi(0)\rangle \in \mathcal{H}$ and some time $t' < t_r$ such that

$$T(|\varphi(t')\rangle, |\varphi(0)\rangle) > \epsilon. \quad (7)$$

Here, we have used $T(|\psi\rangle, |\phi\rangle)$ as a shorthand for $T(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = \sqrt{1 - |\langle\psi|\phi\rangle|^2}$.

In other words, a unitary quantum system is recurrent at some time iff all pure states recur trivially or non-trivially at that time with at least one pure state recurring nontrivially.

Bounds on recurrence time using Dirichlet's approximation theorem—We first consider quantum systems where the evolution is governed by a time-independent Hamiltonian with continuous time. Given a Hamiltonian H (acting on some Hilbert space $\mathcal{H}$), for any state $|\psi(0)\rangle \in \mathcal{H}$, the family of time-evolved states is given by $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$ with $t \in \tau = \mathbb{R}^+$. Here and throughout the Letter, we set $\hbar = 1$. We can obtain a bound for recurrence times using the Dirichlet's approximation theorem.

Theorem 1—Consider a Hamiltonian H with maximum and minimum energy eigenvalues being $E_{\max}$ and $E_{\min}$ respectively. Suppose that the energy spectrum is discrete and finite with $d \geq 2$ distinct energy eigenvalues. Then, the system is ϵ -recurrent (with $0 < \epsilon < 1$) at time t_r such that

$$t_r \leq \frac{2\pi}{E_{\max} - E_{\min}} \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil \right)^{d-2}.$$

The detailed proof of the theorem is presented in the End Matter. We sketch the main idea of the proof here. Suppose the eigenvalues of H are $\{E_k\}_{k=1}^d$ with $E_1 = E_{\min}$ and $E_d = E_{\max}$.

Now, under time evolution for some time t , each respective energy eigenstate gets a phase $e^{-iE_j t}$. For all states to be close to their initial state at time t , we would want the difference between the phases $((E_j - E_k)t)$ to be close to an integer multiple of 2π . More specifically, all possible (pure) states are trivially or nontrivially recurrent at t_r [i.e., $T(|\psi(t_r)\rangle, |\psi(0)\rangle) \leq \epsilon$ for all $|\psi(0)\rangle$] if there exists some integers l_k such that

$$|(E_j - E_k)t_r - 2\pi(l_j - l_k)| \leq 2\epsilon \quad (8)$$

for all $j, k \in \{1, 2, \dots, d\}$.

We let

$$t_0 = \frac{2\pi}{E_d - E_1} = \frac{2\pi}{E_{\max} - E_{\min}}. \quad (9)$$

Then, with $N = \lceil \pi/\epsilon \rceil$, using the simultaneous version of Dirichlet's approximation theorem [Eq. (1)], there exists some integer q satisfying $1 \leq q \leq (2N)^{d-2}$ and some integers l_k , such that

$$\left| (E_k - E_1) \frac{t_0}{2\pi} q - l_k \right| < \frac{1}{2N}, \quad (10)$$

where $k \in \{2, \dots, d-1\}$. Now, note that Eq. (10) also holds true for (i) $k = d$ with $l_d = q$ and (ii) $k = 1$ with $l_1 = 0$ for any q .

It follows that, for any j, k ,

$$\begin{aligned} \left| (E_j - E_k) \frac{t_0}{2\pi} q - (l_j - l_k) \right| &\leq \left| (E_j - E_1) \frac{t_0}{2\pi} q - l_j \right| \\ &+ \left| (E_k - E_1) \frac{t_0}{2\pi} q - l_k \right| < \frac{1}{N}. \end{aligned} \quad (11)$$

Then, letting $t_r = qt_0$, we get

$$t_0 \leq t_r \leq t_0 \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil \right)^{d-2} \quad (12)$$

as well as Eq. (8).

