

Long-Range Nonstabilizerness and Phases of Matter

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Long-range nonstabilizerness can be defined as the amount of nonstabilizerness that cannot be removed by shallow local quantum circuits. We study long-range nonstabilizerness in the context of many-body quantum physics, a task with possible implications for quantum-state preparation protocols and implementation of quantum-error correcting codes. After presenting a simple argument showing that long-range nonstabilizerness is a generic property of many-body states, we restrict to the class of ground states of gapped local Hamiltonians. We focus on one-dimensional (1D) systems and present rigorous results in the context of translation-invariant matrix product states (MPSs), with our analysis extending to all gapped 1D phases under widely accepted physical assumptions. By analyzing the fixed points of the MPS renormalization-group flow, we provide a sufficient condition for long-range nonstabilizerness, which depends entirely on the local MPS tensors. Physically, our condition captures the fact that the mutual information between distant regions of stabilizer fixed points is quantized and remains so after applying shallow quantum circuits.

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Introduction—Stabilizer states and Clifford operations are fundamental tools in quantum information theory [1,2]. Clifford operators generate the stabilizer states and can be simulated efficiently on classical computers [3–5], providing very useful toy models for quantum computation. Because of their special properties, stabilizer states also provide ideal building blocks for the construction of quantum-error-correcting codes [6,7].

A related notion is that of nonstabilizerness [8,9], which quantifies the degree to which a certain state deviates from a stabilizer state. Nonstabilizerness has received increasing attention in the past few years, also due to the recent progress in fault-tolerant quantum computing [10,11], as fault-tolerant implementation of Clifford operations is efficient [7,12–15]. Quantifying nonstabilizerness is thus important from the point of view of quantum simulation: wave functions with large nonstabilizerness generally require more experimental resources to be prepared on a fault-tolerant quantum computer [16].

Recently, nonstabilizerness has been studied from a theoretical perspective in many-body physics [17–20], and much effort has been put into finding corresponding

computable measures [21–27]. An emerging theme [17] is that nonstabilizerness could be a useful tool to characterize many-body states and phases of matter [17–19,22–26], as well as quantum dynamics [28–34]. Several works have also investigated the interplay between nonstabilizerness and other physical properties, such as entanglement [35–41] and quantum chaos [29,42–48].

An interesting development has been the introduction of long-range (LR) nonstabilizerness [17,18]. Originally, it was studied in the context of conformal field theories [17] where it was defined in terms of certain LR correlations, see also Ref. [19]. It was shown that such LR correlations are nonvanishing in the critical ground state (GS) of a $q = 3$ Potts model, thus displaying LR nonstabilizerness. Similar nonlocal correlation functions, written in terms of the so-called stabilizer Rényi entropy [28], were later studied in different critical spin chains [24,49] and quantum dynamics [32,50].

Inspired by the notion of LR entanglement [17,18,23], we can give a more transparent definition of LR nonstabilizerness: it is the obstruction to map a state to a stabilizer state via shallow local quantum circuits. This definition is useful from the point of view of quantum simulation. There, one is often interested in measuring LR correlations, which are not affected by local changes of bases implemented by shallow quantum circuits. Therefore, when a target state does not have LR nonstabilizerness, it is possible to find a state with the same LR correlations but which is a stabilizer state and thus easier to implement fault

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tolerantly. In this sense, LR nonstabilizerness informs us about the difficulty of fault tolerantly implementing a state up to local changes of bases.

LR nonstabilizerness has so far remained largely unexplored. For example, in the context of GS physics, an open question is whether it is possible to find signatures of LR nonstabilizerness in a state wave function. This question is inspired by the established characterization of certain LR entangled states displaying nonzero topological entanglement entropy [51–56].

Here, we formalize and tackle this problem. First, we present a simple argument showing that LR nonstabilizerness is a generic property of many-body states, thus providing a broad characterization of LR nonstabilizerness. Next, we focus on the study of gapped GS wave functions (which are very special and not captured by our typicality argument), and provide rigorous results in the case of one-dimensional (1D) systems described by translation-invariant (TI) matrix product states (MPSs) [57–59]. As we will discuss, although we only provide rigorous results for MPSs, our characterization extends to all gapped 1D phases under widely accepted physical assumptions.

