

Inequalities and realizability constraints between thermodynamic fluctuations in compressible aerodynamic turbulence

G. A. Gerolymos^{✉*} and I. Vallet^{✉†}

Sorbonne Université, Faculty of Science and Engineering, 4 place Jussieu, 75005 Paris, France



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The dilute-gas equation-of-state imposes exact nonlinear relations between variances and correlations of the fluctuations of thermodynamic variables. Correlation coefficients in particular satisfy a nonlinear compatibility relation. Applying the positivity of variances and the covariance inequalities on these exact nonlinear relations yields inequalities that constitute realizability constraints for the thermodynamic turbulence structure, valid for any compressible turbulent dilute-gas flow. These relations are illustrated using direct numerical simulation data for compressible turbulent plane channel and boundary-layer flows. At the limit of moderately compressible turbulence, the exact nonlinear identities and inequalities can be linearized yielding approximate relations that are practical in approximately mapping the thermodynamic turbulence structure in terms of two independent structure parameters. The nonlinear terms that determine the range of validity of the linear moderately compressible regime are assessed in terms of fluctuation amplitudes and order-3 correlation coefficients and correlated with the external flow Mach number.

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I. INTRODUCTION

Air thermodynamics are quite satisfactorily approximated for a wide range of temperatures and pressures [1] by the dilute-gas ($\mathcal{Z} = 1$) equation-of-state (EOS)

$$\mathcal{Z} := \frac{p}{\rho R_g T} = 1 \quad \forall \vec{x}, t \Rightarrow \begin{cases} \bar{p} = \bar{\rho} R_g \bar{T}, \\ \frac{\bar{T}}{\bar{T}} \frac{p'}{\bar{p}} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{\bar{T}} + \frac{\rho' T' - \bar{\rho}' \bar{T}'}{\bar{\rho} \bar{T}}, \end{cases} \quad (1)$$

where $\mathcal{Z}$ is the compressibility factor, p is the pressure, ρ is the density, T is the temperature, R_g is the gas-constant, $(\cdot) = (\cdot) + (\cdot)'$ are Reynolds averages and fluctuations, and $(\cdot) = (\cdot) + (\cdot)''$ are Favre averages and fluctuations. The mathematical relations in the paper do not involve the specific heats and are therefore valid both for constant and variable specific heat at constant pressure c_p . Throughout the paper, in line with previous work [2], we use nondimensional coefficients of variation $CV_{(\cdot)'}'$ and correlation coefficients $c_{(\cdot)'}'[(\cdot)']$, defined in Appendix A.

Inequalities between statistical moments and correlations are essential to understanding the structure of turbulence. Schumann's [3] realizability conditions for the Reynolds stresses and their projection onto Lumley's [4] realizability triangle in the plane of anisotropy invariants are an

*Contact author: georges.gerolymos@sorbonne-universite.fr

†Contact author: isabelle.vallet@sorbonne-universite.fr

example of widely used inequalities related to turbulence structure [5]. These realizability constraints for the Reynolds stresses [3] are the direct consequence of the positivity of the diagonal components of the velocity-covariances tensor for any orientation of the coordinates axes [6]. Lumley [7] also considered the problem of inequalities for third-order mixed moments $\overline{(\cdot)^2[\cdot]}$ between two flow variables.

Most of the work on thermodynamic fluctuations in compressible turbulence focuses on their interaction with the velocity field, either in terms of solenoidal/dilatational splittings of the velocity field [8,9] or in terms of Reynolds-analogy approaches [10], both of which rely on the idea that the thermodynamic fields are subordinated to the velocity field. Regarding direct relations between thermodynamic fluctuations, polytropic approximations are sometimes adopted, e.g., in astrophysical turbulence [11]. Work on the application of polytropic approximations to aerodynamic flows [12] acknowledged the need for two different polytropic exponents, one for (p', ρ') -correlations and another for (ρ', T') -correlations. However, direct numerical simulation (DNS) data for wall-bounded turbulence show large variations of these exponents with wall-normal distance ([2], Fig. 4, p. 459). Blaisdell *et al.* [13] invoked the linear mode-decomposition approximation [14] to justify that away from solid walls we should expect weak correlation between fluctuating pressure and entropy ($c_{s'p'} \approx 0$), where $c_{s'p'} := \overline{s'p'}/(s'_{\text{rms}} p'_{\text{rms}})$. Barre and Bonnet [12] combined this assumption with the generalized polytropic approximation to evaluate the nonisentropic behavior of compressible mixing layers. Notice nonetheless that DNS data of Donzis and Jagannathan [15] for solenoidally forced sustained homogeneous isotropic turbulence (HIT) deviate from the ($c_{s'p'} \approx 0$)-approximation ([2], Fig. 15, p. 471). On the other hand, the ($c_{s'p'} \approx 0$)-approximation seems to hold, except very close to the wall, in compressible turbulent plane channel (TPC) flows ([2], Fig. 14, p. 469). It also holds in a large part of the flow in zero-pressure-gradient (ZPG) turbulent boundary-layers ([16], Fig. 23, p. A4-29), but not in the outer part nor outside of the layer. The ($c_{s'p'} \approx 0$)-approximation was also studied in shock-wave/boundary-layer interactions [17].

There are few works that study specifically the relations between the fluctuations of thermodynamic variables that are imposed by the equation-of-state, and most of these studies focus on the corresponding linearized approximations. Donzis and Jagannathan [15] studied the single-variable probability density functions (PDFs) of $\{p', \rho', T'\}$ in sustained HIT and the pressure variance equation derived from (1) as well as its linearized approximation, highlighting the importance of the $c_{\rho'T'}$ correlation in compressible turbulence. Gerolymos and Vallet [2] studied the linearized relations between variances and correlations of $\{p', \rho', T', s'\}$, evaluated the leading terms of the approximation errors, and assessed the range of validity with increasing Mach number of the linearized approach, using DNS data for compressible TPC flow. They also developed and evaluated relations based on the ($c_{s'p'} \approx 0$)-approximation, which were found to be quite robust with increasing Mach number ([2], Fig. 17, p. 474). This analysis showed that, within the framework of the linearized approximation, all second-order moments and correlations can be expressed in terms of two independent parameters, and led to mapping thermodynamic turbulence structure onto the $(\text{CV}_{T'}/\text{CV}_{\rho'}, c_{\rho'T'})$ -plane ([2], Fig. 14, p. 469), where $\text{CV}_{\cdot}$ represent the coefficients of variation (intensities) and $c_{\rho'T'}$ is the correlation coefficient between density ρ' and temperature T' fluctuations. Hou *et al.* [17] focused specifically on the ($c_{s'p'} \approx 0$)-approximation, assuming that linear [14] acoustic and entropy modes are not coupled. They evaluated, using DNS data [16,17], the approximation error, which is precisely the nonlinear [18] intermodal correlation $c_{s'p'}$. Hou *et al.* [17] preferred using a surrogate of the ratio $(s'_{\text{rms}}/R_g)/\text{CV}_{p'}$ in lieu of $\text{CV}_{T'}/\text{CV}_{\rho'}$ to map thermodynamic turbulence structure.

The paper specifically studies the general relations between fluctuation amplitudes and/or correlation coefficients that are direct consequences of EOS (1), in a way analogous to the realizability analysis of the Reynolds stresses [4]. We concentrate on $\{p', \rho', T'\}$, which satisfy (1). Entropy fluctuations s' are directly expressed in terms of infinite series of $\{p', \rho', T'\}$ [19] and are not studied in this paper. Particular care is taken to distinguish between exact relations (symbols $=$, $\leq$, $\geq$) and approximate relations (symbols $\approx$, $\lesssim$, $\gtrsim$). The fluctuating EOS (1) contains the fluctuation

$(\rho'T')' := \rho'T' - \overline{\rho'T'}$ which is *nonlinear* in (ρ', T') . All the exact equations and inequalities in the paper are consequences of the fluctuating EOS (1) and contain therefore *nonlinear amplitude-dependent* terms whose origin is precisely $(\rho'T')'$.

In Sec. II we summarize the exact relations between second-order moments expressed (Sec. II B) in terms of the nondimensional coefficients of variation (relative fluctuation-level amplitudes or intensities) of and correlation coefficients between thermodynamic fluctuations (Appendix A). We show that these exact nonlinear relations imply that the ratios of fluctuation intensities $\{\text{CV}_{p'}, \text{CV}_{\rho'}, \text{CV}_{T'}\}$ must satisfy realizability inequalities (Sec. III A) graphically identified by realizability boundaries. Furthermore, we obtain in Sec. III B a nonlinear compatibility equation for the correlation coefficients $\{c_{p'\rho'}, c_{p'T'}, c_{\rho'T'}\}$, and inequalities for the signs of and order relations between correlation coefficients or fluctuation intensities (Sec. III C). In Sec. IV we evaluate the importance of the nonlinear terms with increasing flow Mach number (increasing fluctuation levels $\text{CV}_{T'} := T'_{\text{rms}}/\bar{T}$), using DNS data for compressible TPCs and ZPG turbulent boundary-layers (TBLs). In Sec. V we study the linearized approximations of the exact relations and provide detailed mappings of thermodynamic turbulence structure in the $(\text{CV}_{T'}/\text{CV}_{\rho'}, \text{CV}_{\rho'}/\text{CV}_{p'})$ -plane and in the $(\text{CV}_{T'}/\text{CV}_{\rho'}, c_{\rho'T'})$ -plane, where we determine specific regions in terms of relative levels of fluctuation intensities or signs of correlation coefficients. Finally, we provide a comprehensive assessment (Sec. VI) of the linearized approximations of the exact relations, and we summarize the conclusions and perspectives of the paper (Sec. VII). Detailed mathematical calculations are given in the Appendixes.

II. CORRELATION COEFFICIENTS AND THERMODYNAMIC TURBULENCE STRUCTURE

Under EOS (1), the second-order moments involving $\{p', \rho', T'\}$ satisfy a system of three equations (Sec. II A), which are studied in terms of nondimensional coefficients of variation and correlation coefficients (Sec. II B) that define the thermodynamic turbulence structure in terms of five nondimensional quantities (Sec. II C).

A. Second-order moments of thermodynamic fluctuations

Multiplying the fluctuating EOS (1) by $\{p', \rho', T'\}$, respectively, provides upon averaging the exact equations

$$\frac{\tilde{T}}{\bar{T}} \frac{\overline{p'^2}}{\bar{p}} \stackrel{(1)}{=} \frac{\overline{p'\rho'}}{\bar{\rho}} + \frac{\overline{p'T'}}{\bar{T}} + \frac{\overline{p'\rho'T'}}{\bar{\rho}\bar{T}}, \quad (2a)$$

$$\frac{\tilde{T}}{\bar{T}} \frac{\overline{p'\rho'}}{\bar{p}} \stackrel{(1)}{=} \frac{\overline{\rho'^2}}{\bar{\rho}} + \frac{\overline{\rho'T'}}{\bar{T}} + \frac{\overline{\rho'\rho'T'}}{\bar{\rho}\bar{T}}, \quad (2b)$$

$$\frac{\tilde{T}}{\bar{T}} \frac{\overline{p'T'}}{\bar{p}} \stackrel{(1)}{=} \frac{\overline{\rho'T'}}{\bar{\rho}} + \frac{\overline{T'^2}}{\bar{T}} + \frac{\overline{\rho'T'T'}}{\bar{\rho}\bar{T}}, \quad (2c)$$

relating the second-order moments $\{\overline{p'^2}, \overline{\rho'^2}, \overline{T'^2}, \overline{p'\rho'}, \overline{\rho'T'}, \overline{p'T'}\}$ between fluctuations of the thermodynamic variables, $\{p', \rho', T'\}$. Variances are positive,

$$p'_{\text{rms}} := \sqrt{\overline{p'^2}} > 0, \quad \rho'_{\text{rms}} := \sqrt{\overline{\rho'^2}} > 0, \quad T'_{\text{rms}} := \sqrt{\overline{T'^2}} > 0, \quad (3a)$$

where $(\cdot)'_{\text{rms}}$ is the root mean square, and covariances obey the Cauchy-Schwarz inequality

$$\overline{p'\rho'^2} \leq \overline{p'^2} \overline{\rho'^2}, \quad \overline{\rho'T'^2} \leq \overline{\rho'^2} \overline{T'^2}, \quad \overline{p'T'^2} \leq \overline{p'^2} \overline{T'^2}. \quad (3b)$$

We used strict inequalities in (3a). The only exception is a strictly isothermal wall, often used as a boundary condition ($T_w = \text{const}$) in numerical computations, where $[T'_{\text{rms}}]_w = 0$, implying by the fluctuating EOS (1) that $[p'_{\text{rms}}/\bar{p}]_w = [\rho'_{\text{rms}}/\bar{\rho}]_w$ at the wall [[20], (3.5), p. 719].

The three relations between the six moments also involve the third-order moments $\{\overline{p'\rho'T'}, \overline{\rho'\rho'T'}, \overline{\rho'T'^2}\}$. These third-order moments are related by the identity

$$(1) \Rightarrow \frac{\tilde{T}}{\bar{T}} \frac{\overline{p'\rho'T'}}{\bar{\rho}} = \frac{\overline{\rho'^2 T'}}{\bar{\rho}} + \frac{\overline{\rho' T'^2}}{\bar{T}} + \frac{\overline{\rho'^2 T'^2} - \overline{\rho' T'^2}}{\bar{\rho} \bar{T}} \quad (4)$$

obtained by multiplying the fluctuating EOS (1) by $\rho'T'$ and averaging, which contains the fourth-order moment $\overline{\rho'\rho'T'T'}$. There are 10 different third-order moments between $\{p', \rho', T'\}$, related by six equations, but only the three moments $\{\overline{p'\rho'T'}, \overline{\rho'^2 T'}, \overline{\rho' T'^2}\}$ appearing in (2) and related by (4) are necessary in the present work.

Finally, squaring and averaging the fluctuating EOS (1) yields the exact expression for the pressure variance,

$$(1) \Rightarrow \left(\frac{\tilde{T}}{\bar{T}} \right)^2 \frac{\overline{p'^2}}{\bar{\rho}^2} = \frac{\overline{\rho'^2}}{\bar{\rho}^2} + \frac{\overline{T'^2}}{\bar{T}^2} + 2 \frac{\overline{\rho' T'}}{\bar{\rho} \bar{T}} + 2 \frac{\overline{\rho'^2 T'}}{\bar{\rho}^2 \bar{T}} + 2 \frac{\overline{\rho' T'^2}}{\bar{\rho} \bar{T}^2} + \frac{\overline{\rho'^2 T'^2} - \overline{\rho' T'^2}}{\bar{\rho}^2 \bar{T}^2} \quad (5a)$$

$$\stackrel{(4)}{=} \frac{\overline{\rho'^2}}{\bar{\rho}^2} + \frac{\overline{T'^2}}{\bar{T}^2} + 2 \frac{\overline{\rho' T'}}{\bar{\rho} \bar{T}} + \frac{\overline{\rho'^2 T'}}{\bar{\rho}^2 \bar{T}} + \frac{\overline{\rho' T'^2}}{\bar{\rho} \bar{T}^2} + \frac{\tilde{T}}{\bar{T}} \frac{\overline{p'\rho'T'}}{\bar{\rho} \bar{\rho} \bar{T}}, \quad (5b)$$

where (5b) was obtained by using (4) to replace $\overline{\rho'^2 T'^2} - \overline{\rho' T'^2}$ in (5a). The terms involving higher-order correlations on the right-hand side (RHS) of (5a) are expressed exclusively in terms of joint (ρ', T') -statistics, but they contain the fourth-order correlation $\overline{\rho'^2 T'^2}$. On the other hand, the RHS of (5b) does not involve fourth-order correlations but includes the correlation $\overline{p'\rho'T'}$, and as a consequence depends on joint (p', ρ', T') -statistics.

B. Exact nonlinear relations between correlation coefficients

These equations [(2), (4), and (5)], direct consequences of the fluctuating EOS (1), are the starting point of the analysis. Following [2], they can be reexpressed in nondimensional form using coefficients of variation $\{CV_{p'}, CV_{\rho'}, CV_{T'}\}$ (A1a) and correlation coefficients [(A1b)–(A1e)]. The exact relations [(2), (4), and (5)] contain the ratio $\tilde{T}/\bar{T}$ of the Favre-to-Reynolds averaged temperatures. Notice first that $\tilde{T} > 0$, expectedly, since the strictly positive variable T is weighted in the Favre-average by the strictly positive weight ρ ,

$$\rho, T > 0 \forall \vec{x}, t \Rightarrow 0 < \overline{\rho T} = \bar{\rho} \tilde{T} = \bar{\rho} \bar{T} + \overline{\rho' T'} \Rightarrow \frac{\tilde{T}}{\bar{T}} = 1 + \frac{\overline{\rho' T'}}{\bar{\rho} \bar{T}} \stackrel{(A1)}{=} 1 + c_{\rho' T'} CV_{\rho'} CV_{T'} > 0, \quad (6)$$

where $CV_{(\cdot)'} (A1a)$ is the coefficient of variation of the quantity $(\cdot)'$, and $c_{(\cdot)'[(\cdot)']} (A1b)$ is the correlation coefficient between $(\cdot)'$ and $[(\cdot)']$. The inequality (6) is strict. Therefore, the product of the coefficient-of-variation $CV_{\rho'}$ times the ratio $\tilde{T}/\bar{T}$,

$$\Phi_{V_{p'}} := (1 + c_{\rho' T'} CV_{\rho'} CV_{T'}) CV_{p'} = \frac{\tilde{T}}{\bar{T}} CV_{p'} > 0, \quad (7)$$

is strictly positive.