Now, all that is left to be done is to show that there exists some state $|\varphi(0)\rangle$ that recurs nontrivially at t_r . Consider the state $|\varphi(0)\rangle = (|1\rangle + |d\rangle)/\sqrt{2}$. We can show that for any $\epsilon > 0$, there exists a time $t' < t_0 \leq t_r$ such that $T(|\varphi(t')\rangle, |\varphi(0)\rangle) > \epsilon$. This concludes the outline of the proof.

Discrete time unitary evolution—We shall now consider the case when time is discrete, and the system evolves in each time step via a unitary U . For any initial state $|\psi(0)\rangle$, we can write the time-evolved states as $|\psi(n)\rangle = U^n |\psi(0)\rangle$, where $n \in \tau = \mathbb{Z}^+$.

Now, similarly to the continuous time case, we can get a bound for recurrence time when considering discrete time unitary evolution using Dirichlet's approximation theorem. Before we present the bound, we define distance on a circle. For any two points $\theta_1, \theta_2 \in [0, 2\pi)$ on a unit circle, it is natural to define the distance between them as the smallest angle between them,

$$R(\theta_1, \theta_2) := \min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|). \quad (13)$$

Theorem 2—Suppose we have a unitary U , having a discrete and finite spectrum, with $d \geq 2$ distinct eigenvalues $\{e^{i\phi_k}\}$ where $0 \leq \phi_k < 2\pi$. Then, for any $0 < \epsilon \leq 1/2$, the quantum system is ϵ -recurrent at some time $m_r \in \mathbb{N}$ such that

$$m_r \leq \frac{\pi}{\max_{jk} R(\phi_j, \phi_k)} \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil \right)^{d-1}.$$

Here, $\max_{jk} R(\phi_j, \phi_k)$ is like $E_{\max} - E_{\min}$ in the continuous time case. The scaling is roughly $(1/\epsilon)^{d-1}$ in the discrete case as opposed to $(1/\epsilon)^{d-2}$ in the continuous case.

Again, we relegate the detailed proof to the End Matter. For the discrete case, to prove that all states have at least trivial recurrence, we need to show that there exists some integers l_k such that

$$|(\phi_j - \phi_k)m_r - 2\pi(l_j - l_k)| \leq 2\epsilon \quad (14)$$

for all $j, k \in \{1, 2, \dots, d\}$.

Similar to the continuous time case, by using the Dirichlet's approximation theorem, we obtain that for any positive integers m_0 and N , there exists some integer q satisfying $1 \leq q \leq (2N)^{d-1}$ and some integers l_k such that

$$|(\phi_k - \phi_1) \frac{m_0}{2\pi} q - l_k| < \frac{1}{2N}, \quad (15)$$

where $k \in \{1, 2, \dots, d-1\}$. Now, note that Eq. (15) also holds true for $k = 1$ with $l_d = 0$ and any q .

Then, with $N = \lceil \pi/\epsilon \rceil$, Eq. (14) follows from Eq. (15) along with

$$m_0 \leq m_r \leq m_0 \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil \right)^{d-1}. \quad (16)$$

Finally, we need to pick an m_0 , such that at least one state is nontrivially recurrent at time m_r . If we pick $m_0 = \pi/\max_{jk} R(\phi_j, \phi_k)$, then we can verify that for the state $|\varphi(0)\rangle = (|1\rangle + |d\rangle)/\sqrt{2}$, there exists some $m' < m_0 \leq m_r$ such that $T(|\varphi(m')\rangle, |\varphi(0)\rangle) > \epsilon$ for any $\epsilon > 0$. This concludes the outline of the proof.

Approximation of difference of reals by rationals—Now, note that the approach above concerns approximating individual energies and phases by rational approximations [via Eqs. (10) and (15)]. However, the recurrence behavior really depends on energy differences given by

$$|(\phi_j - \phi_k)m_r - 2\pi(l_j - l_k)| \leq 2\epsilon. \quad (14)$$

Here, we show that we can obtain improved bounds by optimizing the rational approximations of such differences more directly.