Long-range nonstabilizerness—We begin by introducing Clifford operations and stabilizer states [2]. We consider a system of N qubits with Hilbert space $\mathcal{H}_N = \bigotimes_{j=1}^N \mathcal{H}_j$, where $\mathcal{H}_j \simeq \mathbb{C}^2$. We denote by σ_j^α the Pauli matrices acting on site j , with $\alpha = 0, 1, 2, 3$ ($\sigma^0 = \mathbb{1}$ is the identity) and by $\{|0\rangle, |1\rangle\}$ the local computational basis. We also denote by $\mathcal{P}_N$ the set of all N -qubit Pauli strings including a global phase $\phi \in [\pm 1, \pm i]$. The Clifford group is the set of unitaries U such that $UPU^\dagger \in \mathcal{P}_N$ for all $P \in \mathcal{P}_N$. The pure stabilizer states are the states generated by applying Clifford gates to the reference state $|0\rangle^{\otimes N}$.

We will consider sequences of states $\{|\Psi_N\rangle\}_{N \in \mathbb{N}}$ defined for increasing N . Assuming that the qubits are arranged over a regular lattice of dimension D (we will mostly focus on $D = 1$), we define the family of local quantum circuits (QCs) as the unitaries $Q_\ell = V_\ell \dots V_2 V_1$, where each “layer” V_n contains arbitrary quantum unitaries acting on disjoint sets of two neighboring qubits (i.e., we require local gates). The integer ℓ is the depth of the circuit. We now define LR nonstabilizerness.

Definition 1—A family of states $\{|\Psi_N\rangle\}_{N \in \mathbb{N}}$ has short-range (SR) nonstabilizerness if for all $\varepsilon_0 > 0$ and all $\alpha > 0$, there exists a local QC Q_{D_N} of depth $D_N = O[\text{polylog}(N)]$ and a stabilizer state $|S_N\rangle$ such that for large enough N

$$\Delta(Q_{D_N}|\Psi_N\rangle, |S_N\rangle) \leq \frac{\varepsilon_0}{N^\alpha} = \varepsilon_N, \quad (1)$$

where $\Delta(|\psi\rangle, |\phi\rangle) = \sqrt{1 - |\langle\psi|\phi\rangle|^2}$ is the trace distance. If the sequence $\{|\Psi_N\rangle\}_{N \in \mathbb{N}}$ does not have SR nonstabilizerness, we say that it has LR nonstabilizerness.

TABLE I. Relation between short- and long-range entanglement (SRE and LRE) and short- and long-range nonstabilizerness (SRN and LRN) in MPSs. While short-range nonstabilizerness is compatible with both short- and long-range entanglement, long-range nonstabilizerness cannot be realized in short-range entangled states. Theorem 1 provides a sufficient condition for long-range nonstabilizerness.

	SRE	LRE
SRN	$ 0\rangle^{\otimes N}$	$(0\rangle^{\otimes N} + 1\rangle^{\otimes N})/\sqrt{2}$
LRN	$\times$	Theorem 1

Compared to previous definitions [18], Eq. (1) allows for some error ε_N that vanishes in the thermodynamic limit. Note that there is some freedom in choosing how fast ε_N approaches zero in N . In our definition, we choose ε_N to decay faster than any power law. This requirement guarantees that expectation values of extensive observables (such as the global spin magnetization around some axis) and their powers cannot distinguish between $Q_{D_N}|\Psi_N\rangle$ and $|S_N\rangle$, due to the general inequality $|\langle\Psi_N|O_N|\Psi_N\rangle - \langle\phi_N|O_N|\phi_N\rangle| \leq \|O_N\|_\infty \Delta(|\Psi_N\rangle, |\phi_N\rangle)$.