The fluctuating EOS (1) relates the instantaneous fluctuations $\{p', \rho', T'\}$ with the consequence that the coefficients of variation $\{CV_{p'}, CV_{\rho'}, CV_{T'}\}$ are generally of the same order of magnitude ([2], Figs. 2 and 3, pp. 456 and 458). Obviously, the sign of $c_{\rho' T'}$ determines whether the ratios

$\tilde{T}/\bar{T} = \Phi_{p'}/CV_{p'}$ are higher or lower than unity,

$$c_{\rho'T'} > 0 \stackrel{(6,7)}{\Rightarrow} \frac{\Phi_{p'}}{CV_{p'}} \stackrel{(7)}{:=} \frac{\tilde{T}}{\bar{T}} > 1, \quad c_{\rho'T'} < 0 \stackrel{(6,7)}{\Rightarrow} \frac{\Phi_{p'}}{CV_{p'}} \stackrel{(7)}{:=} \frac{\tilde{T}}{\bar{T}} < 1. \quad (8)$$

In channel flow, which is characterized by very-cold-wall conditions [21], $c_{\rho'T'} < 0$ near the wall increases in the buffer region to positive values $c_{\rho'T'} > 0$ near the centerline ([2], Fig. 5, p. 463). Therefore, both conditions in (8) are possible, depending on the particular flow-region.

It is straightforward, using the above definitions [(A1) and (7)], to rewrite the exact equations for the correlation coefficients (2) as

$$\frac{\Phi_{p'}}{CV_{p'}} \stackrel{(2a,A1,7)}{=} c_{p'\rho'} + c_{p'T'} \frac{CV_{T'}}{CV_{p'}} + \overbrace{c_{p'\rho'T'} CV_{T'}}^{\text{nonlinear term}} \iff \frac{\Phi_{p'}}{CV_{T'}} = c_{p'\rho'} \frac{CV_{\rho'}}{CV_{T'}} + c_{p'T'} + \overbrace{c_{p'\rho'T'} CV_{\rho'}}^{\text{nonlinear term}}, \quad (9a)$$

$$c_{p'\rho'} \frac{\Phi_{p'}}{CV_{p'}} \stackrel{(2b,A1,7)}{=} 1 + c_{\rho'T'} \frac{CV_{T'}}{CV_{p'}} + c_{\rho'\rho'T'} CV_{T'} \iff c_{p'\rho'} \frac{\Phi_{p'}}{CV_{T'}} = \frac{CV_{\rho'}}{CV_{T'}} + c_{\rho'T'} + c_{\rho'\rho'T'} CV_{\rho'}, \quad (9b)$$

$$c_{p'T'} \frac{\Phi_{p'}}{CV_{p'}} \stackrel{(2c,A1,7)}{=} c_{\rho'T'} + \frac{CV_{T'}}{CV_{p'}} + \underbrace{c_{\rho'T'T'} CV_{T'}}_{\text{nonlinear term}} \iff c_{p'T'} \frac{\Phi_{p'}}{CV_{T'}} = c_{\rho'T'} \frac{CV_{\rho'}}{CV_{T'}} + 1 + \underbrace{c_{\rho'T'T'} CV_{\rho'}}_{\text{nonlinear term}}, \quad (9c)$$

where the alternative equivalent forms of (9) differ by a factor of $CV_{\rho'}/CV_{T'}$. Similarly, Eq. (5) for the pressure variance read

$$\frac{\Phi_{p'}^2}{CV_{p'}^2} \stackrel{(5a,A1,7)}{=} 1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2 c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} + \overbrace{c_{\rho'\rho'T'} CV_{T'}}^{\text{nonlinear term}} + CV_{T'} \left(2 c_{\rho'\rho'T'} + 2 c_{\rho'T'T'} \frac{CV_{T'}}{CV_{\rho'}} \right) + (c_{\rho'\rho'T'T'} - c_{\rho'T'}^2) CV_{T'}^2, \quad (9d)$$

$$\stackrel{(5b,A1,7)}{=} 1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2 c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} + \underbrace{CV_{T'} \left(c_{\rho'\rho'T'} + c_{\rho'T'T'} \frac{CV_{T'}}{CV_{\rho'}} + c_{p'\rho'T'} \frac{\Phi_{p'}}{CV_{\rho'}} \right)}_{\text{nonlinear term } c_{\text{NLN}} CV_{T'}}. \quad (9e)$$

The expressions [(9d) and (9e)] for the nonlinear terms are equivalent because of the identity (4) for the triple correlations

$$(4) \stackrel{(A1)}{\iff} c_{p'\rho'T'} \Phi_{p'} = c_{\rho'\rho'T'} CV_{\rho'} + c_{\rho'T'T'} CV_{T'} + (c_{\rho'\rho'T'T'} - c_{\rho'T'}^2) CV_{T'} CV_{\rho'}. \quad (10)$$

The linearized approximation of the exact nonlinear relation (10) has been reported in [2] [(A1a), p. 476].

The linear part of [(9d) and (9e)] corresponds to the well-known linearized approximation $CV_{p'}^2 \cong CV_{\rho'}^2 + CV_{T'}^2 + 2 c_{\rho'T'} CV_{T'} CV_{\rho'}$ [2,15]. The coefficient of $CV_{T'}$ in the nonlinear term of [(9d) and (9e)] is a structure parameter (function of correlation coefficients and amplitude ratios) defined by

$$c_{\text{NLN}} := c_{\rho'\rho'T'} + c_{\rho'T'T'} \frac{CV_{T'}}{CV_{\rho'}} + c_{p'\rho'T'} \frac{\Phi_{p'}}{CV_{\rho'}} \stackrel{(10)}{=} 2 c_{p'\rho'T'} \frac{\Phi_{p'}}{CV_{\rho'}} - (c_{\rho'\rho'T'T'} - c_{\rho'T'}^2) CV_{T'} \quad (11a)$$

so that (9d) and (9e) read

$$\frac{\Phi_{p'}^2}{CV_{\rho'}^2} (9d, 9e, 11a)_1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2 c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} + c_{NLN} CV_{T'}. \quad (11b)$$

C. Thermodynamic turbulence structure

When considering turbulence structure [22,23], we are interested essentially in amplitude-independent quantities, correlation-coefficients, and ratios of coefficients-of-variation, so that by *thermodynamic turbulence structure* (TTS) we refer to one of the sets of five quantities,

$$\text{TTS} := \left\{ \frac{CV_{p'}}{CV_{\rho'}}, \frac{CV_{T'}}{CV_{\rho'}}, c_{\rho'T'}, c_{p'\rho'}, c_{p'T'} \right\}; \text{TT\$} := \left\{ \frac{\Phi_{p'}}{CV_{\rho'}}, \frac{CV_{T'}}{CV_{\rho'}}, c_{\rho'T'}, c_{p'\rho'}, c_{p'T'} \right\}, \quad (12)$$

which contain, by definition (7), the same information. The second set, TT\$, lumps the amplitude-dependent term $\tilde{T}/\bar{T} = 1 + c_{\rho'T'} CV_{\rho'} CV_{T'}$ into $\Phi_{p'}$, and is therefore more practical for general exact relations, which remain valid in the strongly compressible régime. The choice of $CV_{\rho'}$ (instead of $CV_{T'}$) in the denominator was made to avoid division by 0 at strictly isothermal walls.

The exact relations (9) involve a part expressed in terms of elements of TT\$ (12) which we will call *linear*. It is amplitude-independent in the sense that it involves ratios of coefficients of variation only. The remaining part is $\propto CV_{T'}$. This amplitude-dependent part originates from the nonlinear term $(\rho'T')' := \rho'T' - \bar{\rho}'\bar{T}'$ in the fluctuating EOS (1) and will therefore be called *nonlinear*. Notice that the nonlinear terms are $\propto CV_{T'}$ because of the choice of $CV_{\rho'}$ to scale $\{\Phi_{p'}, CV_{T'}\}$ in TT\$ (12).

III. EXACT NONLINEAR REALIZABILITY AND COMPATIBILITY

The fluctuating EOS (1) imposes constraints on the fluctuation intensities, defining the generally valid nonlinear realizability domain of thermodynamic turbulence (Sec. III A), a nonlinear compatibility relation between correlation coefficients (Sec. III B), and several exact inequalities between correlation coefficients or fluctuation amplitudes.

A. Nonlinear realizability

Using the covariance inequality (3b) in the identity (11b) yields

$$\begin{aligned} \Phi_{p'}^2 &\stackrel{(11b)}{=} CV_{\rho'}^2 + CV_{T'}^2 \pm 2CV_{T'}CV_{\rho'} + 2(c_{\rho'T'} \mp 1)CV_{T'}CV_{\rho'} + c_{NLN}CV_{T'}CV_{\rho'}^2 \\ &= (CV_{\rho'} \pm CV_{T'})^2 + 2(c_{\rho'T'} \mp 1)CV_{T'}CV_{\rho'} + c_{NLN}CV_{T'}CV_{\rho'}^2 \\ c_{\rho'T'} - 1 \leq 0 &\Rightarrow \begin{cases} \Phi_{p'}^2 \leq (CV_{\rho'} + CV_{T'})^2 + c_{NLN}CV_{T'}CV_{\rho'}^2 \\ \Phi_{p'}^2 \geq (CV_{\rho'} - CV_{T'})^2 + c_{NLN}CV_{T'}CV_{\rho'}^2 \end{cases} \end{aligned} \quad (13a)$$

These inequalities can be combined, upon dividing by $CV_{\rho'}^2$, as

$$(11b, 13a) \Rightarrow 0 \leq \left(\frac{CV_{T'}}{CV_{\rho'}} - 1 \right)^2 \leq \left(\frac{\Phi_{p'}}{CV_{\rho'}} \right)^2 - c_{NLN} CV_{T'} \leq \left(\frac{CV_{T'}}{CV_{\rho'}} + 1 \right)^2 \quad (13b)$$

implying the exact amplitude-dependent inequality

$$(13b) \Rightarrow \left(\frac{\Phi_{p'}}{CV_{\rho'}} \right)^2 - c_{NLN} CV_{T'} \geq 0, \quad (14)$$

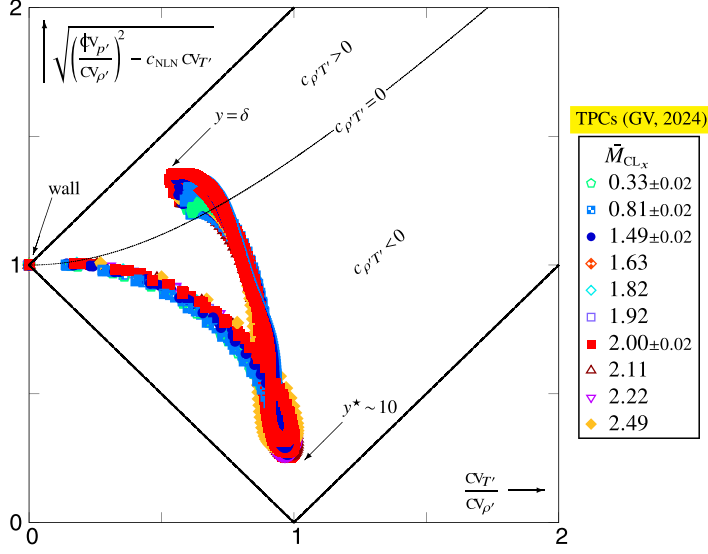


FIG. 1. Nonlinear realizability boundaries defined by the exact inequalities (15), exact locus $c_{\rho'T'} = 0$ (17e), and loci in the $(\frac{CV_{T'}}{CV_{\rho'}}, \sqrt{\frac{(\Phi V_{p'})^2}{CV_{\rho'}^2} - c_{NLN} CV_{T'}})$ -plane of DNS data for compressible turbulent plane channel flow (TPC) [24], covering the range $97 \leq Re_{\tau^*} \leq 985$ and $0.32 \leq \bar{M}_{CL,x} \leq 2.49$; the same symbol is used for all Re_{τ^*} data at the same $\bar{M}_{CL,x}$.

which leads to the final *realizability relation*

$$\left| \frac{CV_{T'}}{CV_{\rho'}} - 1 \right| \leq \sqrt{\left(\frac{\Phi V_{p'}}{CV_{\rho'}} \right)^2 - c_{NLN} CV_{T'}} \leq \left(\frac{CV_{T'}}{CV_{\rho'}} + 1 \right) \quad (15)$$

containing the structure parameter c_{NLN} (11a). In the $(\frac{CV_{T'}}{CV_{\rho'}}, \sqrt{\frac{(\Phi V_{p'})^2}{CV_{\rho'}^2} - c_{NLN} CV_{T'}})$ -plane, under EOS (1), realizable fields of turbulent fluctuations of the thermodynamic variables lie inside the region defined by the inequalities (15). This region is bounded in the $(\frac{CV_{T'}}{CV_{\rho'}}, \sqrt{\frac{(\Phi V_{p'})^2}{CV_{\rho'}^2} - c_{NLN} CV_{T'}})$ -plane by straight lines (Fig. 1), which are mutually orthogonal at points (0,0) and (1,0). All amplitude-dependent nonlinear effects, combining the ratio $\tilde{T}/\bar{T}$ in the definition (7) of $\Phi V_{p'}$ and the nonlinear term $-c_{NLN} CV_{T'}$, which contains the coefficient c_{NLN} (11a), are included in the definition (15) of the ordinates, whereas the abscissas are simply the ratio $CV_{T'}/CV_{\rho'}$. In the linear régime (moderately compressible turbulence) this mapping of the thermodynamic turbulence structure simplifies to the $(\frac{CV_{T'}}{CV_{\rho'}}, \frac{CV_{p'}}{CV_{\rho'}})$ -plane (Sec. VC).