To make the situation a bit more general, for any set of real numbers $\{\alpha_i\}_{i=1}^d$ and any natural number N , we want to find integers $\{l_i\}_{i=1}^d$ and an integer q such that [20]

$$|(\alpha_i - \alpha_j) - (l_i - l_j)/q| < 1/Nq. \quad (17)$$

Furthermore, we want to find the smallest integer M (dependent on N and d) such that $1 \leq q \leq M$. Using the simultaneous version of Dirichlet's approximation theorem and following a similar approach to that used earlier, we can obtain the bound $M = (2N)^{d-1}$.

How can we do better? Note that one approach to proving the Dirichlet's approximation theorem involves the pigeonhole principle [21]. The approach considered earlier, focussing on approximating each α_i individually, corresponds to dividing the parameter space $[0, 1)^{d-1}$ into pigeonholes which are small hypercubes $[0, 1/(2N))^{d-1}$. However, when optimized to approximate differences between the rational approximations, it is helpful to use pigeonholes of different shapes, as shown in Fig. 1. These shapes are larger than the original hypercubes while still fulfilling certain constraints that are necessary for the proof to work. In particular, the shapes are chosen such that for any two points in a pigeonhole, the differences between them for each coordinate are similar and small. It turns out that M corresponds to the number of pigeonholes. Then, using these larger pigeonholes, M is reduced. This leads to improved bounds for situations in which N is sufficiently large (corresponding to good approximation), as given below.

Proposition 1—For any $d \geq 2$ distinct real numbers $\alpha_1, \alpha_2, \dots, \alpha_d$ and any positive integer N , there exists some integers $l_1, l_2, \dots, l_d$ and $q \geq 1$ with

$$\begin{aligned} q &\leq (2N)^{d-1}, \\ q &\leq N(2N+1)^{d-2}, \\ q &\leq \frac{(2N+2)^{d-1}}{d}, \end{aligned}$$

such that

$$|(\alpha_i - \alpha_j) - (l_i - l_j)/q| < 1/Nq$$

for all i, j .

The detailed proof of this proposition is presented in Supplemental Material [22]. Finally, if we are also given that for some given n and m , $\alpha_n - \alpha_m$ is an integer, then we

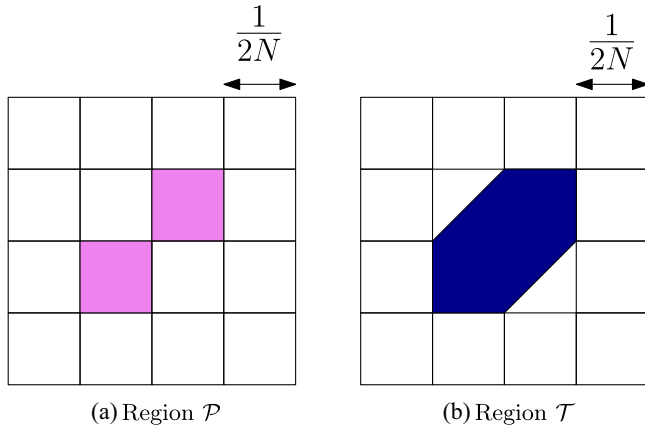


FIG. 1. The pigeonholes used to cover $[0, 1)^{d-1}$ when $d = 3$. For the simple application of the Dirichlet approximation theorem, the corresponding pigeonhole would be a square of side length $1/(2N)$. (a) Region $\mathcal{P}$. (b) Region $\mathcal{T}$.

can reduce d by one in the upper bounds for q . Then, this modified result holds for $d \geq 3$.

Better bounds for recurrence time—Now, let us go back to the proof outline of Theorem 1. We wished to find integers q and l_k such that the following equation holds for all j, k :

$$|(E_j - E_k)t_r - 2\pi(l_j - l_k)| \leq 2\epsilon. \quad (18)$$

Writing $t_r = t_0 q$, with $\alpha_k = E_k t_0 / (2\pi)$ and $N = \lceil \pi/\epsilon \rceil$, we can now use the modified version of Proposition 1 to get tighter bounds on t_r .