Definition 1 is analogous to that of LR entanglement, with the difference that LR entangled states cannot be mapped to product states by a shallow circuit Q_N . Since any product state is locally equivalent to the stabilizer state $|0\rangle^{\otimes N}$, LR nonstabilizerness is stronger than LR entanglement. In fact, there are states with LR entanglement but not LR nonstabilizerness, cf. Table I. We also note that Definition 1 differs from the one given in Refs. [60–62]. For a bipartition of the system into regions A and B , these references considered unitaries of the form $U_A \otimes U_B$ and not QCs. It is also worth repeating that Definition 1 is a statement about the thermodynamic limit.

Typicality of LR nonstabilizerness—The first natural question is whether LR nonstabilizerness exists. We give an heuristic argument showing that, in fact, LR nonstabilizerness is typical. To this end, we count the number of quantum states in a Hilbert space of dimension $\mathcal{D} = 2^N$ that can be distinguished up to an error ε_N . This is equivalent to the number of ε_N balls in the Hilbert space, scaling as [63] $n_B \sim e^{(1-\varepsilon_N^2)2^{N-1}}$. Next, we count the number of stabilizer states to which we apply a circuit of depth D_N . First, we recall that the number of stabilizer states n_S scales as [64] $n_S \sim 2^{N^2/2}$. Second, we count the number of circuits of depth D_N . Recall that a quantum circuit of m two-qubit gates can be approximated to an error ε_N (in operator norm) by a quantum circuit of $O[m \text{polylog}(m/\varepsilon_N)]$ gates chosen from a universal gate set [2] of a finite number of elements n_g . Since the total number of space-time positions for a gate is $\sim ND_N \text{polylog}(ND_N/\varepsilon_N)$, and in each position we can choose n_g gates, the total number of such local quantum circuits is $n_C \sim e^{ND_N \text{polylog}(ND_N/\varepsilon_N)}$. In conclusion, for $D_N \sim \text{polylog}(N)$, $\varepsilon_N = \varepsilon_0/N^\alpha$, we have

$$\lim_{N \rightarrow \infty} \frac{n_C n_S}{n_B} \rightarrow 0, \quad (2)$$

meaning that LR nonstabilizerness is typical.

LR nonstabilizerness and phases of matter—After establishing the typicality of LR nonstabilizerness, we turn to characterizing LR nonstabilizerness in GSs of local gapped Hamiltonians. Once again, a strong motivation to tackle this problem comes from the study of entanglement. Indeed, the characterization of LR entanglement in GSs of local gapped Hamiltonians has played a pivotal role in our understanding of topological phases of matter [54,65–68], revealing important hidden structure of the corresponding wave functions.

We consider the simplest case of 1D systems, in the context of MPSs. We therefore only provide a rigorous characterization of LR nonstabilizerness in local gapped Hamiltonians with exact MPSs as GSs. Under widely accepted assumptions, however, our characterization extends to all GSs of gapped local Hamiltonians. This can be justified by a physically motivated (but not mathematically rigorous) argument, which we provide below.

First, recall the definition of topological phases [54]: two gapped local Hamiltonians are in the same phase if they can be adiabatically deformed one into the other, without closing the gap. Equivalently, one can define topological phases in terms of local transformations: if two GSs can be mapped one onto the other (up to some error ε) by a local circuit whose depth is polylogarithmic in N/ε , the corresponding Hamiltonians are in the same phase [69]. Definition 1 then implies that the presence of LR nonstabilizerness is invariant along the adiabatic paths connecting Hamiltonians within a phase.

On the other hand, it is widely accepted that, in 1D, every topological phase has a Hamiltonian representative with exact MPSs as GSs: this is because there is no 1D topological invariant except from the trivial one, the GS degeneracy [71–76]. Therefore, a given GS of a gapped local Hamiltonian has LR nonstabilizerness if and only if so does the exact MPS adiabatically connected to it. This concludes our argument justifying our restriction to exact MPSs.