To illustrate the mapping of actual turbulent flows onto the nonlinear realizability plane (Fig. 1), we consider compressible turbulent plane channel (TPC) flow (Fig. 2), for which all trivariate moments of $\{p', \rho', T'\}$ up to order 3 are available in the database [24]. The loci of DNS data for TPC flow over a wide range of Reynolds and Mach numbers [24] map, on the nonlinear realizability plane (Fig. 1), the wall-normal evolution of the thermodynamic turbulence structure of this class of very-cold-wall flows. Because of the strictly isothermal wall boundary-condition ($T_w = \text{const } \forall \vec{x}, t \Rightarrow CV_{T'}|_w = 0$) the wall maps exactly by (11b) onto the point (0,1). Moving away from the wall, $\Phi V_{p'}/CV_{\rho'}$ diminishes rapidly [Fig. 2(e)], whereas $CV_{T'}/CV_{\rho'}$ remains close to 0.9 [Fig. 2(d)], so that the locus on the realizability plane approaches the point $(\frac{1}{4}, 1)$ (Fig. 1), near $y^* \approx 10$ [Figs. 2(d) and 2(e)], then moves upwards (Fig. 1) towards the centerline where $\Phi V_{p'} > CV_{\rho'}$

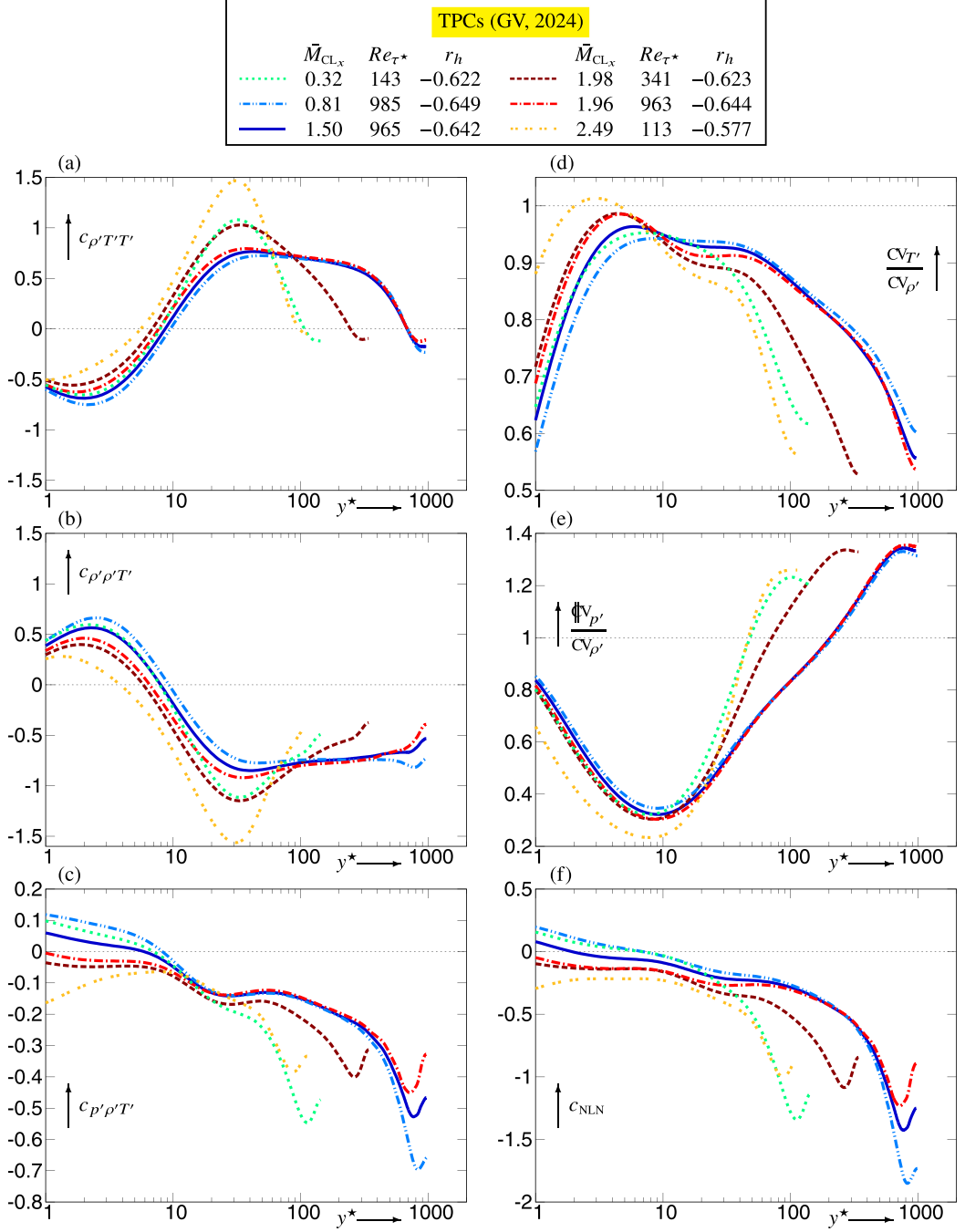


FIG. 2. Triple correlation coefficients, $c_{\rho'T'T'}$ (a), $c_{\rho'\rho'T'}$ (b), $c_{\rho'\rho'T'}$ (c), ratios of fluctuation intensities, $CV_{T'}/CV_{\rho'}$ (d) and $\Phi_{V_{\rho'}}/CV_{\rho'}$ (e), and structure parameter c_{NLN} (f) in the nonlinear term of (9d) and (9e), plotted against the HCB-scaled [25] wall-distance y^* (log scale), for selected flows from the compressible turbulent plane channel (TPC) flows DNS database [24] (GV, 2024), representative of the available range $97 \leq Re_{\tau}^* \leq 985$ and $0.32 \leq \bar{M}_{CL,x} \leq 2.49$.

[Fig. 2(e)]. The system (9a)–(9c) can be solved for the correlation coefficients $\{c_{p'\rho'}, c_{p'T'}, c_{\rho'T'}\}$ in terms of the amplitude ratios $\{CV_{T'}/CV_{\rho'}, \Phi_{V_{p'}}/CV_{\rho'}\}$, yielding after some algebra

$$c_{p'\rho'} \stackrel{(9,11a)}{=} \frac{\left(\frac{\Phi_{V_{p'}}}{CV_{\rho'}}\right)^2 - \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 + 1 - \overbrace{\left(c_{\text{NLN}} - 2c_{\rho'T'}\right)CV_{T'}}^{\text{nonlinear term}}}{2\frac{\Phi_{V_{p'}}}{CV_{\rho'}}}, \quad (16a)$$

$$c_{p'T'} \stackrel{(9,11a)}{=} \frac{\left(\frac{\Phi_{V_{p'}}}{CV_{\rho'}}\right)^2 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 - 1 - \overbrace{\left(c_{\text{NLN}} - 2c_{\rho'T'T'}\right)\frac{CV_{T'}}{CV_{\rho'}}}^{\text{nonlinear term}}}{2\frac{\Phi_{V_{p'}}}{CV_{\rho'}}\frac{CV_{T'}}{CV_{\rho'}}}, \quad (16b)$$

$$c_{\rho'T'} \stackrel{(9,11a)}{=} \frac{\left(\frac{\Phi_{V_{p'}}}{CV_{\rho'}}\right)^2 - \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 - 1 - \overbrace{c_{\text{NLN}}CV_{T'}}^{\text{nonlinear term}}}{2\frac{CV_{T'}}{CV_{\rho'}}} = \frac{\left(\left(\frac{\Phi_{V_{p'}}}{CV_{\rho'}}\right)^2 - c_{\text{NLN}}CV_{T'}\right)}{2\frac{CV_{T'}}{CV_{\rho'}}} - \frac{1 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2}{2\frac{CV_{T'}}{CV_{\rho'}}}, \quad (16c)$$

where we used the definition (11a) of the structure parameter c_{NLN} . The identity (10) can be used to reexpress the nonlinear terms in (16), providing the leading-order expression of the nonlinear terms given in [[2], (4.2), p. 460]. The equation (16c) for $c_{\rho'T'}$ defines an explicit mapping between the nonlinear realizability $(CV_{T'}/CV_{\rho'}, \sqrt{\Phi_{V_{p'}}^2/CV_{\rho'}^2 - c_{\text{NLN}}CV_{T'}})$ -plane and the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane (Sec. III D).

The signs of the correlation coefficients are readily identified from (16),

$$c_{p'\rho'} \stackrel{(16a)}{\gtrless} 0 \iff \left(\frac{\Phi_{V_{p'}}}{CV_{\rho'}}\right)^2 \gtrless \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 - 1 + \overbrace{\left(c_{\text{NLN}} - 2c_{\rho'T'}\right)CV_{T'}}^{\text{nonlinear term}}, \quad (17a)$$

$$c_{p'T'} \stackrel{(16b)}{\gtrless} 0 \iff \left(\frac{\Phi_{V_{p'}}}{CV_{\rho'}}\right)^2 \gtrless 1 - \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 + \overbrace{\left(c_{\text{NLN}} - 2c_{\rho'T'T'}\right)\frac{CV_{T'}}{CV_{\rho'}}}^{\text{nonlinear term}} CV_{T'}, \quad (17b)$$

$$c_{\rho'T'} \stackrel{(16c)}{\gtrless} 0 \iff \left(\frac{\Phi_{V_{p'}}}{CV_{\rho'}}\right)^2 \gtrless 1 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 + \overbrace{c_{\text{NLN}}CV_{T'}}^{\text{nonlinear term}}. \quad (17c)$$

Finally, the order relation between $c_{p'T'}$ and $c_{p'\rho'}$ is readily obtained from (9b) and (9c),

$$c_{p'T'} \stackrel{(9b, 9c)}{\gtrless} c_{p'\rho'} \iff \frac{CV_{T'}}{CV_{\rho'}} \stackrel{(9b, 9c)}{\gtrless} 1 + \underbrace{\frac{c_{\rho'\rho'T'} - c_{\rho'T'T'}}{1 - c_{\rho'T'}} CV_{T'}}_{\text{nonlinear term}} \quad (17d)$$

so that we expect the order relation between $(c_{p'T'}, c_{p'\rho'})$ to be the same as the order relation between $(CV_{T'}, CV_{\rho'})$ (relative fluctuation amplitude $T'_{\text{rms}}/\bar{T}$ higher than the relative fluctuation amplitude $\rho'_{\text{rms}}/\bar{\rho}$ implies $c_{p'T'} > c_{p'\rho'}$ and *vice versa*), with a slight adaptation for the nonlinear term at conditions of strongly compressible turbulence.

Notice that (17c) can be represented exactly in the nonlinear realizability plane (Fig. 1) because we can rewrite

$$c_{\rho'T'} \leq 0 \stackrel{(16c)}{\iff} \sqrt{\left(\frac{\Phi V_{p'}}{CV_{\rho'}}\right)^2 - c_{NLN} CV_{T'}} \leq \sqrt{1 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2}. \quad (17e)$$

On the contrary, the exact loci $c_{p'\rho'}=0$ and $c_{p'T'}=0$ in the nonlinear realizability plane (Fig. 1) are amplitude-dependent [(17a) and (17b)].

B. Exact nonlinear compatibility relations

Equations (9) represent the constraints imposed by the fluctuating EOS (1) on the thermodynamic turbulence structure. Each couple of relations {(9a, 9b), (9b, 9c), (9c, 9a)}, can be solved for the ratios of relative fluctuation levels $\{\Phi V_{p'}/CV_{\rho'}, CV_{T'}/CV_{\rho'}\}$, yielding the exact relations

$$0 < \frac{\Phi V_{p'}}{CV_{\rho'}} \stackrel{(9a,9b)}{=} \frac{c_{p'T'} - c_{p'\rho'} c_{\rho'T'}}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}} + \overbrace{CV_{T'} \frac{c_{p'T'} c_{\rho'\rho'T'} - c_{p'\rho'T'} c_{\rho'T'}}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}}}_{\text{nonlinear } O(CV_{T'}) \text{ term}}, \quad (18a)$$

$$\stackrel{(9a,9c)}{=} \frac{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}}{1 - c_{p'T'}^2} + CV_{T'} \frac{c_{p'\rho'T'} - c_{p'T'} c_{\rho'T'T'}}{1 - c_{p'T'}^2}, \quad (18b)$$

$$\stackrel{(9b,9c)}{=} \frac{1 - c_{\rho'T'}^2}{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}} + CV_{T'} \frac{c_{\rho'\rho'T'} - c_{p'T'} c_{\rho'T'T'}}{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}}, \quad (18c)$$

$$0 < \frac{CV_{T'}}{CV_{\rho'}} \stackrel{(9a,9b)}{=} \frac{1 - c_{p'\rho'}^2}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}} + CV_{T'} \frac{c_{\rho'\rho'T'} - c_{p'\rho'} c_{p'\rho'T'}}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}}, \quad (18d)$$

$$\stackrel{(9a,9c)}{=} \frac{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}}{1 - c_{p'T'}^2} + CV_{T'} \frac{c_{p'T'} c_{p'\rho'T'} - c_{\rho'T'T'}}{1 - c_{p'T'}^2}, \quad (18e)$$

$$\stackrel{(9b,9c)}{=} \frac{c_{p'T'} - c_{p'\rho'} c_{\rho'T'}}{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}} + \underbrace{CV_{T'} \frac{c_{p'T'} c_{\rho'\rho'T'} - c_{p'\rho'} c_{\rho'T'T'}}{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}}}_{\text{nonlinear } O(CV_{T'}) \text{ term}}. \quad (18f)$$

These different expressions for $\Phi V_{p'}/CV_{\rho'}$ or $CV_{T'}/CV_{\rho'}$ are of course compatible. Equating every pair of expressions for $\Phi V_{p'}/CV_{\rho'}$, {(18a, 18b), (18b, 18c), (18c, 18a)}, or $CV_{T'}/CV_{\rho'}$, {(18d, 18e), (18e, 18f), (18f, 18e)}, yields the same nonlinear compatibility relation

$$1 = c_{p'\rho'}^2 + c_{p'T'}^2 + c_{\rho'T'}^2 - 2 c_{p'\rho'} c_{p'T'} c_{\rho'T'} + \underbrace{CV_{T'} ((c_{p'\rho'} - c_{p'T'} c_{\rho'T'}) c_{p'\rho'T'} - (1 - c_{p'T'}^2) c_{\rho'\rho'T'} - (c_{p'\rho'} c_{p'T'} - c_{\rho'T'}) c_{\rho'T'T'})}_{\text{nonlinear } O(CV_{T'}) \text{ term}}. \quad (19)$$

Alternatively, each couple of relations {(9a, 9b), (9b, 9c), (9c, 9a)}, can be solved for the ratios of relative fluctuation levels $\{\Phi V_{p'}/CV_{T'}, CV_{\rho'}/CV_{T'}\}$, yielding the exact compatible relations

$$0 < \frac{\Phi V_{p'}}{CV_{T'}} \stackrel{(9a,9b)}{=} \frac{c_{p'T'} - c_{p'\rho'} c_{\rho'T'}}{1 - c_{p'\rho'}^2} + \overbrace{CV_{\rho'} \frac{c_{p'\rho'T'} - c_{p'\rho'} c_{\rho'\rho'T'}}{1 - c_{p'\rho'}^2}}^{\text{nonlinear } O(CV_{\rho'}) \text{ term}}, \quad (20a)$$

$$\stackrel{(9a,9c)}{=} \frac{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}} + CV_{\rho'} \frac{c_{p'\rho'} c_{\rho'T'T'} - c_{\rho'T'} c_{p'\rho'T'}}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}}, \quad (20b)$$

$$\stackrel{(9b,9c)}{=} \frac{1 - c_{\rho'T'}^2}{c_{p'T'} - c_{p'\rho'} c_{\rho'T'}} + CV_{\rho'} \frac{c_{\rho'T'T'} - c_{\rho'T'} c_{\rho'\rho'T'}}{c_{p'T'} - c_{p'\rho'} c_{\rho'T'}}, \quad (20c)$$

$$0 < \frac{CV_{\rho'}}{CV_{T'}} \stackrel{(9a,9b)}{=} \frac{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}}{1 - c_{p'\rho'}^2} + CV_{\rho'} \frac{c_{p'\rho'} c_{p'\rho'T'} - c_{\rho'\rho'T'}}{1 - c_{p'\rho'}^2}, \quad (20d)$$

$$\stackrel{(9a,9c)}{=} \frac{1 - c_{p'T'}^2}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}} + CV_{\rho'} \frac{c_{\rho'T'T'} - c_{p'T'} c_{p'\rho'T'}}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}}, \quad (20e)$$

$$\stackrel{(9b,9c)}{=} \frac{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}}{c_{p'T'} - c_{p'\rho'} c_{\rho'T'}} + \underbrace{CV_{\rho'} \frac{c_{p'\rho'} c_{\rho'T'T'} - c_{p'T'} c_{\rho'\rho'T'}}{c_{p'T'} - c_{p'\rho'} c_{\rho'T'}}}_{\text{nonlinear } O(CV_{\rho'}) \text{ term}}, \quad (20f)$$

whose nonlinear term is $\propto CV_{\rho'}$, in line with the alternative equivalent expressions in (9). Again, equating every pair of expressions for $\mathbb{V}_{p'}/CV_{T'}$, $\{(20a, 20b), (20b, 20c), (20c, 20a)\}$, or $CV_{\rho'}/CV_{T'}$, $\{(20d, 20e), (20e, 20f), (20f, 20e)\}$, yields the same nonlinear compatibility relation

$$1 = c_{p'\rho'}^2 + c_{p'T'}^2 + c_{\rho'T'}^2 - 2 c_{p'\rho'} c_{p'T'} c_{\rho'T'} + \underbrace{CV_{\rho'} ((c_{p'T'} - c_{p'\rho'} c_{\rho'T'}) c_{p'\rho'T'} - (c_{p'T'} c_{p'\rho'} - c_{\rho'T'}) c_{\rho'\rho'T'} - (1 - c_{p'\rho'}^2) c_{\rho'T'T'})}_{\text{nonlinear } O(CV_{\rho'}) \text{ term}}. \quad (21)$$

The two forms [(19) and (21)] of the compatibility equation differ only in the expression of the nonlinear term, either $\propto CV_{T'}$ in (19) or $\propto CV_{\rho'}$ in (21). The underlying inherent nonlinearity in the relation between second-order and third-order correlations is discussed in Appendix B, where three alternative cyclic forms (B4) of the compatibility relation are also reported.

C. Exact nonlinear inequalities

By the covariance inequalities (A1c) the denominators in (18b), (18e), (20d), and (20a) are positive, implying the following generally valid inequalities between correlation coefficients:

$$\stackrel{(18b)}{c_{p'T'} c_{\rho'T'}} \leq c_{p'\rho'} + \underbrace{CV_{T'} (c_{p'\rho'T'} - c_{p'T'} c_{\rho'T'T'})}_{\text{nonlinear term}}, \quad (22a)$$

$$\stackrel{(18e)}{c_{\rho'T'}} \leq c_{p'\rho'} c_{p'T'} + CV_{T'} (c_{p'T'} c_{p'\rho'T'} - c_{\rho'T'T'}), \quad (22b)$$

$$\stackrel{(20d)}{c_{\rho'T'}} \leq c_{p'T'} c_{p'\rho'} + CV_{\rho'} (c_{p'\rho'} c_{p'\rho'T'} - c_{\rho'\rho'T'}), \quad (22c)$$

$$\stackrel{(20a)}{c_{p'\rho'} c_{\rho'T'}} \leq c_{p'T'} + \underbrace{CV_{\rho'} (c_{p'\rho'T'} - c_{p'\rho'} c_{\rho'\rho'T'})}_{\text{nonlinear term}}, \quad (22d)$$

where (22b) and (22c) can be merged by considering the minimum of the nonlinear terms.