Theorem 3—Consider a Hamiltonian H with the maximum and minimum energy eigenvalues being $E_{\max}$ and $E_{\min}$ respectively. Suppose that the energy spectrum is discrete and finite with number of distinct energy eigenvalues being $d \geq 3$. Then, the system is ϵ -recurrent (with $0 < \epsilon < 1$) at some time t_r such that

$$\begin{aligned} t_r &\leq \frac{2\pi}{E_{\max} - E_{\min}} \left\lceil \frac{\pi}{\epsilon} \right\rceil \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil + 1 \right)^{d-3}, \\ t_r &\leq \frac{2\pi}{E_{\max} - E_{\min}} \frac{1}{d-1} \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil + 2 \right)^{d-2}. \end{aligned}$$

Note that the second bound also holds true for $d = 2$.

Similarly, we can use Proposition 1 to get the tighter bounds on recurrence time for discrete time evolution.

Theorem 4—Suppose we have a unitary U , having a discrete and finite spectrum, with $d \geq 2$ distinct eigenvalues $\{e^{i\phi_k}\}$ where $0 \leq \phi_k < 2\pi$. Then, for any $0 < \epsilon \leq 1/2$, the quantum system is ϵ -recurrent at some time $m_r \in \mathbb{N}$ such that

$$\begin{aligned} m_r &\leq \frac{\pi}{\max_{jk} R(\phi_j, \phi_k)} \left\lceil \frac{\pi}{\epsilon} \right\rceil \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil + 1 \right)^{d-2}, \\ m_r &\leq \frac{\pi}{\max_{jk} R(\phi_j, \phi_k)} \frac{1}{d} \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil + 2 \right)^{d-1}. \end{aligned}$$

Again, in comparison to the continuous time case, $\max_{jk} R(\phi_j, \phi_k)$ corresponds to $E_{\max} - E_{\min}$, and $d + 1$ replaces d .

Tightness of the bounds—By constructing explicit Hamiltonians with long uniform recurrence times, we can show that the bounds in the continuous time case are relatively tight. In particular, we have the following theorem.

Theorem 5—For any given positive η , $\epsilon < 1$, we can always find a Hamiltonian H such that the system is ϵ -recurrent at a time t_r with

$$t_r \geq \frac{2\pi - 2 \arcsin(\epsilon)}{E_{\max} - E_{\min}} \left(\frac{\pi}{\arcsin(\epsilon)} - 1 - \eta \right)^{d-2},$$

where d is the number of distinct energy eigenvalues, and $E_{\max}$ and $E_{\min}$ are the maximum and minimum energy eigenvalues, respectively.

Moreover, for $d = 2$, the recurrence time is exactly given by

$$t_r = \frac{2\pi - 2\arcsin(\epsilon)}{E_{\max} - E_{\min}}. \quad (19)$$

The proof of the above results is presented in Supplemental Material [22].

Now, for $\epsilon \ll 1$ and $\eta \ll 1$, we have approximately

$$t_r \gtrsim \frac{2\pi}{E_{\max} - E_{\min}} \left(\frac{\pi}{\epsilon}\right)^{d-2}, \quad (20)$$

and for $d = 2$, we have

$$t_r \approx \frac{2\pi}{E_{\max} - E_{\min}}. \quad (21)$$

In comparison, we have the following upper bound from Theorem 3,

$$t_r \lesssim \frac{2\pi}{E_{\max} - E_{\min}} \frac{1}{d-1} \left(\frac{2\pi}{\epsilon}\right)^{d-2}. \quad (22)$$

Comparing the above equations, we conclude that we have found an approximately tight bound for $d = 2$ and $d = 3$. For higher dimensions, the scaling of the two bounds is very similar. However, our upper bound differs from the lower bound by a factor of $2^{d-2}/(d-1)$.