For a lattice of local Hilbert-space dimension d , MPSs are defined by a single tensor $A_{\alpha,\beta}^i$, where $i = 1, \dots, d$, $\alpha, \beta = 1, \dots, \chi$, their wave function taking the form $|v^{(N)}(A)\rangle = \sum_{i_1, \dots, i_N=1}^d \text{tr}(A^{i_1} \dots A^{i_N}) |i_1, \dots, i_N\rangle$. Here we view A^i as a $\chi \times \chi$ matrix (χ is called the bond dimension), which also defines the transfer matrix $\mathbb{E}_A = \sum_{i=1}^d A^i \otimes A^{*i}$. Next, we recall two key concepts: the blocking procedure and the MPS canonical form (CF) [58]. Given an MPS with tensor A^i , one can construct a new tensor by grouping (blocking) q sites, such that the corresponding matrices are obtained by the product $B^i = A^{i_1} \dots A^{i_q}$, where i denotes all possible sets of indices $\{i_1, \dots, i_q\}$. The new tensor B has physical

dimension d^q , the same bond dimension χ , and transfer matrix $\mathbb{E}_B = \mathbb{E}_A^q$.

In order to introduce the CF, we first define normal tensors. A tensor A is called normal if (i) it is irreducible, i.e., A^i have no common nontrivial invariant subspace, and (ii) $\mathbb{E}_A$ has a unique largest eigenvalue $\lambda_1 = 1$ and no other eigenvalue with the same magnitude. MPSs defined by a normal tensor display short-range correlation functions (and thus SR entanglement), with correlation length ξ determined by the first subleading eigenvalue $\lambda_2(\mathbb{E}_A)$ via $\xi = -1/\ln|\lambda_2|$.

A fundamental result is that, after blocking and possibly up to a gauge transformation $A^i \rightarrow WA^iW^{-1}$ [58], every tensor A^i can be expressed in terms of normal tensors in the CF $A^i = \bigoplus_{j=1}^b \text{diag}(\mu_{j,1}, \dots, \mu_{j,m_j}) \otimes A_j^i$, where $\mu_{j,k}$ are complex numbers, with $|\mu_{j,k}| \leq 1$ and at least one of them having magnitude exactly 1. Here, A_j^i are normal tensors generating MPSs that are mutually orthogonal in the thermodynamic limit. It follows that any normalized TI MPS can be written as

$$|\phi_N\rangle = \frac{1}{c_N} \sum_{j=1}^b \beta_j^{(N)} |v^{(N)}(A_j)\rangle, \quad (3)$$

where $\beta_j^{(N)} = \sum_{k=1}^{m_j} \mu_{j,k}^N$, while c_N is the normalization. States of the form (3) may represent LR entangled states, an example being the superposition of symmetry-broken GSs of a symmetric local Hamiltonian.

RG flow and LR nonstabilizerness—A powerful tool in the MPS framework is the possibility to define a real-space RG flow [58]. The RG iteration consists of blocking two sites, $A^i \mapsto B^i = A^{i_1} A^{i_2}$, followed by a polar decomposition $B = VC$, where V is an isometry $V^\dagger V = \mathbb{1}_{\chi^2}$. The RG flow maps the tensor A into C , implementing a coarse-graining procedure.

The form of RG fixed points is known [58]. In order to define them, consider a system of N sites, each hosting a left and right qudit L_n, R_n . Then, up to local isometries acting on adjacent sites L_n, R_n , they read [58]

$$|\tilde{\phi}_N\rangle = \sum_{j=1}^b \alpha_j^{(N)} |\Omega_j\rangle, \quad (4)$$

where $|\Omega_j\rangle = \bigotimes_{i=1}^N |\omega_j\rangle_{R_i, L_{i+1}}$ ($\alpha_j^{(N)}$ may depend on N). Here $|\omega_j\rangle_{R_i, L_{i+1}} = \sum_{m_j=1}^{d_j} \sqrt{\lambda_{m_j}} |m_j\rangle_{R_i} \otimes |m_j\rangle_{L_{i+1}}$, with $\langle m_j | m_{j'} \rangle_{R_i, L_{i+1}} = \delta_{j,j'}$ for all i . This condition states the local orthogonality of the RG fixed points.