D. Mapping onto the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane

In [2] we chose the couple $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ as independent variables to map the thermodynamic turbulence structure. Equation (16c) explicitly maps the nonlinear realizability $(CV_{T'}/CV_{\rho'}, \sqrt{CV_{p'}^2/CV_{\rho'}^2 - c_{NLN} CV_{T'}})$ -plane onto the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane, $CV_{T'}/CV_{\rho'}$ being the common abscissa of these two representations. The expression of the remaining correlation

coefficients in terms of $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ is readily obtained by solving (9b) and (9c) for $\{c_{p'\rho'}, c_{p'T'}\}$, respectively, yielding

$$c_{p'\rho'} \stackrel{(9)}{=} \frac{1 + c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} + c_{\rho'\rho'T'} CV_{T'}}{\frac{\mathbb{V}_{p'}}{CV_{\rho'}}} \stackrel{(11b)}{=} \frac{1 + c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} + c_{\rho'\rho'T'} CV_{T'}}{\sqrt{1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} + c_{NLN} CV_{T'}}}, \quad (23a)$$

$$c_{p'T'} \stackrel{(9)}{=} \frac{c_{\rho'T'} + \frac{CV_{T'}}{CV_{\rho'}} + c_{\rho'T'T'} CV_{T'}}{\frac{\mathbb{V}_{p'}}{CV_{\rho'}}} \stackrel{(11b)}{=} \frac{c_{\rho'T'} + \frac{CV_{T'}}{CV_{\rho'}} + c_{\rho'T'T'} CV_{T'}}{\sqrt{1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} + c_{NLN} CV_{T'}}}, \quad (23b)$$

where the structure parameter c_{NLN} was defined in (11a). With the present choice of independent variables there is no longer a simple separation between linear and nonlinear (amplitude-dependent) terms. Because of the positivity of variances [(3a) and (A1a)], the sign of $\{c_{p'\rho'}, c_{p'T'}\}$ is readily determined from (23),

$$c_{p'\rho'} \leq 0 \stackrel{(23a)}{\iff} c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} \leq -1 - \overbrace{c_{\rho'\rho'T'} CV_{T'}}^{\text{nonlinear term}}, \quad (24a)$$

$$c_{p'T'} \leq 0 \stackrel{(23b)}{\iff} c_{\rho'T'} \leq -\frac{CV_{T'}}{CV_{\rho'}} - \underbrace{c_{\rho'T'T'} CV_{T'}}_{\text{nonlinear term}}. \quad (24b)$$

Inequalities (24) are generally valid, also in the case of strongly compressible turbulence where the nonlinear terms $\propto CV_{T'}$ cannot be neglected. The order relation between $c_{p'\rho'}$ and $c_{p'T'}$ was given in (17d), and can also be obtained from (23a) and (23b), which have the same positive denominator. By (11b) we can also define the conditions for the order relations of fluctuation intensities $\{\mathbb{V}_{p'}, CV_{\rho'}, CV_{T'}\}$,

$$\mathbb{V}_{p'} \leq CV_{\rho'} \stackrel{(11b)}{\iff} c_{\rho'T'} \leq -\frac{1}{2} \frac{CV_{T'}}{CV_{\rho'}} - \underbrace{\frac{1}{2} c_{NLN} CV_{\rho'}}_{\text{nonlinear term}}, \quad (25a)$$

$$\mathbb{V}_{p'} \leq CV_{T'} \stackrel{(11b)}{\iff} c_{\rho'T'} \leq -\frac{1}{2} \frac{1}{\frac{CV_{T'}}{CV_{\rho'}}} - \underbrace{\frac{1}{2} c_{NLN} CV_{\rho'}}_{\text{nonlinear term}}, \quad (25b)$$

with nonlinear terms $\propto CV_{\rho'}$. Because of the nonlinear terms, which involve third-order moments, the exact inequalities (24) and (25) cannot be represented graphically in the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane. They are useful in determining the approximation error of their linearized counterparts which define specific regions in the plane (Sec. VD).

IV. LINEAR AND NONLINEAR REGIMES IN COMPRESSIBLE WALL TURBULENCE

The amplitude-dependent *nonlinear* terms appearing in the original exact relations (9) and in all equations and inequalities developed in Sec. III are $\propto CV_{T'}$ or $\propto CV_{\rho'}$. As discussed in Sec. III B, the equivalent expressions of the nonlinear term in the compatibility relation, (19) which is $\propto CV_{T'}$ and (21) which is $\propto CV_{\rho'}$, support the idea that $CV_{T'}$ and $CV_{\rho'}$ are of the same order of magnitude. This is illustrated by the DNS data for TPC flow [Fig. 2(d)]. We use DNS data from channels and boundary-layers to assess the limit below which the nonlinear terms in the equations can be dropped with reasonable accuracy (*linear regime* of *moderately compressible turbulence*) and to identify conditions of *strongly compressible turbulence* where the nonlinear terms must be retained (*nonlinear regime*). This limit naturally correlates with the level of $CV_{T'}$ (Sec. IV A) which in turn depends on the flow Mach number (Sec. IV B). The above distinction between compressible turbulence regimes is purely local and is only concerned with the accuracy of linearized approximations,

independently of the presence or not of compressibility-induced fluctuating-pressure waves or other nonlocal structures [26,27].

A. Increasing nonlinearity with the level of $CV_{T'}$

The importance of the nonlinear terms can be assessed by evaluating the approximation error of the linearization of the exact equations. The nonlinear realizability boundaries (Fig. 1) were determined from the amplitudes identity (11) and many of the exact inequalities in Sec. III were obtained from the nonlinear compatibility equation (19). Checks (not reported here for conciseness) of available DNS data [24] of compressible turbulent plane channel (TPC) flow satisfy both the exact relation (11b) and the exact compatibility relation (19) with error $< 10^{-5}$. This DNS database [24] of compressible turbulent plane channel (TPC) flows includes all correlations between thermodynamic fluctuations up to third-order and was used to assess the importance of the nonlinear terms in these equations (11) and (19). The approximation error of the linearized approximations (D2) and (31), obtained by neglecting the nonlinear term in these two identities (11) and (19), increases with increasing flow Mach number (Fig. 3).

The approximation error of the widely used [2,15] approximation $CV_{p'}^2 \approx CV_{p'}^2 + CV_{T'}^2 + 2c_{p'\rho'T'}CV_{p'}CV_{T'}$ (D2) obtained by neglecting the nonlinear terms in (11b) is to leading order

$$\frac{\sqrt{CV_{p'}^2 + CV_{T'}^2 + 2c_{p'\rho'T'}CV_{p'}CV_{T'}}}{CV_{p'}} \stackrel{(11b)}{=} \frac{\Phi_{p'}}{CV_{p'}} \sqrt{1 - c_{NLN} \frac{CV_{p'}^2}{\Phi_{p'}^2} CV_{T'}} \stackrel{(\sim)}{\sim} 1 - \frac{1}{2} c_{NLN} \frac{CV_{p'}^2}{\Phi_{p'}^2} CV_{T'} + \dots \quad (26)$$

The departure from unity of the expression on the LHS of (26) is small [Fig. 3(a)] up to low-supersonic Mach numbers ($\bar{M}_{CL_x} \leq 1.5$) but the linearization error (26) increases, locally, to $\sim 5\%$ at $\bar{M}_{CL_x} = 2$ reaching $\sim 20\%$ at $\bar{M}_{CL_x} = 2.5$ [Fig. 3(a)]. This is a pessimistic view, because the wall-normal location of the maximum linearization error [Fig. 3(a)] is also the neighborhood of low levels of p' , $\frac{2}{10} \leq CV_{p'}/CV_{p'} \leq \frac{4}{10}$ [Fig. 2(e)], so that this maximum error [Fig. 3(a)] is calculated relative to small levels of $CV_{p'}$ compared to the local intensities of the other thermodynamic variables, $CV_{p'}$ and $CV_{T'}$ [Figs. 2(d) and 2(e)]. Plotting the linearization error against the local temperature-fluctuation intensity $CV_{T'}(y)$ shows that this increase is not linear with $CV_{T'}$ [Fig. 3(c)] but probably increases as $CV_{T'}^2$. This is explained by the dependence of the factor c_{NLN} (11a) in the nonlinear term (11b) both on the triple correlation coefficients $\{c_{p'\rho'T'}, c_{p'T'T'}, c_{p'\rho'T'}\}$ and on the amplitude ratios $\{\Phi_{p'}/CV_{p'}, CV_{T'}/CV_{p'}\}$, all of which vary (Fig. 2) with varying $(Re_{\tau^*}, \bar{M}_{CL_x})$. The coefficient c_{NLN} increases in absolute value with increasing $\bar{M}_{CL_x}$ in the region $5 \lesssim y^* \lesssim 30$ [Fig. 2(f)] where the maximum linearization error occurs [Fig. 3(a)]. The similarity of the c_{NLN} -profiles [Fig. 2(f)] with the $c_{p'\rho'T'}$ -profiles [Fig. 2(c)] is easily explained by the linearized form of (11a) $c_{NLN} \sim 2c_{p'\rho'T'}CV_{p'}/CV_{p'} + O(CV_{T'})$. Systematic examination of the data at constant Reynolds numbers (not shown) suggests that the nonlinear effects are stronger at low Re, so that the $\propto CV_{T'}^2$ curve-fit [Fig. 3(c)] is an average trend of the Re-dependent data. Notice that comparison of the profiles (Fig. 2) at nearly constant $\bar{M}_{CL_x} \approx 2$ for $Re_{\tau^*} \in \{341, 963\}$ (Re-effect) and at nearly constant $Re_{\tau^*} \approx 1000$ for $\bar{M}_{CL_x} \in \{0.81, 1.50, 1.96\}$ (M-effect) clearly identifies a near-wall region ($y^* \lesssim 20$) where the profiles are essentially Mach-dependent and an outer region ($y^* \gtrsim 40$) where they are mostly Reynolds-dependent. An exception to this observation is the correlation coefficient $c_{p'\rho'T'}$ for which substantial Mach-dependence is observed for the $Re_{\tau^*} \approx 1000$ data in the wake region [Fig. 2(c)]. The c_{NLN} -profiles because of their relation $c_{NLN} \sim 2c_{p'\rho'T'}CV_{p'}/CV_{p'} + O(CV_{T'})$ (11a) with the $c_{p'\rho'T'}$ -profiles also exhibit this Mach-dependence in the wake region [Fig. 2(f)].

The linearized compatibility relation $c_{p'\rho'}^2 + c_{p'T'}^2 + c_{\rho'T'}^2 - 2c_{p'\rho'T'}c_{p'T'}c_{\rho'T'} \approx 1$ (31) obtained by dropping the nonlinear terms in (19) is a more accurate approximation [Figs. 3(b) and 3(d)] than the previously studied linearized amplitudes relation [Figs. 3(a) and 3(c)]. With respect to the wall-normal distance, the approximation error is maximal around $y^* \approx 10$ [Fig. 3(c)] and

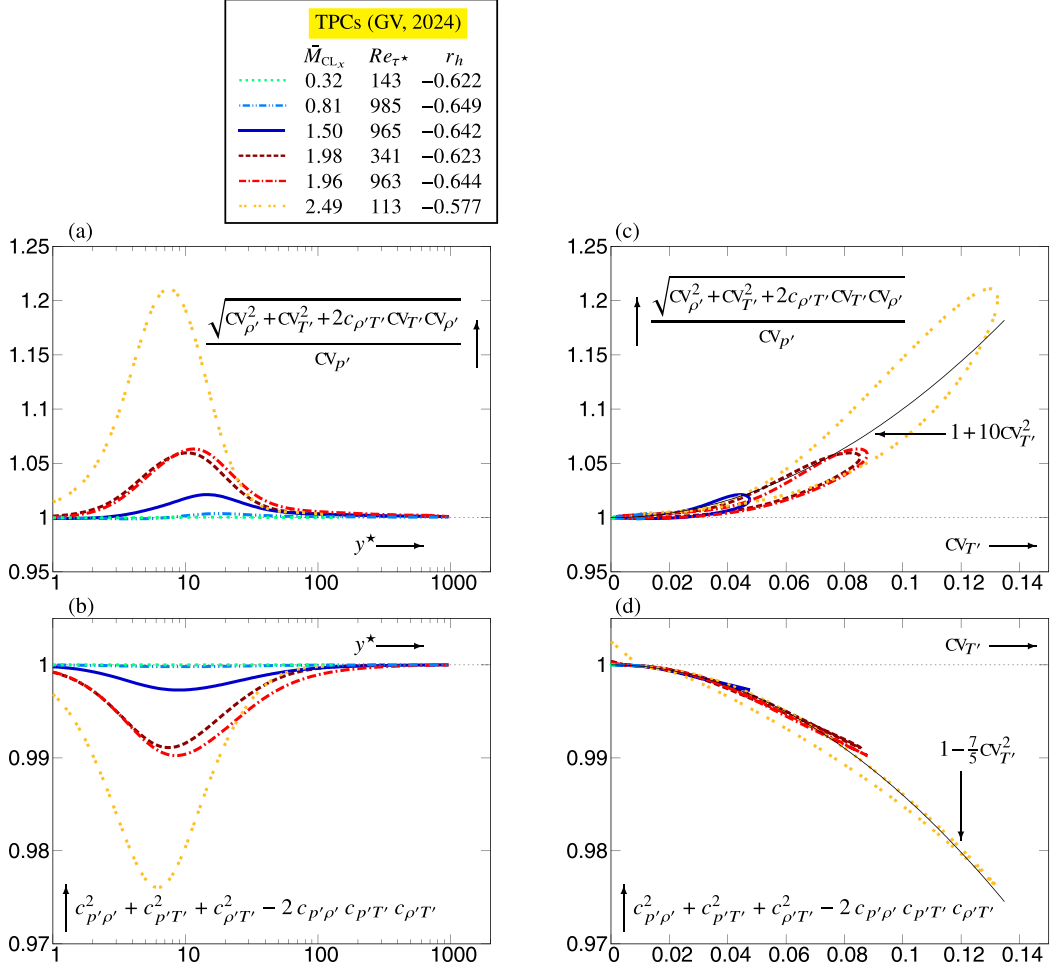


FIG. 3. Approximation error (departure from unity) of (a,c) the linearized amplitudes relation $CV_{p'}^2 \approx CV_{p'}^2 + CV_{T'}^2 + 2c_{p'T'} CV_{T'} CV_{p'}$ (D2) for $CV_{p'}$ and of (b,d) the RHS of the linearized compatibility relation $1 \approx c_{p'\rho'}^2 + c_{p'T'}^2 + c_{\rho'T'}^2 - 2c_{p'\rho'} c_{p'T'} c_{\rho'T'}$ (31), plotted (a,b) against the HCB-scaled [25] wall-distance y^* (log scale), and (c,d) against the local temperature-fluctuation level $CV_{T'} := T'_{rms}/\bar{T}$ (A1a), for selected flows from the compressible turbulent plane channel (TPC) flows DNS database [24] (GV, 2024), representative of the available range $97 \leq Re_{\tau^*} \leq 985$ and $0.32 \leq \bar{M}_{CL,x} \leq 2.49$.

increases with increasing Mach number, reaching $\sim 2.5\%$ at $\bar{M}_{CL,x} = 2.49$ [Fig. 3(c)]. Plotting the linearization error against the local temperature-fluctuation intensity $CV_{T'}(y)$ shows that it increases approximately $\propto CV_{T'}^2$ [Fig. 3(d)]. This implies that the combination of correlation coefficients (2CCs and 3CCs) which multiplies $CV_{T'}$ in the nonlinear term of the exact compatibility relation (19) increases nonlinearly $\propto CV_{T'}$.