Comparison with related work—The recent independent work of Kotowski and Oszmaniec [15] has many interesting results, with the main focus of the paper being on individual state recurrence. In the Supplemental Material (Theorem S1) [22], they obtain the following bound for system recurrence in continuous time using a different approach,

$$t_r < \frac{\pi\epsilon}{E_{\max} - E_{\min}} \left(\frac{4\pi}{\epsilon} + 1\right)^{d-1} \lesssim \frac{4\pi^2}{(E_{\max} - E_{\min})} \left(\frac{4\pi}{\epsilon}\right)^{d-2}, \quad (23)$$

where d is the dimension of the quantum system. For a nondegenerate system, the above bound gives the same scaling $((1/\epsilon)^{d-2})$ as in Theorem 1. We also obtain tighter prefactors in Theorems 1 and 3 with the same scaling.

Discussion—For any unitarily evolving finite dimensional quantum system, we can find a time at which all states are arbitrarily close to their initial configuration, and at least one state has deviated significantly over this time interval. We have obtained bounds on the recurrence time for this notion of system recurrence, in both continuous and discrete time. These scale approximately like $(1/\epsilon)^{d-2}$ or $(1/\epsilon)^{d-1}$, respectively, where d is the number of distinct energies for the system, and ϵ is the accuracy of recurrence. The reason for the difference is that we consider a discrete

set of times even in the continuous case, but we select the time interval such that the relative phase between the highest and lowest energy states returns exactly to zero in each interval [i.e., $(E_{\max} - E_{\min})\delta t = 2\pi$], whereas we do not have this freedom in the discrete case.

Our approach also introduced some mathematical techniques to approximate the differences between real numbers via rationals, which may be of interest in other circumstances.

Although we would not expect similar recurrences to occur in general for infinite dimensional systems (even when their spectra are discrete), we would expect that similar results could be obtained by placing a restriction on initial states and Hamiltonians (e.g., a finite number of energy eigenvalues in any finite energy interval, and a bound on the ground state energy of the Hamiltonian and the expected energy of the initial state), as in [4]. It would also be interesting to consider the recurrence of open quantum systems or general quantum channels (e.g., systems evolving via Lindblad equations or nonunitary CPTP maps) in a similar context. While we would expect that such evolutions would not yield recurrences for all ϵ due to contractivity, it would be good to obtain bounds on the recurrence behavior in such cases.

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Data availability—This is a purely mathematical work and no data was created or analyzed in this study.

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- [19] Note that we do not require all states to recur non-trivially, as some states might be stationary (e.g., energy eigestates for Hamiltonian evolution).
- [20] Note that this is a strict inequality, and hence, a slightly stronger condition than Eq. (C2).
- [21] If you have $n + 1$ items places in n pigeonholes, one of the pigeonholes must contain at least two items.
- [22] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/392k-bgpt> for further details on rational approximation for difference of reals using tiling.

End Matter

Proof of Lemma 1—Given some arbitrary $t \in \tau$, it is sufficient to show that (i) $T[\rho(t), \rho(0)] \leq \epsilon$ for all $\rho(0)$ iff $T[|\psi(t)\rangle, |\psi(0)\rangle] \leq \epsilon$ for all $|\psi(0)\rangle$, and (ii) $T[\sigma(t), \sigma(0)] > \epsilon$ for some $\sigma(0)$ iff $T[|\varphi(t)\rangle, |\varphi(0)\rangle] > \epsilon$ for some $|\varphi(0)\rangle$.

For (i), the “only if” direction is trivial. For the “if” direction, consider some arbitrary density operator $\rho(0) = \sum_i p_i |\psi_i(0)\rangle \langle \psi_i(0)|$. For unitary time evolution, we can write $\rho(t) = \sum_i p_i |\psi_i(t)\rangle \langle \psi_i(t)|$. Then, we have

$$T(\rho(t), \rho(0)) \leq \sum_i p_i T(|\psi_i(t)\rangle, |\psi_i(0)\rangle) \leq \epsilon, \quad (\text{A1})$$

where the first inequality follows from joint convexity of the trace distance and the second follows from assumption.