RG fixed points have a simple form, allowing us to make progress on the characterization of LR nonstabilizerness. To this end, we make use of recent results constructing efficient preparation protocols for MPSs [77,78]. Given an MPS $|\phi_N\rangle$ in CF (3) and any $\varepsilon_N > 0$ (possibly depending

on N), it was shown in Ref. [78] that there exists an RG fixed-point MPS $|\tilde{\phi}_N\rangle$ and a QC of depth $D_N = O[\log(N/\varepsilon_N)]$ such that $\Delta(|\phi_N\rangle, Q_{D_N}|\tilde{\phi}_N\rangle) \leq \varepsilon_N$.

Note that $|\tilde{\phi}_N\rangle$ can be entirely determined based on the knowledge of the local tensors of $|\phi_N\rangle$. Hence, since any MPS is related to an MPS in the form (4) by a shallow QC, Definition 1 implies that an MPS $|\phi_N\rangle$ has LR nonstabilizerness if and only if the corresponding state $|\tilde{\phi}_N\rangle$ does [79]. We can therefore restrict to studying LR nonstabilizerness in the class of RG fixed points.

A sufficient condition for LR nonstabilizerness—We are left with the task of characterizing LR nonstabilizerness of RG fixed points. Clearly, LR nonstabilizerness requires $b > 1$ in Eq. (4); otherwise, the state has SR entanglement and thus SR nonstabilizerness. When $b > 1$, the problem of characterizing LR nonstabilizerness becomes nontrivial. Indeed, consider the fixed points

$$|\tilde{\phi}_N[\alpha, \beta]\rangle = \alpha|0\rangle^{\otimes N} + \beta|1\rangle^{\otimes N}, \quad (5)$$

where we take α and β to be N independent and with $|\alpha|^2 + |\beta|^2 = 1$ [80]. When $\alpha = 0, 1$, $|\tilde{\phi}_N[\alpha, \beta]\rangle$ is a stabilizer state, while for $\alpha = 2^{-1/2}$, $|\tilde{\phi}_N[\alpha, \beta]\rangle$ has SR nonstabilizerness. Indeed, in this case $\beta = e^{i\varphi}/\sqrt{2}$ for some φ . By applying the unitary $U_1 = e^{i\varphi(\sigma_1^z - 1)/2}$ on the first site, we obtain the stabilizer state $(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$. Therefore, $|\tilde{\phi}_N[1/\sqrt{2}, \beta]\rangle$ has SR nonstabilizerness. However, it is not clear if $|\tilde{\phi}_N[\alpha, \beta]\rangle$ has LR nonstabilizerness for $\alpha \neq 0, 1, 2^{-1/2}$. This turns out to be the case, as follows from our second main result.

Theorem 1—A sufficient condition for an MPS to have LR nonstabilizerness is that its RG fixed point (4) satisfies

$$\lim_{N \rightarrow \infty} H(\{|\alpha_j^{(N)}|^2\}) \notin \mathbb{N}, \quad (6)$$

where we introduced the Shannon entropy $H(\{p_j\}) = -\sum_j p_j \log_2(p_j)$ and assumed $\sum_j |\alpha_j^{(N)}|^2 = 1$.