B. Intensity of temperature fluctuations $CV_{T'}$ in channels and boundary-layers

The above investigation (Fig. 3) of the linearization error of exact equations was limited to TPC flow because not all of the correlations that are necessary for this analysis were available in databases of other flows. This limitation notwithstanding, the temperature fluctuations intensity $CV_{T'}$ was clearly identified [Figs. 3(c) and 3(d)] as an appropriate sensor for increasing nonlinear effects. It

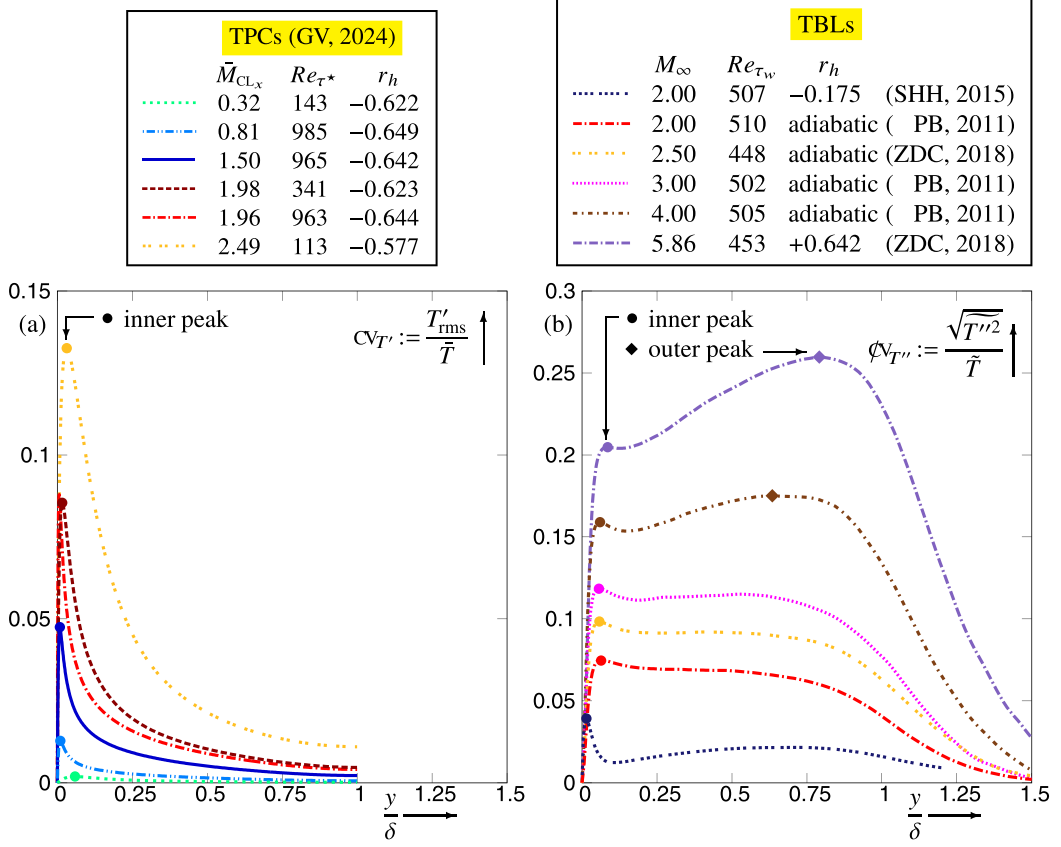


FIG. 4. Typical profiles of $CV_{T'} := T'_{rms}/\bar{T}$ for turbulent plane channel flow (TPC) from the DNS datasets of Gerolymos and Vallet [24] (GV, 2024) and of $\phi_{V_{T''}} := (\sqrt{T''^2})/\bar{T}$ for zero-pressure-gradient (ZPG) turbulent boundary-layers (TBLs), both from the adiabatic-wall ($T_w = T_r$) DNS datasets of Pirozzoli and Bernardini [28] (PB, 2011) and from the cooled-wall ($T_w < T_r$) DNS datasets of Zhang *et al.* [30] (ZDC, 2018) and of Shadloo *et al.* [29] (SHH, 2015), plotted against the outer-scaled wall-distance y/δ [δ is the half-channel height for TPCs and the location $\tilde{u}(\delta) = 0.99 u_\infty$ for TBLs].

is therefore worthwhile, in order to connect the mathematical developments in the paper with the behavior of actual flows, to examine data for $CV_{T'}$ in wall turbulence, channels and boundary-layers (Fig. 4).

A difficulty encountered in the compilation of compressible wall-turbulence data for this analysis is that many of the available databases contain data for $\{\tilde{T}, \sqrt{\tilde{T}''^2}\}$ and not for $\{\bar{T}, T'_{rms}\}$, so that in these cases we will use the pseudo-CV

$$\phi_{V_{T''}} := \frac{\sqrt{\tilde{T}''^2}}{\bar{T}} \quad (27)$$

in lieu of the coefficient of variation $CV_{T'}$. The differences between $CV_{T'}$ and $\phi_{V_{T''}}$ are studied in Appendix C where it is shown that, to leading order, $CV_{T'} \sim \phi_{V_{T''}}[1 + O(CV_{\rho'})]$ (C2b) in contrast to $\tilde{T} \sim \bar{T}[1 + O(CV_{\rho'} CV_{T'})]$ (C2c). Therefore, the difference between Favre and Reynolds fluctuation intensities [Fig. 10(b)] increases faster with Mach number than the difference between averages [Fig. 10(a)]. In channels the departure from unity of the ratio $CV_{T'}/\phi_{V_{T''}}$ increases with $CV_{T'}$ (Fig. 11) consistently with (C2b), reaching 5% at $\bar{M}_{CLx} = 2.5$.

The wall-normal profiles of temperature fluctuation intensity behave differently in compressible turbulent plane channels (TPCs) [24] and in zero-pressure-gradient (ZPG) turbulent boundary-layers (TBLs) [28–30] (Fig. 4). The wall thermal condition is best described [21] by the nondimensional enthalpy difference across the wall-layer,

$$r_h := \begin{cases} \frac{\bar{h}_w - h_\infty}{\frac{1}{2} u_\infty^2} & \text{ZPGTBLs,} \\ \frac{\bar{h}_w - h_{CL}}{\frac{1}{2} u_{CL}^2} & \text{TPCs,} \end{cases} \quad (28)$$

where $(\cdot)_{CL}$ denotes centerline (TPCs), $(\cdot)_\infty$ denotes conditions at infinity (ZPGTBLs), and $(\cdot)_w$ denotes the wall. This quantity, which appears directly [[21], (3.3), p. A25-9] in the generally accepted [10] quadratic $\bar{h}(\bar{u})$ relation, is better adapted than the wall-to-recovery temperature ratio $\bar{T}_w/T_r$, which strongly varies with the external flow Mach number M_∞^2 . In TPC flow, $CV_{T'}$ has an inner-peak near the wall, $CV_{T'_{\text{PEAK}}} := \max_y(CV_{T'})$, and then diminishes monotonically to smaller values at the centerline [Fig. 4(a)]. The Mach number has no particular influence on the shape the $CV_{T'}$ -profile in TPCs [Fig. 4(a)], it only determines the peak-value. The $\mathcal{CV}_{T''}$ -profiles of ZPG TBLs over an adiabatic wall [Fig. 4(b)] differ substantially from the channel profiles. Notice first that the thermal boundary-layer is thicker than the dynamic (velocity) boundary-layer [δ is the thickness defined by the velocity-profile; Fig. 4(b)]. This is the expected behavior when the effective Prandtl-number $Pr_{\text{eff}} < 1$ ([31], Fig. 1.6, p. 10), and DNS data indicate that, except very near the wall, the turbulent Prandtl number, $Pr_T < 1$ both in channels ([21], Fig. 5, p. A25-14) and in boundary-layers ([32], Fig. 17, p. A3-30), decreasing with increasing wall distance. In contrast to channels [Fig. 4(a)], the $\mathcal{CV}_{T''}$ -profile in adiabatic boundary-layers forms a plateau after the near-wall (inner) peak [Fig. 4(b)], and decreases to small values in the range $0.8 \lesssim y/\delta \lesssim 1.5$. With increasing M_∞ , the value of $\mathcal{CV}_{T''}$ in the outer part of the dynamic boundary-layer increases, and at $M_\infty \approx 4$ there appears an outer peak which is higher than the inner peak, and further increases with increasing M_∞ . Not only are the thermal and dynamic boundary-layer thicknesses different in TBLs, but the thermal wall condition is also very different in channels ($-0.65 \leq r_h \leq -0.58$ [21]) and in adiabatic boundary-layers ($r_h = r_f \approx 0.89$ [28–30]). The data of Shadloo *et al.* [29] include a ZPGTBL at $M_\infty = 2$ with a cold wall ($r_h = -0.18$) showing substantial difference in the $CV_{T'}$ -profile compared to the adiabatic ZPGTBL at the same $M_\infty = 2$ [Fig. 4(b)], with a much sharper inner peak of $CV_{T'}$ and lower values in the outer part of the boundary-layer. To the authors' knowledge, there are no available boundary-layer data in the even lower r_h range of channel flows.

The maximal values in the profile, $\max_y CV_{T'}$ or $\max_y \mathcal{CV}_{T''}$, increase with increasing flow Mach number, $\bar{M}_{CL_x}$ or M_∞ (Fig. 5). For $M_\infty < 4$ the maximal value corresponds to the inner peak, whereas for $M_\infty \geq 4$ it corresponds to the outer peak (Fig. 4), as shown in the adiabatic TBL data of Pirozzoli and Bernardini [28] at $M_\infty = 4$ for which both peaks are plotted (Figs. 4 and 5). The channel data, for the entire range $100 \lesssim Re_{\tau^*} \lesssim 1000$, are well grouped together, showing an increase $\propto \bar{M}_{CL_x}^{(2+0.073\bar{M}_{CL_x})}$ very similar to the peak of entropy fluctuations ([19], Fig. 2, p. 5), satisfying very closely the empirical relation $CV_{T'_{\text{PEAK}}} \approx \max_y(s'_{\text{rms}})/c_p$. A closer examination of the data ([19], Fig. 2, p. 5) shows a weak increase with increasing Re_{τ^*} . The inner peak for supersonic boundary-layers ($M_\infty \leq 4$) follows a different slower increasing curve, and the data of Pirozzoli and Bernardini [28] at $M_\infty = 2$ show a decrease of $\mathcal{CV}_{T'_{\text{PEAK}}}$ with increasing Re_{τ_w} (Fig. 5). Notice also the strong effect of r_h on the value of $CV_{T'_{\text{PEAK}}}$ in the $M_\infty = 2$ boundary-layer data of Shadloo *et al.* [29] showing a substantial decrease for the $r_h = -0.18$ cold-wall case (Fig. 5). In the hypersonic régime ($M_\infty \geq 5$) the outer peak increases strongly, although available data [30] are not yet sufficiently dense in $\{Re_{\tau_w}, M_\infty, r_h\}$ to determine a precise correlation for the increase of the outer peak with M_∞ (Fig. 4). Generally, for $M_\infty \leq 2$ (or $\bar{M}_{CL_x} \leq 2$), $\mathcal{CV}_{T''} \lesssim 0.1$ (or $CV_{T'} \lesssim 0.1$), and thermodynamic turbulence is expected to be in the linear régime, whereas at higher Mach numbers, regions of the flow in the nonlinear régime are expected, in particular near the boundary-layer edge for hypersonic Mach numbers (Figs. 4 and 5).

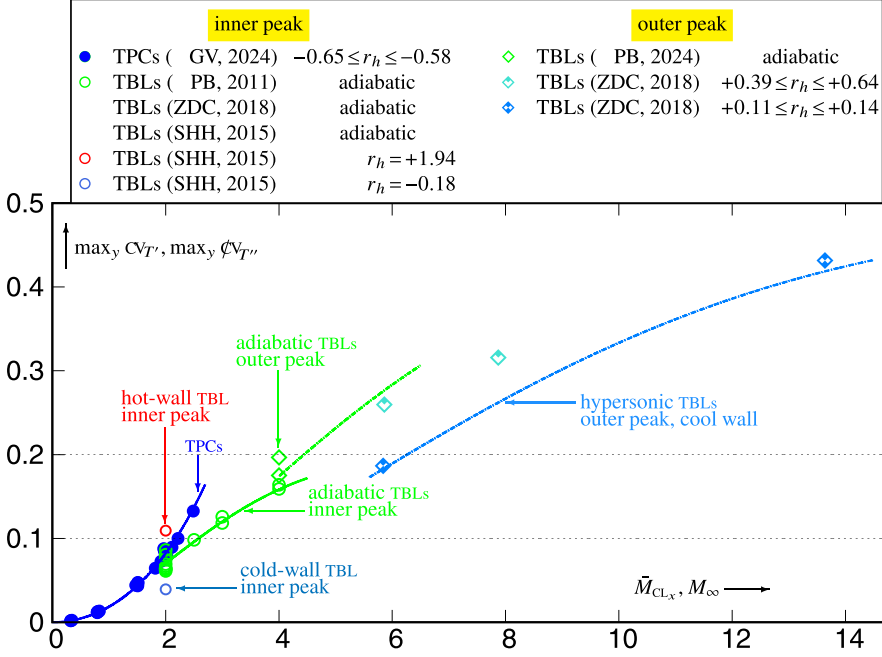


FIG. 5. Maximal values of the profiles of $CV_{T'} := T'_{rms}/\bar{T}$ for turbulent plane channel flow (TPC) from the DNS datasets of Gerolymos and Vallet [24] (GV, 2024; $97 \leq Re_{\tau^*} \leq 985$) and of $(\sqrt{\hat{T}''^2})/\bar{T}$ for zero-pressure-gradient (ZPG) turbulent boundary-layers (TBLs; $200 \leq Re_{\tau_w} \leq 1100$), both from the adiabatic-wall ($T_w = T_r$) DNS datasets of Pirozzoli and Bernardini [28] (PB, 2011) and from the nonadiabatic-wall ($T_w \neq T_r$) DNS datasets of Zhang *et al.* [30] (ZDC, 2018) and Shadloo *et al.* [29] (SHH, 2015), plotted against the centerline streamwise Mach number $\tilde{M}_{CL,x}$ for TPCs or the external flow Mach number M_∞ for TBLs.

V. APPROXIMATE LINEARIZED REALIZABILITY, COMPATIBILITY AND MAPPINGS

The linear régime (moderately compressible turbulence) corresponds to flows or flow-regions where the relative fluctuation amplitudes of the thermodynamic quantities, quantified by the coefficients of variation (A1a) are sufficiently small, formally

$$CV_{p'} \ll 1; \quad CV_{\rho'} \ll 1; \quad CV_{T'} \ll 1 \quad \stackrel{(7,C1)}{\Rightarrow} \quad \begin{cases} \Phi_{p'} \approx CV_{p'} \\ \Phi_{T'} \approx CV_{T'}. \end{cases} \quad (29)$$

Then, the nonlinear terms $\propto CV_{T'}$ in (9) can be neglected with acceptable accuracy. The analysis of the linearization error for the pressure variance equation [Fig. 3(c)] suggests that the limit of the linear régime can be set to $\max(CV_{p'}, CV_{\rho'}, CV_{T'}) \lesssim 10\%$, although this limit depends on the specific equation that is linearized, some approximations being more robust (valid for higher fluctuation amplitudes) than others [2].

The detailed mathematical proofs of approximate equalities and inequalities in the linear régime are given in Appendix D. We use below, when necessary, equations and inequalities demonstrated in Appendix D to obtain the final linearized relations. The straightforward linearization of the basic exact nonlinear equations is summarized for completeness in (D1)–(D3). Notice that dropping the nonlinear terms from the inequalities allows many further developments leading to the definition of specific regions, both in the linear realizability $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane (Sec. V C) or in the $(CV_{T'}/CV_{\rho'}, c_{\rho T'})$ -plane (Sec. V D).

A. Approximate linearized inequalities between correlation coefficients

Dropping the nonlinear terms in the inequalities (17) and (22) provides general approximate inequalities between correlation coefficients,

$$\begin{aligned} & \text{(22b,22c,29)} \\ c_{\rho'T'} & \lesseqgtr c_{p'\rho'} c_{p'T'}, \end{aligned} \quad (30a)$$

$$\begin{aligned} & \text{(22d,29)} \\ c_{p'\rho'} c_{\rho'T'} & \lesseqgtr c_{p'T'}, \end{aligned} \quad (30b)$$

$$\begin{aligned} & \text{(22a,29)} \\ c_{p'T'} c_{\rho'T'} & \lesseqgtr c_{p'\rho'}, \end{aligned} \quad (30c)$$

$$\begin{aligned} & \text{(17d,29)} \\ c_{p'T'} \lesseqgtr c_{p'\rho'} & \iff CV_{T'} \lesseqgtr CV_{\rho'}, \end{aligned} \quad (30d)$$

$$\begin{aligned} & \text{(17a,29)} \\ c_{p'\rho'} \lesseqgtr 0 & \iff \left(\frac{CV_{p'}}{CV_{\rho'}} \right)^2 \lesseqgtr \left(\frac{CV_{T'}}{CV_{\rho'}} \right)^2 - 1, \end{aligned} \quad (30e)$$

$$\begin{aligned} & \text{(17b,29)} \\ c_{p'T'} \lesseqgtr 0 & \iff \left(\frac{CV_{p'}}{CV_{\rho'}} \right)^2 \lesseqgtr 1 - \left(\frac{CV_{T'}}{CV_{\rho'}} \right)^2, \end{aligned} \quad (30f)$$

$$\begin{aligned} & \text{(17c,29)} \\ c_{\rho'T'} \lesseqgtr 0 & \iff \left(\frac{CV_{p'}}{CV_{\rho'}} \right)^2 \lesseqgtr 1 + \left(\frac{CV_{T'}}{CV_{\rho'}} \right)^2. \end{aligned} \quad (30g)$$

Inequalities (30a)–(30c) are obtained from the linearization of (22) which are a by-product of the proof of the nonlinear compatibility relation (Sec. III C) and can also be obtained from the corresponding linearized proof (D5). It will be seen in the following that they are essential to the analysis of the linearized mapping of the thermodynamic turbulence structure, underscoring the importance of the compatibility relation.