For (ii), the “if” direction is trivial. For the “only if” direction, we write $\sigma(0) = \sum_i p_i |\varphi_i(0)\rangle \langle \varphi_i(0)|$, and $\sigma(t) = \sum_i p_i |\varphi_i(t)\rangle \langle \varphi_i(t)|$ (unitary evolution). Then, we have

$$\sum_i p_i T(|\varphi_i(t)\rangle, |\varphi_i(0)\rangle) \geq T(\sigma(t), \sigma(0)) > \epsilon, \quad (\text{A2})$$

where, again, the first inequality follows from joint convexity of the trace distance and the second from assumption.

Noting that the first expression in the above equation is an average, we can deduce that there must exist some j such that $T[|\varphi_j(t)\rangle, |\varphi_j(0)\rangle] > \epsilon$.

Proof of Theorem 1—Suppose we are dealing with a Hilbert space of dimension D . Let the eigenvalues and eigenstates of H be $\{E_k\}_{k=1}^D$ and $\{|k\rangle\}_{k=1}^D$, respectively.

As H has d distinct eigenvalues, $D \geq d$. We arrange the eigenvalues such that (i) $E_1 = E_{\min}$ and $E_d = E_{\max}$, and (ii) $E_1, E_2, \dots, E_d$ are distinct. Thus, for the degenerate case, for any $k > d$, $E_k = E_j$ for some $j \leq d$.

Let

$$|\psi(0)\rangle = \sum_{k=1}^D c_k |k\rangle \quad (\text{B1})$$

be some arbitrary initial state.

We have, after time t ,

$$|\psi(t)\rangle = \sum_{k=1}^D e^{-iE_k t} c_k |k\rangle. \quad (\text{B2})$$

Now, the trace distance, between the two states is given by

$$\begin{aligned} T(|\psi(t)\rangle, |\psi(0)\rangle)^2 &= 1 - |\langle \psi(t) | \psi(0) \rangle|^2 \\ &= 1 - \sum_{j,k} |c_j|^2 |c_k|^2 e^{-i(E_j - E_k)t} \\ &= \sum_{j,k} |c_j|^2 |c_k|^2 (1 - \cos[(E_j - E_k)t]). \end{aligned} \quad (\text{B3})$$

Now, suppose we can find a time t_r such that

$$|(E_j - E_k)t_r - 2\pi(l_j - l_k)| \leq 2\epsilon \quad (\text{B4})$$

for some integers l_k and $j, k \in \{1, 2, \dots, d\}$. (It automatically follows that the above equation holds for $j, k = 1, 2, \dots, D$.) Defining $\alpha_k = E_k t_r - 2\pi l_k$, we can rewrite the above equation as

$$|(E_j - E_k)t_r - 2\pi(l_j - l_k)| = |(\alpha_j - \alpha_k)| \leq 2\epsilon. \quad (\text{B5})$$

Then, it follows from Eq. (B3) that

$$\begin{aligned} T(|\psi(t_r)\rangle, |\psi(0)\rangle)^2 &= \sum_{j,k} |c_j|^2 |c_k|^2 (1 - \cos[(\alpha_j - \alpha_k)t]) \\ &\leq \frac{1}{2} \sum_j \sum_k |c_j|^2 |c_k|^2 (\alpha_j - \alpha_k)^2, \end{aligned} \quad (\text{B6})$$

where we have used the periodicity of cosine in the first line and the inequality $1 - \cos(x) \leq x^2/2$ in the second line.