Theorem 1 relies on the special entanglement features of stabilizer states [51,81]. Let $|\tilde{\phi}_N\rangle$ be an RG fixed point (4) and assume that there exists Q_{D_N} with $D_N = O[\text{polylog}(N)]$ such that

$$\Delta(Q_{D_N}|\tilde{\phi}_N\rangle, |S_N\rangle) \leq \varepsilon_N, \quad (7)$$

where $|S_N\rangle$ are stabilizer states. Consider a partition of the system into three regions, A, B, C , where A and B are disconnected intervals, while C is the complement of $A \cup B$, cf. Fig. 1. We take the size of each connected component of C to be $O(N)$, and $|A| = |B|$ with

$$4D_N + 4 \leq |A \cup B| \leq 8D_N. \quad (8)$$

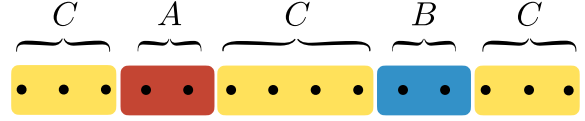


FIG. 1. Partition considered in Theorem 1. The 1D chain is divided into disjoint regions A, B , and C , where A and B are sufficiently separated, while C is the complement of $A \cup B$.

For any $|\psi\rangle$, we define the mutual information [2] $I_{A,B}[\psi] = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, where $\rho_R = \text{Tr}_{\bar{R}}(|\psi\rangle\langle\psi|)$ ($\bar{R}$ is the complement of the region R), while $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy. The proof of Theorem 1 requires the following facts, which are rigorously stated and proven in Supplemental Material [82]: (1) The mutual information of $|\psi_N\rangle = Q_{D_N}|\tilde{\phi}_N\rangle$ is $I_{A,B}[\psi_N] = H[\{|\alpha_j^{(N)}|^2\}]$, where $\alpha_j^{(N)}$ are the coefficients in Eq. (4). This is a consequence of the fact that, for suitably chosen partitions $A \cup C \cup B$, the mutual information of RG fixed points cannot be changed by shallow circuits.

(2) The mutual information of any stabilizer state $|S_N\rangle$ is quantized, $I_{A,B}[S_N] = r, r \in \mathbb{N}$.

(3) $I_{A,B}[\nu]$ is continuous in $|\nu\rangle$, which follows from the Fannes inequality [85]: given two states ρ_R and σ_R over R , with trace distance $\delta = \|\rho - \sigma\|_1/2$, we have

$$|S(\rho_R) - S(\sigma_R)| \leq \delta|R| + H_{\text{bin}}(\delta), \quad (9)$$

where $H_{\text{bin}}(\varepsilon) = -\varepsilon \log_2 \varepsilon - (1 - \varepsilon) \log_2 (1 - \varepsilon)$.

Now, Eq. (7) implies that the trace distance between the reduced density matrices of $|\psi_N\rangle$ and of $|S_N\rangle$ on any region R vanishes in the thermodynamic limit, due to the contractivity of the trace distance [2]. Then, because of the bound (9) and the fact that $\varepsilon_N|A \cup B|$ vanishes for $N \rightarrow \infty$ [thanks to Eq. (8)], the mutual information of $|\psi_N\rangle$ converges to that of $|S_N\rangle$. Using the facts 1 and 2 above, we establish Eq. (6), completing the proof [82].

Restricting to Eq. (5), Theorem 1 completely classifies the α, β for which the state has LR nonstabilizerness. Indeed, $H_{\text{bin}}[|\alpha|^2] \in \mathbb{N}$ if and only if $|\alpha|^2 = 0, 1, 1/2$, so that Eq. (6) is a necessary and sufficient condition for LR nonstabilizerness. However, we expect that this is not always the case. As an example, consider

$$|\tilde{\phi}_N(t)\rangle = \alpha_1(t)|00\rangle^{\otimes N/2} + \alpha_2(t)|01\rangle^{\otimes N/2} + \alpha_3(t)|10\rangle^{\otimes N/2} + \alpha_4(t)|11\rangle^{\otimes N/2}, \quad (10)$$

where $\alpha_1^2(t) = 10^{-1}$, $\alpha_2^2(t) = 3^{-1/4}10^{-1}$, $\alpha_3^2(t) = t$, and $\alpha_4^2(t) = 1 - t - 10^{-1} - 3^{-1/4}10^{-1}$. By inspection, we see that $H[\{|\alpha_j(t^*)|^2\}] = 1$ for $t^* \approx 0.023$. In this case, despite the mutual information being an integer, we conjecture that $|\tilde{\phi}_N(t^*)\rangle$ has LR nonstabilizerness. Although we cannot prove it, we can show a weaker statement, namely, that there is no QC Q_{D_N} with $D_N = O[\text{polylog}(N)]$ such that

$Q_{D_N}|\tilde{\phi}_N(t^*)\rangle = |S_N\rangle$, where $|S_N\rangle$ is a stabilizer state. This statement is a consequence of the following result.