Inequalities (30d)–(30g) define regions in the linearized realizability $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane with respect to the signs of the correlation coefficients which are studied in Sec. V C. By (30d), in the linear regime, the order relation between $(c_{p'T'}, c_{p'\rho'})$ follows the order relation between $(CV_{T'}, CV_{\rho'})$.

B. Approximate linearized compatibility relation

The approximate compatibility relation, obtained by dropping the nonlinear term in (19) and (21), reads

$$c_{p'\rho'}^2 + c_{p'T'}^2 + c_{\rho'T'}^2 - 2 c_{p'\rho'} c_{p'T'} c_{\rho'T'} \stackrel{(19, 21, 29)}{\cong} 1 \quad (31)$$

which can also be proven directly from the linearized equations (D5).

Equation (31) can be considered as a quadratic equation for each correlation coefficient in terms of the other two, and can be solved successively for $\{c_{p'\rho'}, c_{p'T'}, c_{\rho'T'}\}$. In each case, the general inequalities (30) determine uniquely the correct root,

$$\begin{aligned} & \text{(31,30a)} \\ c_{\rho'T'} & \cong c_{p'\rho'} c_{p'T'} - \sqrt{(1 - c_{p'\rho'}^2)(1 - c_{p'T'}^2)} \leq c_{p'\rho'} c_{p'T'}, \end{aligned} \quad (32a)$$

$$\begin{aligned} & \text{(31,30b)} \\ c_{p'T'} & \cong c_{p'\rho'} c_{\rho'T'} + \sqrt{(1 - c_{p'\rho'}^2)(1 - c_{\rho'T'}^2)} \geq c_{p'\rho'} c_{\rho'T'}, \end{aligned} \quad (32b)$$

$$\begin{aligned} & \text{(31,30c)} \\ c_{p'\rho'} & \cong c_{p'T'} c_{\rho'T'} + \sqrt{(1 - c_{p'T'}^2)(1 - c_{\rho'T'}^2)} \geq c_{p'T'} c_{\rho'T'}, \end{aligned} \quad (32c)$$

where the radicals are obviously non-negative by the covariance inequality (A1c). These equations (32) are specific to the linear régime because the choice of the roots is based to the linearized

inequalities (32), showcasing the additional relations that can be obtained for moderately compressible turbulence and the implications of the compatibility relation. The approximation accuracy of (32) is assessed using TPC-flow DNS data in Sec. VI B.

C. Approximate linearized realizability in the $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane

By (29), the realizability inequality (15) linearizes to

$$(15, 29) \Rightarrow \left| 1 - \frac{CV_{T'}}{CV_{\rho'}} \right| \lesssim \frac{CV_{p'}}{CV_{\rho'}} \lesssim 1 + \frac{CV_{T'}}{CV_{\rho'}} \quad (33)$$

so that the linearized realizability boundary in the $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane consists of three straight lines [Fig. 6(a)], on which the approximate correlation coefficients calculated from (D3) are ± 1 (D4).

Combining the linearized realizability inequality (33) with the inequalities (30d)–(30g) defines four distinct regions in the linearized realizability $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane,

$$\textcircled{A} : \frac{CV_{T'}}{CV_{\rho'}} \leq 1; 0 < 1 - \frac{CV_{T'}}{CV_{\rho'}} \leq \frac{CV_{p'}}{CV_{\rho'}} \leq \sqrt{1 - \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2}, \quad (34a)$$

$$\textcircled{B} : \frac{CV_{T'}}{CV_{\rho'}} \geq 1; 0 < \frac{CV_{T'}}{CV_{\rho'}} - 1 \leq \frac{CV_{p'}}{CV_{\rho'}} \leq \sqrt{\left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 - 1}, \quad (34b)$$

$$\textcircled{C} : \sqrt{\left| 1 - \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 \right|} \leq \frac{CV_{p'}}{CV_{\rho'}} \leq \sqrt{1 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2}, \quad (34c)$$

$$\textcircled{D} : \sqrt{1 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2} \leq \frac{CV_{p'}}{CV_{\rho'}} \leq 1 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right). \quad (34d)$$

In each of these regions (34) specific inequalities (D6) hold regarding the correlation coefficients. Combining (D6) with the general inequalities (30a)–(30c) leads, after some calculations detailed in (D7) and (D8) to the following conditions for the correlation coefficients in each of the zones $\{\textcircled{A}, \textcircled{B}, \textcircled{C}, \textcircled{D}\}$:

$$\textcircled{A} \quad (\text{D6a}, \text{D7a}, \text{D8a}) \Rightarrow c_{\rho'T'} \gtrsim c_{p'T'} \gtrsim 0 \gtrsim c_{p'\rho'}; \quad |c_{p'T'}| \gtrsim |c_{p'\rho'}|, \quad (35a)$$

$$\textcircled{B} \quad (\text{D6b}, \text{D7b}, \text{D8b}) \Rightarrow c_{\rho'T'} \gtrsim c_{p'\rho'} \gtrsim 0 \gtrsim c_{p'T'}; \quad |c_{p'\rho'}| \gtrsim |c_{p'T'}|, \quad (35b)$$

$$\textcircled{C} \quad (\text{D6c}, \text{D7a}, \text{D8c}) \Rightarrow c_{\rho'T'} \gtrsim 0 \gtrsim c_{p'\rho'} c_{p'T'} \gtrsim \min(c_{p'\rho'}, c_{p'T'}); \quad c_{p'\rho'} \gtrsim 0; \quad c_{p'T'} \gtrsim 0, \quad (35c)$$

$$\textcircled{D} \quad (\text{D6d}, \text{D7a}, \text{D8c}) \xRightarrow{(30a)} 0 \gtrsim c_{\rho'T'} \gtrsim c_{p'\rho'} c_{p'T'} \gtrsim \min(c_{p'\rho'}, c_{p'T'}); \quad c_{p'\rho'} \gtrsim 0; \quad c_{p'T'} \gtrsim 0 \quad (35d)$$

In line with the partitioning of the nonlinear realizability plane [Fig. 1(a)], in the linear régime, $c_{\rho'T'} \gtrsim 0$ in region $\textcircled{D}$ adjoining the upper realizability boundary [Fig. 6(a)], and $c_{\rho'T'} \lesssim 0$ in regions $\{\textcircled{A}, \textcircled{B}, \textcircled{C}\}$. In the nonlinear case, the partitioning $\{\textcircled{A}, \textcircled{B}, \textcircled{C}\}$ was not strictly possible, because of the remaining nonlinear terms in the exact inequalities (17a), (17b), and (17d). In the regions adjoining the lower realizability boundary, either $c_{p'T'} \lesssim 0 \lesssim c_{p'\rho'}$ in $\textcircled{A}$ or $c_{p'\rho'} \lesssim 0 \lesssim c_{p'T'}$ in $\textcircled{B}$ [Fig. 6(a)]. In the linear régime, under EOS (1), turbulence where both $c_{p'T'}$ and $c_{p'\rho'}$ are negative is not possible. Examples of this mapping of thermodynamic turbulence structure for channels and boundary-layers in the linearized realizability $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane are given in Sec. VI A.

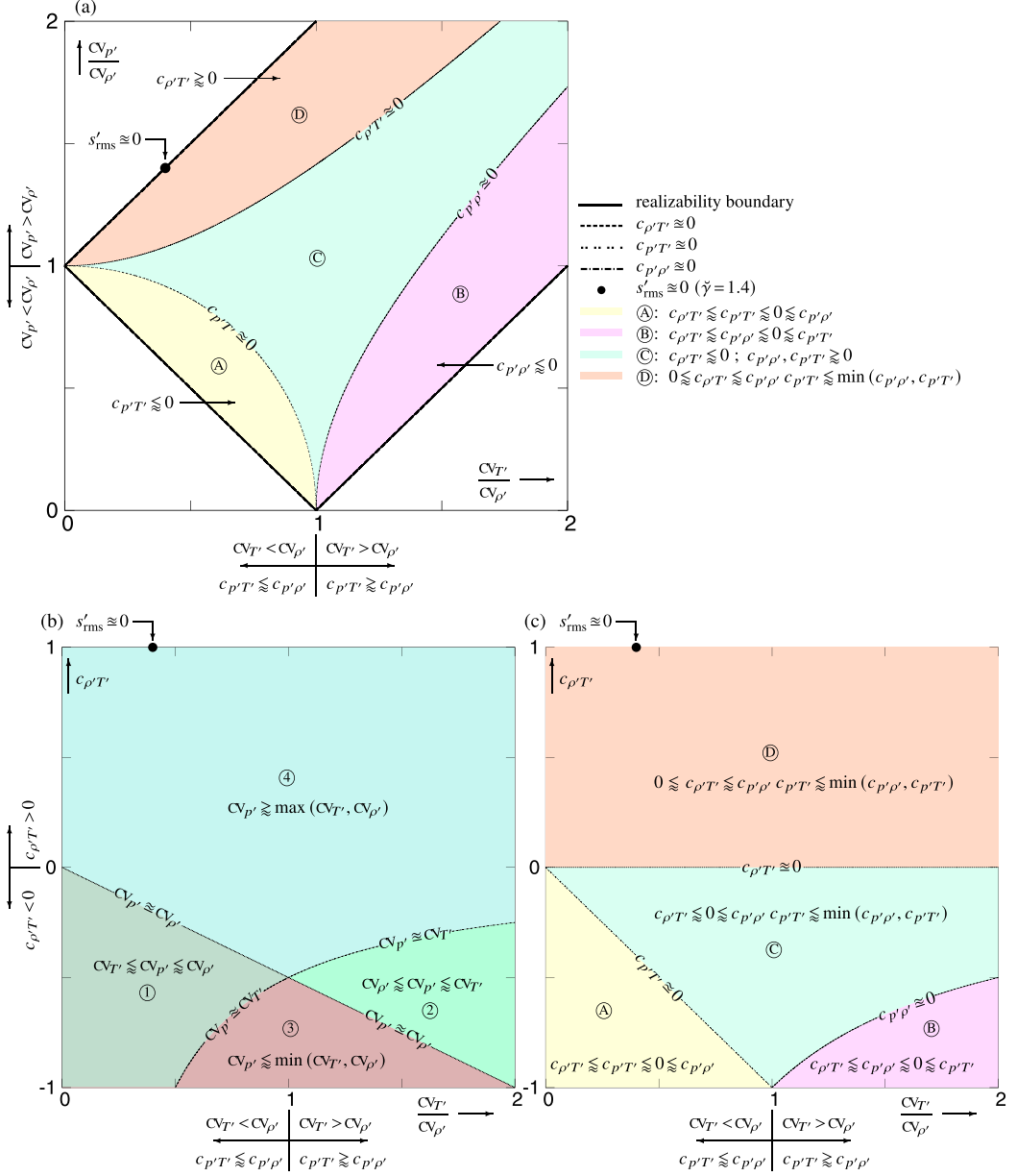


FIG. 6. Approximate linearized partitionings of (a) the $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane with the realizability boundaries (33) and of (b,c) the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane, by (a,c) the loci of $\{c_{p'T'} \approx 0, c_{p'T'} \approx 0, c_{p'\rho'} \approx 0\}$ determining the four distinct regions $\{(A), (B), (C), (D)\}$ [(34) and (36)] regarding the signs and order relations (35), and (b) by the loci of $\{CV_{p'} \approx CV_{\rho'}, CV_{p'} \approx CV_{T'}\}$ determining the four distinct regions $\{(1), (2), (3), (4)\}$ (37) with respect to the order relations between fluctuation intensities $\{CV_{p'}, CV_{\rho'}, CV_{T'}\}$; also indicated the isolated point of linearized isentropic turbulence [[2], (6.1), p. 468].

D. Approximate linearized mapping onto the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane

By the exact nonlinear equation (16c), the nonlinear realizability plane (Fig. 6) can be mapped onto the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane. It is straightforward to linearize (16c) as (D3c) and to apply this

mapping to the linear régime. The $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane was used in [2] to map the thermodynamic turbulence structure. The relevant calculations (D10)–(D13) are detailed in Appendix D.

The regions $\{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}\}$ defined in (34) with respect to the signs and order relations (35) between the correlation coefficients $\{c_{p'\rho'}, c_{p'T'}, c_{\rho'T'}\}$ map onto the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane [Fig. 6(c)] as

$$\mathbb{A} : \frac{CV_{T'}}{CV_{\rho'}} \leq 1; \quad -1 \leq c_{\rho'T'} \leq -\frac{CV_{T'}}{CV_{\rho'}} \xRightarrow{(D13,D6)} (35a), \quad (36a)$$

$$\mathbb{B} : \frac{CV_{T'}}{CV_{\rho'}} \geq 1; \quad -1 \leq c_{\rho'T'} \leq -\frac{1}{\frac{CV_{T'}}{CV_{\rho'}}} \xRightarrow{(D13,D6)} (35b), \quad (36b)$$

$$\mathbb{C} : \max\left(-\frac{CV_{T'}}{CV_{\rho'}}, -\frac{1}{\frac{CV_{T'}}{CV_{\rho'}}}\right) \leq c_{\rho'T'} \leq 0 \xRightarrow{(D13,D6)} (35c), \quad (36c)$$

$$\mathbb{D} : 0 \leq c_{\rho'T'} \leq 1 \xRightarrow{(29,D6)} (35d). \quad (36d)$$

By (D4), the lower-realizability boundary [Fig. 6(a)] is mapped to the $(c_{\rho'T'} = -1)$ -line, and the upper realizability boundary is mapped to the $(c_{\rho'T'} = +1)$ -line, so that the entire $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane is realizable.

Regions with respect to order relations between fluctuation intensities are trivial in the $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane because its axes are precisely the ratios of fluctuation intensities. This is no longer the case in the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane. The inequalities (D12) define four distinct regions in the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane,

$$\textcircled{1} : \max\left(-1, -\frac{1}{2} \frac{1}{\frac{CV_{T'}}{CV_{\rho'}}}\right) \leq c_{\rho'T'} \leq -\frac{1}{2} \frac{CV_{T'}}{CV_{\rho'}} \xRightarrow{(D12)} CV_{T'} \lesssim CV_{p'} \lesssim CV_{\rho'}, \quad (37a)$$

$$\textcircled{2} : -\frac{1}{2} \frac{CV_{T'}}{CV_{\rho'}} \leq c_{\rho'T'} \leq -\frac{1}{2} \frac{1}{\frac{CV_{T'}}{CV_{\rho'}}} \xRightarrow{(D12)} CV_{\rho'} \lesssim CV_{p'} \lesssim CV_{T'}, \quad (37b)$$

$$\textcircled{3} : -1 \leq c_{\rho'T'} \leq \min\left(-\frac{1}{2} \frac{CV_{T'}}{CV_{\rho'}}, -\frac{1}{2} \frac{1}{\frac{CV_{T'}}{CV_{\rho'}}}\right) \xRightarrow{(D12)} CV_{p'} \lesssim \min(CV_{T'}, CV_{\rho'}), \quad (37c)$$

$$\textcircled{4} : \max\left(-\frac{1}{2} \frac{CV_{T'}}{CV_{\rho'}}, -\frac{1}{2} \frac{1}{\frac{CV_{T'}}{CV_{\rho'}}}\right) \leq c_{\rho'T'} \leq 1 \xRightarrow{(D12)} CV_{p'} \gtrsim \max(CV_{T'}, CV_{\rho'}), \quad (37d)$$

with respect to the relative levels of fluctuation intensities. These regions $\{\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}\}$ define a different partitioning [Fig. 6(b)] of the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane, with respect to the relative levels of fluctuation intensities $\{CV_{p'}, CV_{\rho'}, CV_{T'}\}$. Examples of application of these mappings of thermodynamic turbulence structure for TPC flow are given in Sec. VIA.

VI. APPLICATION AND ACCURACY OF THE LINEARIZED APPROXIMATIONS

The usefulness of the approximate relations obtained in Sec. V to the analysis of the thermodynamic turbulence structure is illustrated by examples of application to channel and boundary-layer flows (Sec. VIA). These approximate relations represent the mathematically correct asymptotic behavior of low-Mach aerodynamic turbulence where the low-intensity conditions (29) are fully satisfied. The limit of their applicability with increasing Mach number is expected to be flow-dependent. This remark notwithstanding, the approximation error of the different linearized relations is assessed for TPC flow at $\bar{M}_{CL_x} = 2$ (Sec. VIB).