If we let A be a random variable with outcomes α_j and the respective probabilities being $p_j = |c_j|^2$, we get

$$\begin{aligned} \frac{1}{2} \sum_{j,k} p_j p_k (\alpha_j - \alpha_k)^2 &= \frac{1}{2} \sum_{j,k} p_j p_k (\alpha_j^2 + \alpha_k^2 - 2\alpha_j \alpha_k) \\ &= \langle A^2 \rangle - \langle A \rangle^2 = \text{Var}(A). \end{aligned} \quad (\text{B7})$$

Thus, we have

$$T(|\psi(t_r)\rangle, |\psi(0)\rangle)^2 \leq \text{Var}(A). \quad (\text{B8})$$

Then, using Popoviciu's inequality on variances, we have

$$\text{Var}(A) \leq \frac{(\max_k \alpha_k - \min_k \alpha_k)^2}{4} \leq \frac{(2\epsilon)^2}{4} = \epsilon^2. \quad (\text{B9})$$

Finally, we have $T(|\psi(t_r)\rangle, |\psi(0)\rangle) \leq \epsilon$. Now, we have reduced the problem to proving Eq. (B4) with t_r following the appropriate bounds.

We can rewrite Eq. (B4) as

$$\left| (E_j - E_k) \frac{t_0}{2\pi} q - (l_j - l_k) \right| \leq \frac{\epsilon}{\pi}, \quad (\text{B10})$$

where

$$t_0 = \frac{2\pi}{E_d - E_1} = \frac{2\pi}{E_{\max} - E_{\min}} \quad (\text{B11})$$

and we let $t_r = qt_0$.

With $N = \lceil \pi/\epsilon \rceil$, using the simultaneous version of Dirichlet's approximation theorem [Eq. (1)], there exists some integer q satisfying $1 \leq q \leq (2N)^{d-2}$ and some integers l_k , such that

$$\left| (E_k - E_1) \frac{t_0}{2\pi} q - l_k \right| < \frac{1}{2N}, \quad (\text{B12})$$

where $k \in \{2, \dots, d-1\}$. Now, note that Eq. (B12) also holds true for (i) $k = d$ with $l_d = q$ and (ii) $k = 1$ with $l_1 = 0$ for any q . Thus, we can say that there exists some

integers l_k and q with $1 \leq q \leq (2N)^{d-2}$ such that Eq. (B12) holds for all k .

Now, for any j, k , consider

$$\begin{aligned} \left| (E_j - E_k) \frac{t_0}{2\pi} q - (l_j - l_k) \right| &\leq \left| (E_j - E_1) \frac{t_0}{2\pi} q - l_j \right| \\ &+ \left| (E_k - E_1) \frac{t_0}{2\pi} q - l_k \right| < \frac{1}{N}. \end{aligned} \quad (\text{B13})$$

Thus, we have obtained Eq. (B10).

Also, noting that $t_r = qt_0$, we get

$$t_0 \leq t_r \leq t_0 \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil \right)^{d-2}. \quad (\text{B14})$$

Now, we need to show that there exists at least one state that has nontrivial recurrence. Consider the state

$$|\varphi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |d\rangle). \quad (\text{B15})$$

Then, we have, for any time t ,

$$T(|\varphi(t)\rangle, |\varphi(0)\rangle)^2 = \sin^2((E_d - E_1)t/2). \quad (\text{B16})$$

Now, we can always find a t' with $0 \leq t'(E_d - E_1)/2 < \pi$ such that $\sin^2((E_d - E_1)t'/2) > \epsilon^2$ (for any $\epsilon < 1$). Thus, we have

$$T(|\varphi(t')\rangle, |\varphi(0)\rangle) = |\sin((E_d - E_1)t'/2)| > \epsilon. \quad (\text{B17})$$

Note that $t' < t_0 \leq t_r$. This concludes our proof.

Proof of Theorem 2—Suppose that the dimension of the system is D . Then, suppose U has eigenvalues $\{e^{i\phi_k}\}_{k=1}^D$ and eigenstates $\{|k\rangle\}_{k=1}^D$, with $\phi_k \in [0, 2\pi)$. As U has d distinct eigenvalues, we have $D \geq d$. Moreover, we can arrange the eigenvalues such that $\phi_1, \phi_2, \dots, \phi_d$ are all distinct.