Theorem 2—A necessary condition for the RG fixed point (4) to have exact SR nonstabilizerness [i.e., to satisfy Eq. (1) with $\varepsilon_0 = 0$] is that, for all $i \neq j$,

$$|\alpha_i|^4/|\alpha_j|^4 \in \mathbb{Q}. \quad (11)$$

The proof relies on the fact that, for any stabilizer state $|S\rangle$ with reduced density matrix $\rho_{A,B} = \text{Tr}_C[|S\rangle\langle S|]$, it must be $(\rho_{A,B}^{T_A})^2 \propto (\rho_{A,B}^{T_A})^4$ [86], where $(\cdot)^{T_A}$ denotes partial transpose over the region A . Using this property, we show in Supplemental Material [82] that $|\alpha_i(t^*)|^4/|\alpha_j(t^*)|^4 \in \mathbb{Q}$ for $i \neq j$ is a necessary condition for $Q_{D_N}|\tilde{\phi}_N(t^*)\rangle = |S_N\rangle$.

Since Eq. (11) is not satisfied by (10), we conclude that there is no shallow QC mapping $|\tilde{\phi}_N(t^*)\rangle$ exactly into a stabilizer state. Going beyond, we should prove that there is no QC for which the approximate equality (1) holds. Technically, this appears to be challenging, as the trace distance generally increases exponentially in system size under partial transpose [87].

Outlook—Our Letter raises several questions. First, while our sufficient condition for LR nonstabilizerness is quite powerful, it would be important to find a condition that is also necessary. This would allow us to resolve our conjecture about the LR nonstabilizerness of the states (10). Second, it is natural to ask whether one could establish a connection between LR nonstabilizerness and the behavior of suitably defined nonlocal correlation functions. Finally, an interesting problem is whether states with SR nonstabilizerness can be viewed as the trivial phase with respect to an equivalence relation and study the corresponding classes.

As a possible future direction, an extension of our results would be the characterization of LR symmetry-protected topological (SPT) nonstabilizerness [18] in the context of MPSs. We expect that progress could be made by combining some of our ideas with classical results on long-range SPT entanglement of MPSs [71,72,88]. Similarly, another direction is the characterization of gapless SPT phases [89–92], although this would likely require new techniques. It is also natural to ask whether one can define LR nonstabilizerness of unitary operators, paralleling the topological classification of quantum cellular automata [93–95], or extend the notion of LR nonstabilizerness by allowing for local measurements and nonlocal classical communication [77,96].

Finally, while we only considered 1D systems, we expect a much richer picture in higher dimensions. For instance, since stabilizer Hamiltonians cannot realize non-Abelian order [97], it would be interesting to study signatures of LR nonstabilizerness in the GS wave functions of non-Abelian topologically ordered models.

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- states $|1\rangle^{\otimes N}$, we obtain a TI MPS of the form (5) with $\alpha = n/\sqrt{n^2 + m^2}$ and $\beta = m/\sqrt{n^2 + m^2}$, which can approximate any pair of positive real numbers. When $\alpha \in \mathbb{C}$ (we can always assume that $\beta \in \mathbb{R}^+$ up to a global phase), we can sum product states defined by local tensors with an additional phase $e^{i\varphi}$, yielding the state $e^{iN\varphi}|0\rangle^{\otimes N}$. Choosing $\varphi = 2\pi r/s$ with $r, s \in \mathbb{N}$ and restricting to $N \equiv 1 \pmod{s}$, we obtain states of the form (5) where α has a phase angle $\varphi = 2\pi r/s$.
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