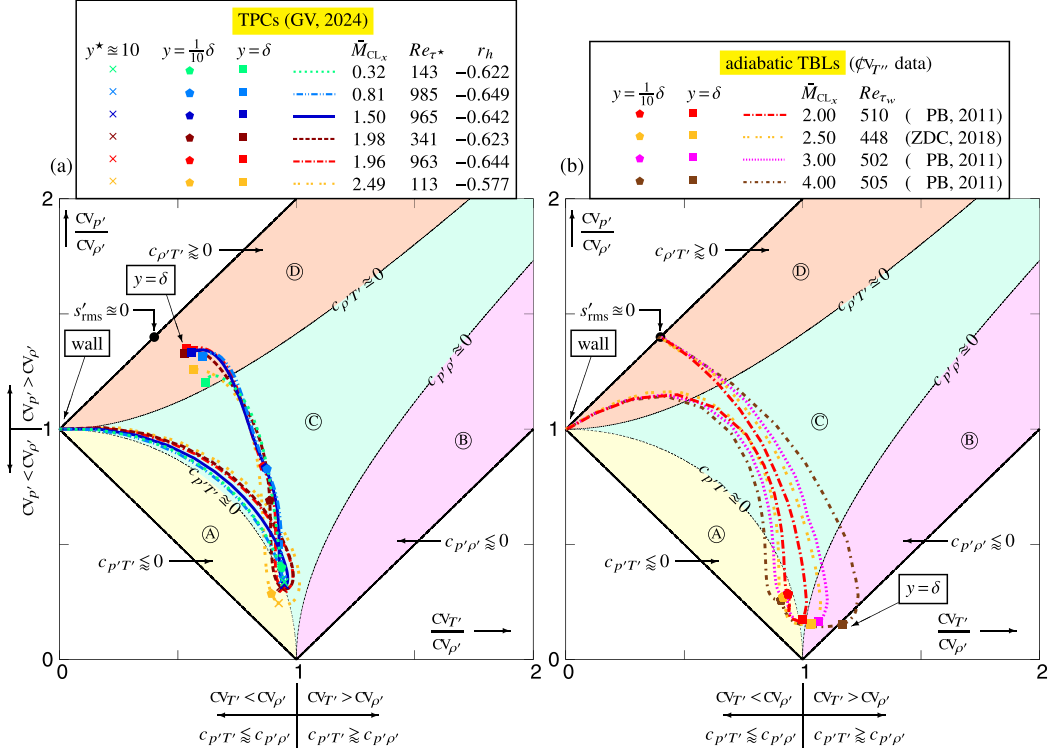


FIG. 7. Approximate linearized mapping on the $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane [Fig. 6(a)] of representative DNS data for (a) turbulent plane channel flow from the DNS datasets of Gerolymos and Vallet [24] (GV, 2024) and for (b) adiabatic-wall zero-pressure-gradient (ZPG) turbulent boundary-layers (TBLs) from the datasets of Pirozzoli and Bernardini [28] (PB, 2011) and of Zhang *et al.* [30] (ZDC, 2018).

A. Application of the linearized mappings

The significant differences between channels and boundary-layers, already observed in the wall-normal distributions (Fig. 4) and in the peak value (Fig. 5) of $CV_{T'}$, also appear in the realizability plane (Fig. 7). The difference in the thermal wall-condition affects the correlation coefficients in the near-wall sublayer, with $c_{\rho'T'} < 0$ and $c_{p'T'}$ close to 0 for the very-cold-wall channels [Fig. 7(a)], and $c_{\rho'T'} > 0$ and $c_{p'T'}$ away of the 0-line for the hot-wall adiabatic condition in the boundary-layers [Fig. 7(b)]. In TPC flow [Fig. 7(a)], moving from the sublayer towards $y^* \approx 10$ the locus in the linearized realizability plane moves towards the lower boundary reaching low levels of $CV_{p'}/CV_{\rho'}$ [Fig. 2(d)], then moves upwards towards positive $c_{\rho'T'} > 0$ at centerline, where values $CV_{p'}/CV_{\rho'} > 1$ are observed [Fig. 2(d)]. Comparison of the $y^* \approx 10$ and $y/\delta \approx \frac{1}{10}$ locations along the locus in the realizability plane clearly shows that the wall-distance of the $(CV_{p'}/CV_{\rho'})$ -minimum location is inner-scaled. In adiabatic-wall TBLs [Fig. 7(b)], moving away from the sublayer, the flow also reaches low $CV_{p'}/CV_{\rho'}$ levels with negative $c_{\rho'T'} < 0$, but contrary to channels, these conditions of low $CV_{p'}/CV_{\rho'}$ and negative $c_{\rho'T'} < 0$ persist up to the boundary-layer edge. For $y > \delta$, the loci move upwards towards higher $CV_{p'}/CV_{\rho'} > 1$ and positive $c_{\rho'T'} > 0$ [Fig. 7(b)], ultimately approaching nearly isentropic turbulence.

Notice that with increasing M_∞ the locus in the $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane moves closer to the lower realizability boundary near $y = \delta$, and at $M_\infty = 4$ crosses it [Fig. 7(b)], indicating that the linearized approximation becomes inaccurate because of the high $CV_{T'}$ levels that are observed near the boundary-layer edge [Fig. 4(b)]. Furthermore, regarding boundary-layers, DNS data are for $CV_{T''}$

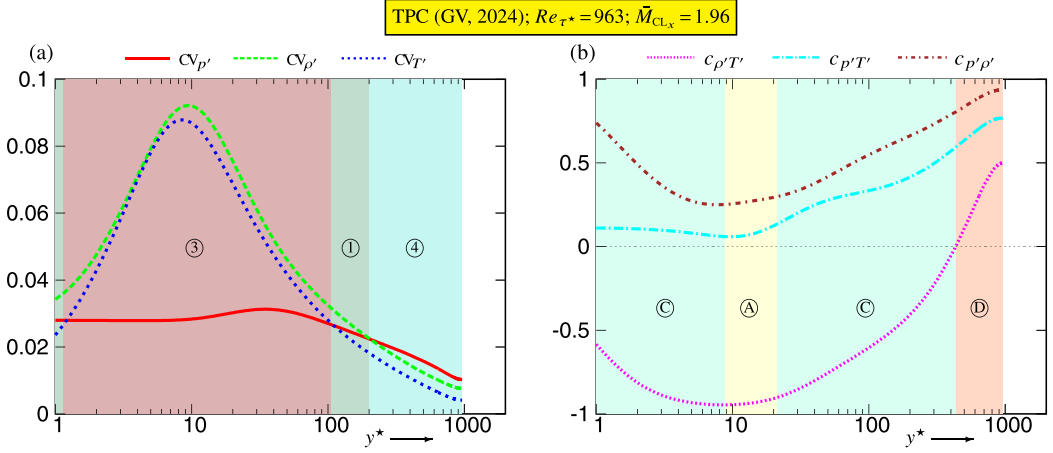


FIG. 8. DNS data for compressible turbulent plane channel flow at $(Re_{\tau^*}, \bar{M}_{CL_x}) = (963, 1.96)$ [24], plotted against the HCB-scaled [25] wall-distance y^* (log scale), of (a) fluctuation amplitudes $\{CV_{\rho'}, CV_{\rho'}, CV_{T'}\}$ with background highlighted according to the approximate regions $\{\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}\}$ [Fig. 6(b)] of order relations between fluctuation intensities (37) and of (b) the correlation coefficients $\{c_{\rho'\rho'}, c_{\rho'T'}, c_{\rho'\rho'}\}$ with background highlighted according to the approximate regions $\{\textcircled{A}, \textcircled{B}, \textcircled{C}, \textcircled{D}\}$ [Fig. 6(c)] of signs and order relations between correlation coefficients (34) and (35).

and not for $CV_{T'}$, and this could further increase the approximation error. The expected linearization errors are analyzed in Appendix C but, unfortunately, the relevant correlations are not available in the TBL data.

In the linear régime (29), the partitionings of the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane [(36) and (37)] are very helpful in understanding the interdependence of the thermodynamic turbulence structure (TTS) parameters (12). Both these partitionings use $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ as inputs. Partitioning $\{\textcircled{A}, \textcircled{B}, \textcircled{C}, \textcircled{D}\}$ [Fig. 8(a)] is based on the evaluation of the amplitude ratio $CV_{\rho'}/CV_{\rho'}$ from the linearized approximation (D2) whereas partitioning $\{\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}\}$ [Fig. 8(b)] monitors the signs of the correlation coefficients $\{c_{\rho'\rho'}, c_{\rho'T'}\}$ evaluated from the linearized relations (D10).

For TPC flow [Fig. 8(a)], because of the strictly isothermal wall boundary-condition, $CV_{T'} \leq CV_{\rho'} \leq CV_{\rho'}$, rapidly ($y^* \gtrsim 1.1$) switching to $CV_{\rho'} \leq CV_{T'} \leq CV_{\rho'}$ which holds in the buffer layer and up to $y^* \approx 100$. Further away from the wall, both $CV_{T'}$ and $CV_{\rho'}$ decrease much faster than $CV_{\rho'}$, so that in the outer part of the flow ($y^* \gtrsim 200$) $CV_{T'} \leq CV_{\rho'} \leq CV_{\rho'}$ [Fig. 8(a)]. The robust (Sec. VIB) linearized approximation (D2) predicts correctly the succession of zones $\{\textcircled{1}, \textcircled{3}, \textcircled{1}, \textcircled{4}\}$ [Fig. 8(a)] moving away from the wall. Notice that in this flow $CV_{T'}/CV_{\rho'} < 1 \forall y$ so that zone $\textcircled{2}$ is not encountered. Regarding the correlation coefficients [Fig. 8(b)], the DNS data indicate that both $c_{\rho'\rho'} > 0 \forall y$ and $c_{\rho'T'} > 0 \forall y$ in this flow so that only zones $\textcircled{C}$ and $\textcircled{D}$ should be encountered. However, in the region $10 \lesssim y^* \lesssim 20$, the linearization error in (D10b) predicts small negative values for $c_{\rho'T'}$ for $10 \lesssim y^* \lesssim 20$, instead of small positive values, erroneously identifying the flow in the neighboring zone $\textcircled{A}$ [Fig. 8(b)].

B. Accuracy of the linearized approximations

In the linear régime (29), the exact equations (9) between order-2 moments and the compatibility identities (19) and (21) were linearized providing several approximations [(D2), (D3), (D5), (32), and (D10)], which were used to map the thermodynamic turbulence structure of compressible turbulent flows (Fig. 6) These approximate relations were invariably obtained from the linearization of a corresponding exact relation thereby precisely determining the approximation error. A comprehensive assessment of the expected accuracy of these approximations is obtained (Fig. 9) by

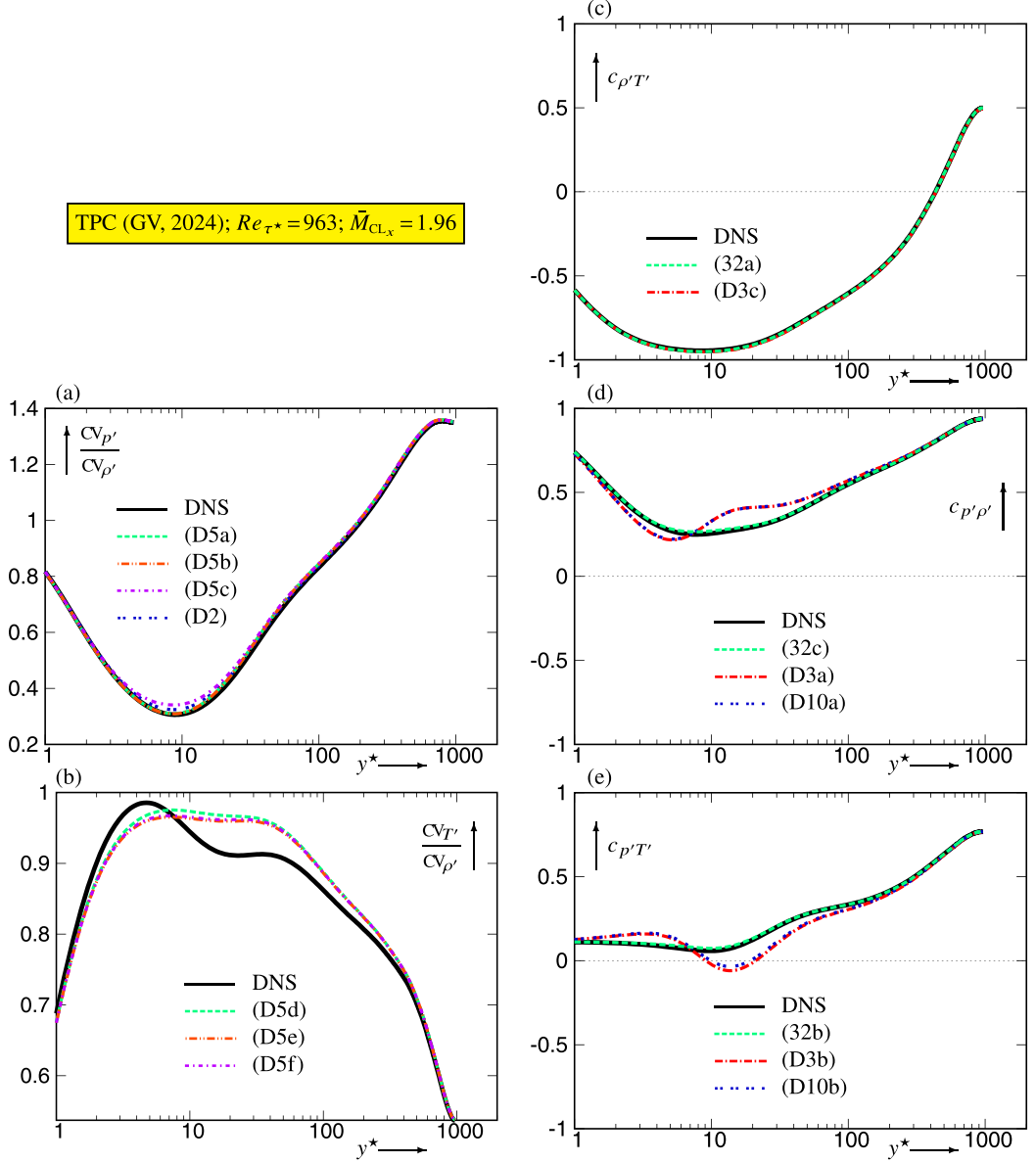


FIG. 9. Comparison of DNS data for compressible turbulent plane channel flow at $(Re_{\tau^*}, \bar{M}_{CL_x}) = (963, 1.96)$ [24], plotted against the HCB-scaled [25] wall-distance y^* (log scale), with linearized approximations, from the correlation coefficients $\{c_{\rho'\rho'}, c_{\rho'T'}, c_{\rho'\rho'T'}\}$ [(D5) and (32)], from the fluctuation intensities $\{CV_{\rho'}, CV_{\rho'}, CV_{T'}\}$ (D3), or from $\{CV_{T'}/CV_{\rho'}, c_{\rho'T'}\}$ [(D2) and (D10)].

comparison of the approximate relations with DNS data for compressible turbulent plane channel flow at $(Re_{\tau^*}, \bar{M}_{CL_x}) = (963, 1.96)$ [24]. The choice of $\bar{M}_{CL_x} \approx 2$ for the accuracy assessment was made as this seems to be a safe limit for the validity of the linearized approximation, based on the analysis of DNS data (Appendix C).