Suppose we have an arbitrary initial state $|\psi(0)\rangle = \sum_k c_k |k\rangle$. Then, we have $|\psi(m)\rangle = \sum_k c_k e^{i\phi_k m} |k\rangle$ for any $m \in \mathbb{Z}^+$. Now, similar to the proof of Theorem 1, we have

$$T(|\psi(0)\rangle, |\psi(m)\rangle) = \sum_{j,k} |c_j|^2 |c_k|^2 (1 - \cos[(\phi_j - \phi_k)m]) \quad (\text{C1})$$

for any time m .

Suppose we can find a time m_r such that

$$|(\phi_j - \phi_k)m_r - 2\pi(l_j - l_k)| \leq 2\epsilon \quad (\text{C2})$$

for all j, k and where l_j are some integers. Then, following similar steps as before, we get

$$T(|\psi(0)\rangle, |\psi(m_r)\rangle) \leq \epsilon. \quad (\text{C3})$$

Let m_0 be an arbitrary positive integer. With $N = \lceil \pi/\epsilon \rceil$, using Dirichlet's approximation theorem [Eq. (1)], there exists some integer q satisfying $1 \leq q \leq (2N)^{d-1}$ and some integers l_k such that

$$\left| (\phi_k - \phi_1) \frac{m_0}{2\pi} q - l_k \right| < \frac{1}{2N}, \quad (\text{C4})$$

where $k \in \{1, 2, \dots, d-1\}$. Now, note that Eq. (C4) also holds true for $k = 1$ with $l_d = 0$ and any q , we have

$$\begin{aligned} \left| (\phi_j - \phi_k) \frac{m_0}{2\pi} q - (l_j - l_k) \right| &\leq \left| (\phi_j - \phi_1) \frac{m_0}{2\pi} q - l_j \right| \\ &+ \left| (\phi_k - \phi_1) \frac{m_0}{2\pi} q - l_k \right| < \frac{1}{N}. \end{aligned} \quad (\text{C5})$$

Thus, we have obtained Eq. (C2) with $m_r = m_0 q$.

To summarize, we have shown that for any positive integer m_0 , we can find an m_r such that

$$m_0 \leq m_r \leq m_0 \left(2 \left\lceil \frac{\pi}{\epsilon} \right\rceil \right)^{d-1} \quad (\text{C6})$$

and all states are (trivially or nontrivially) recurrent at m_r .

We now need to pick an m_0 such that we have at least one state having nontrivial recurrence. Consider the state

$$|\varphi(0)\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle), \quad (\text{C7})$$

where $|1\rangle$ and $|2\rangle$ are any arbitrary eigenstates of U with distinct eigenvalues. For some time $m \in \mathbb{Z}$, we have

$$\begin{aligned} T(|\varphi(m)\rangle, |\varphi(0)\rangle)^2 &= \sin^2((\phi_2 - \phi_1)m/2) \\ &= \sin^2(R(\phi_1, \phi_2)m/2). \end{aligned} \quad (\text{C8})$$

Now, note that there always exists an integer m' such that $m'R(\phi_1, \phi_2) \in (\pi/3, \pi]$ for all $0 < R(\phi_1, \phi_2) \leq \pi$. Thus, for such an m' , we have

$$\begin{aligned} T(|\varphi(m')\rangle, |\varphi(0)\rangle)^2 &= \sin^2(R(\phi_1, \phi_2)m'/2) \\ &> \sin^2(\pi/6) = 1/4 \geq \epsilon^2, \end{aligned} \quad (\text{C9})$$

and

$$m' < \frac{\pi}{R(\phi_1, \phi_2)}. \quad (\text{C10})$$

Now, we pick ϕ_1 and ϕ_2 such that $R(\phi_1, \phi_2) = \max_{jk} R(\phi_j, \phi_k)$. Thus, we have

$$m' < \frac{\pi}{\max_{jk} R(\phi_j, \phi_k)}. \quad (\text{C11})$$

Finally, picking $m_0 = \pi/\max_{jk} R(\phi_j, \phi_k)$ and substituting in Eq. (C6), we obtain the bound given in the theorem.