Approximations using as inputs second-order correlation coefficients only [(D5) and (32)] are obtained from the linearized compatibility equation (31). The approximation of each correlation

coefficient $\{c_{p'\rho'}, c_{p'T'}, c_{\rho'T'}\}$ from the other two [(32a), (32b), and (32c)] which were obtained by combining the linearized compatibility equation (31) with the linearized inequalities between 2CCs (30) is very accurate [Figs. 9(c), 9(d), and 9(e)]. This is in line with the small approximation error of the linearized compatibility equation [$\sim 1\%$ at $\bar{M}_{CL_x} = 2$; Figs. 3(b) and 3(d)], and suggests that approximations [(32a), (32b), and (32c)] are quite robust (remain accurate at relatively high Mach numbers). Regarding the estimate of the intensities' ratio $CV_{p'}/CV_{\rho'}$ from the 2CCs, all of the three approximations [(D5a), (D5b), and (D5c)] are different functions of all three correlation coefficients $\{c_{p'\rho'}, c_{p'T'}, c_{\rho'T'}\}$ and perform quite well, although (D5c) is slightly less accurate than the other two near the low levels of $CV_{p'}/CV_{\rho'}$ [$5 \lesssim y^* \lesssim 25$, Fig. 9(a)]. There is no obvious explanation for this difference in accuracy, as the approximation errors of all three relations [(D5a), (D5b), and (D5c)] which are obtained by the corresponding exact nonlinear equations (18a), (18b), and (18c) are $\propto CV_{T'}$ with different factors. Contrary to these quite accurate approximations based on the linearized compatibility equation (D5), the corresponding approximations of the intensities' ratio $CV_{T'}/CV_{\rho'}$ from the 2CCs [(D5d), (D5e), and (D5f)] are substantially less accurate [Fig. 9(b)], all of the three approximations [(D5d), (D5e), and (D5f)] predicting a nearly constant plateau of $CV_{T'}/CV_{\rho'} \approx 0.95$ in the buffer region $5 \lesssim y^* \lesssim 40$ with a $\pm 10\%$ approximation error. Although not plotted here, the approximation improves substantially with decreasing $\bar{M}_{CL_x}$ but the linearized approximation systematically predicts a plateau in the buffer region, implying that the variation of the ratio $CV_{T'}/CV_{\rho'}$ in the buffer region is related to the nonlinear $\propto CV_{T'}$ term in the exact nonlinear relations (18d), (18e), and (18f) and is amplitude-dependent. This is corroborated by the DNS data for $CV_{T'}/CV_{\rho'}$ [Fig. 2(d)]. At $Re_{\tau^*} \approx 1000$ the ratio $CV_{T'}/CV_{\rho'}$ has indeed a plateau in the buffer region, but with increasing Mach number there clearly appears a bump of an increasing maximum of $CV_{T'}/CV_{\rho'}$ moving towards the wall. The data at $(Re_{\tau^*}, \bar{M}_{CL_x}) = (113, 2.49)$ show that at higher $\bar{M}_{CL_x}$ this maximum still increases above $CV_{T'}/CV_{\rho'} = 1$ and moves closer to the wall [Fig. 2(d)].

In the linear régime (29), the exact nonlinear realizability plane is approximated by the $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane. The linearized approximation (D3c) of $c_{\rho'T'}$ using as input the amplitude ratios $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ is very accurate, in line with the findings in [2] (Fig. 5, p. 463), whereas the corresponding approximations for $c_{p'\rho'}$ (D3a) and $c_{p'T'}$ (D3b) perform less satisfactorily in the high- $CV_{T'}$ buffer region $5 \lesssim y^* \lesssim 40$ [Figs. 9(d) and 9(e)].

Finally, in the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane, the approximation (D2) of the amplitude ratio $CV_{p'}/CV_{\rho'}$ is very robust [Fig. 9(a)], whereas the approximations with input $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ of $c_{p'\rho'}$ (D3a) and $c_{p'T'}$ (D3b) perform almost identically with the approximations (D3a) and (D3b), which use the amplitude ratios $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ as input [Figs. 9(d) and 9(e)]. This is easily explained by the very high accuracy of the $c_{\rho'T'}$ $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ approximation [Fig. 9(c)].

The levels of $CV_{T'}$, for external flow Mach number $\bar{M}_e \leq 2$, are quite comparable in channels and boundary-layers (Figs. 4 and 5). Therefore, the conclusions of these comparisons of different linearized approximations with DNS data (Fig. 9) can probably be extended to ZPGTBLs for $\bar{M}_e \leq 2$.

VII. CONCLUSIONS

The dilute-gas equation-of-state, which is an accurate model of air thermodynamics for a wide range of aerodynamic flows, imposes three exact equations between the members of the set

$$\mathcal{T}_{TS} := \{CV_{p'}, CV_{\rho'}, CV_{T'}, c_{\rho'T'}, c_{p'\rho'}, c_{p'T'}; c_{\rho'\rho'T'}, c_{\rho'T'T'}, c_{p'\rho'T'}\}$$

which contains all six second-order moments between the basic thermodynamic variables $\{p', \rho', T'\}$, in the form of turbulence intensities measured by the coefficients of variation $\{CV_{p'}, CV_{\rho'}, CV_{T'}\}$, of second-order correlation coefficients $\{c_{\rho'T'}, c_{p'\rho'}, c_{p'T'}\}$ and the three of the ten third-order correlation coefficients which involve simultaneously ρ' and T' , $\{c_{\rho'\rho'T'}, c_{\rho'T'T'}, c_{p'\rho'T'}\}$. These equations contain a linear (amplitude-independent) term involving only amplitude-ratios and correlation coefficients $\{\mathbb{P}_{p'}/CV_{\rho'}, CV_{T'}/CV_{\rho'}, c_{\rho'T'}, c_{p'\rho'}, c_{p'T'}\}$ and a nonlinear (amplitude-dependent) term $\propto CV_{T'}$, and they can be combined to calculate the amplitude-ratios in terms of various equivalent expressions of the correlation coefficients. The equivalence

between these expressions leads to an exact nonlinear (amplitude-dependent) compatibility relation between the correlation coefficients. Applying the positivity of variances in these expressions yields three inequalities involving the correlation coefficients which are generally valid (they contain an amplitude-dependent nonlinear term).

The covariance inequality $|c_{\rho'T'}| \leq 1$ can be used in the exact equations to show that the turbulence intensities must satisfy specific inequalities, which can be combined to show that realizable (physically possible) states of thermodynamic turbulence lie inside a realizability domain in the $(CV_{T'}/CV_{\rho'}, \sqrt{CV_{\rho'}^2/CV_{\rho'}^2 - c_{NLN} CV_{T'}})$ -plane (where c_{NLN} is a structure parameter). The realizability domain in this plane is bounded by three straight lines. The exact equations establish a bijective (one-to-one) mapping between the nonlinear realizability plane and the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane, and the nonlinear inequalities can be used to define specific regions (partitions) in these planes.

Nonlinear (amplitude-dependent) effects in the above analysis are generally represented by a structure parameter multiplied by $CV_{T'}$ or $CV_{\rho'}$. Comparison of available DNS data of $CV_{T'}$ shows significant differences between compressible turbulent plane channel flows (TPCs) and zero-pressure-gradient turbulent boundary-layers (ZPGTBLs). In particular, with increasing $M_\infty \geq 4$ in adiabatic and slightly-cold-wall ZPGTBLs, an outer peak of $CV_{T'}$ appears and replaces the near-wall peak as the maximum of fluctuating-temperature intensity. This difference in behavior between channels and boundary-layers is in part attributed to the very-cold-wall condition in TPCs, far colder than for the available ZPGTBL data. These important differences notwithstanding, analysis of available DNS data for TPCs and ZPGTBLs provides a sufficiently representative cartography of the increase of $CV_{T'}$ with increasing external flow Mach number (M_∞ or M_{CL_x}). Available detailed DNS data of the thermodynamic turbulence structure for TPC flows were used to quantify the nonlinear effects both for the exact relation between variances and for the compatibility equation, suggesting that $CV_{T'} = 10\%$, locally, is a plausible threshold below which linearization of the exact equations is expected to be sufficiently accurate (linear régime of moderately compressible turbulence), roughly corresponding to $\bar{M}_e \leq 2$ both for channels and boundary-layers.

In the linear régime, the nonlinear terms in all the exact equalities and inequalities can be dropped, yielding simpler relations involving only five parameters,

$$\text{TTS} := \left\{ \frac{CV_{\rho'}}{CV_{\rho'}}, \frac{CV_{T'}}{CV_{\rho'}}, c_{\rho'T'}, c_{\rho'\rho'}, c_{\rho'T'} \right\}.$$

In particular, the nonlinear realizability plane simplifies to the $(CV_{T'}/CV_{\rho'}, CV_{\rho'}/CV_{\rho'})$ -plane with corresponding realizability boundaries and a bijection into the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane. The linearized approximate version of the exact inequalities between amplitude ratios and correlation coefficients defines regions in both these planes, with respect to the signs of the correlation coefficients or to the relative levels of fluctuation intensities of the thermodynamic variables. This is particularly useful, not only in understanding the thermodynamic turbulence structure, but also in analyzing data where few of the members of $\mathcal{T}_{TS}$ are available. Notice in particular that, in the linear régime, thermodynamic turbulence where all of the three correlation coefficients $\{c_{\rho'T'}, c_{\rho'\rho'}, c_{\rho'T'}\}$ are negative is physically impossible, and that negative $c_{\rho'\rho'}$ or $c_{\rho'T'}$ implies also negative $c_{\rho'T'}$. Furthermore, $CV_{T'} \gtrless CV_{\rho'} \Rightarrow c_{\rho'T'} \gtrless c_{\rho'\rho'}$. Several other inequalities and order relations are detailed in the paper. These inequalities also guide the choice of the correct root of the linearized compatibility relation to express each of the correlation coefficients $\{c_{\rho'T'}, c_{\rho'\rho'}, c_{\rho'T'}\}$ as a function of the other two.

The mathematical analysis in the paper is valid independently of the particular flow conditions. It was applied in this paper to the analysis of canonical wall-bounded flows, channels, and boundary-layers, and it is believed that it will be useful for the analysis of more complex flows. Detailed DNS data for the thermodynamic turbulence structure in boundary-layers are needed for a more thorough comparison with channel flows, and in particular to enhance our understanding of the influence of the wall thermal condition on turbulence structure, including some still open questions on pressure fluctuations and on Reynolds analogy.

Applications of the relations developed in the paper are similar to the applications of Lumley's realizability triangle regarding Reynolds-stresses, which is used to analyze turbulence structure [33] or to identify specific flow regions [34]. The investigation of differences in thermodynamic turbulence structure between channels and boundary-layers is an example of such applications. Furthermore, in exact analogy with the realizability requirements for Reynolds-stress models [4], the realizability and compatibility constraints formulated in the paper are intended for use in the development of consistent compressible turbulence models. Such models should include density and temperature variances and should satisfy compatibility relations between the fluxes $\{\overline{p'u'_i}, \overline{\rho'u'_i}, \overline{\rho T''u''_i}\}$ of transport of the thermodynamic fluctuations by the fluctuating velocity components. Finally, the linearized approximations become very accurate at low-Mach flow and can be applied to the study of the Mach-independent quasi-incompressible limit of aerodynamic turbulence.

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The authors contributed equally to this work. The authors of this paper were ordered alphabetically.

DATA AVAILABILITY

The data that support the findings of this article are openly available [24].

APPENDIX A: FLUCTUATION-LEVELS AND CORRELATION-COEFFICIENTS

As we are interested in turbulence structure and inequalities, it is practical to use nondimensional quantities, coefficients-of-variation,

$$CV_{p'} := \frac{p'_{\text{rms}}}{\bar{p}} > 0, \quad CV_{\rho'} := \frac{\rho'_{\text{rms}}}{\bar{\rho}} > 0, \quad CV_{T'} := \frac{T'_{\text{rms}}}{\bar{T}} > 0, \quad (\text{A1a})$$

and correlation-coefficients

$$c_{p'\rho'} := \frac{\overline{p'\rho'}}{p'_{\text{rms}} \rho'_{\text{rms}}}, \quad c_{p'T'} := \frac{\overline{p'T'}}{p'_{\text{rms}} T'_{\text{rms}}}, \quad c_{\rho'T'} := \frac{\overline{\rho'T'}}{\rho'_{\text{rms}} T'_{\text{rms}}}, \quad (\text{A1b})$$

which by the the covariance inequality (3b) are bounded in the interval $[-1, +1]$

$$(3b) \Rightarrow |c_{p'\rho'}| \leq 1, \quad |c_{p'T'}| \leq 1, \quad |c_{\rho'T'}| \leq 1. \quad (\text{A1c})$$

The coefficients-of-variation (A1a) determine the fluctuations' amplitude relative to their mean-value, and represent therefore nondimensional levels (turbulence intensities) of the thermodynamic fluctuations. Higher-order correlation coefficients

$$c_{p'\rho'T'} := \frac{\overline{p'\rho'T'}}{p'_{\text{rms}} \rho'_{\text{rms}} T'_{\text{rms}}}; \quad c_{\rho'\rho'T'} := \frac{\overline{\rho'\rho'T'}}{\rho'_{\text{rms}} \rho'_{\text{rms}} T'_{\text{rms}}}; \quad c_{\rho'T'T'} := \frac{\overline{\rho'T'T'}}{\rho'_{\text{rms}} T'_{\text{rms}} T'_{\text{rms}}};$$

$$c_{\rho'\rho'T'T'} := \frac{\overline{\rho'\rho'T'T'}}{\rho'_{\text{rms}} \rho'_{\text{rms}} T'_{\text{rms}} T'_{\text{rms}}} \quad (\text{A1d})$$

also appear in the amplitude-dependent part of the exact relations between fluctuation amplitudes and correlation coefficients, and in the relation between Reynolds and Favre fluctuation amplitudes [(C1) and (C2)]. General statistical bounds for higher-order correlation coefficients are not very

sharp [7], as they usually involve the univariate or multivariate kurtosis [35], which are often large. We will only note the positivity of

$$\overline{(\rho'T' - \overline{\rho'T'})^2} \geq 0 \iff \overline{\rho'^2 T'^2} - \overline{\rho'T'}^2 \geq 0 \stackrel{(A1d, A1b)}{\Rightarrow} c_{\rho'\rho'T'T'} - c_{\rho'T'}^2 \geq 0 \quad (A1e)$$

whose lower bound can be sharpened to $c_{\rho'\rho'T'T'} - c_{\rho'T'}^2 \geq \max(c_{p'\rho'T'}^2, c_{\rho'\rho'T'}^2, c_{\rho'T'T'}^2)$ [[35], (2.8,2.10), pp. 13–14], this being alternatively an upper bound for the absolute value of the triple correlations.

APPENDIX B: INHERENT NONLINEARITY

Compatibility relations (19) and (21) have identical expressions for the linear term, but different expressions for the nonlinear term,

$$\begin{aligned} \text{NLNT}_{\text{CR}} &:= \frac{O(\text{CV}_{T'})}{\text{CV}_{T'} \left((c_{p'\rho'} - c_{p'T'} c_{\rho'T'}) c_{p'\rho'T'} - (1 - c_{p'T'}^2) c_{\rho'\rho'T'} - (c_{p'\rho'} c_{p'T'} - c_{\rho'T'}) c_{\rho'T'T'} \right)} \\ &\stackrel{(19, 21)}{=} \frac{\text{CV}_{\rho'} \left((c_{p'T'} - c_{p'\rho'} c_{\rho'T'}) c_{p'\rho'T'} - (c_{p'T'} c_{p'\rho'} - c_{\rho'T'}) c_{\rho'\rho'T'} - (1 - c_{p'\rho'}^2) c_{\rho'T'T'} \right)}{O(\text{CV}_{\rho'})} \end{aligned} \quad (B1a)$$

$$\Rightarrow O(\text{CV}_{\rho'}) \sim O(\text{CV}_{T'}), \quad (B1b)$$

which is another justification of the generally accepted idea that fluctuation intensities of ρ' and T' are expected to be of the same order-of-magnitude [2]. The equality (B1a) of the nonlinear terms in (19) and (21) implies the exact identity

$$0 < \frac{\text{CV}_{T'}}{\text{CV}_{\rho'}} \stackrel{(19, 21)}{=} \frac{(c_{p'T'} - c_{p'\rho'} c_{\rho'T'}) c_{p'\rho'T'} - (c_{p'T'} c_{p'\rho'} - c_{\rho'T'}) c_{\rho'\rho'T'} - (1 - c_{p'\rho'}^2) c_{\rho'T'T'}}{(c_{p'\rho'} - c_{p'T'} c_{\rho'T'}) c_{p'\rho'T'} - (1 - c_{p'T'}^2) c_{\rho'\rho'T'} - (c_{p'\rho'} c_{p'T'} - c_{\rho'T'}) c_{\rho'T'T'}} \quad (B2)$$

which is valid for any intensity of the thermodynamic fluctuations and does not contain an explicit nonlinear term. Identity (B2) is an amplitude-independent structure relation between the correlation coefficients of order-2 (2CCs), $\{c_{p'\rho'}, c_{p'T'}, c_{\rho'T'}\}$, the correlation coefficients of order-3 (3CCs), which contain both ρ' and T' , $\{c_{p'\rho'T'}, c_{\rho'\rho'T'}, c_{\rho'T'T'}\}$, and the ratio $\text{CV}_{T'}/\text{CV}_{\rho'}$ of fluctuation amplitudes, again highlighting the nonlinear relation between 2CCs and 3CCs.

The amplitude information contained in this relation is better seen by considering (19) and (21) separately. It is straightforward to solve (19) for $\text{CV}_{T'}$,

$$0 < \text{CV}_{T'} \stackrel{(19)}{=} \frac{1 - c_{p'\rho'}^2 - c_{p'T'}^2 - c_{\rho'T'}^2 + 2 c_{p'\rho'} c_{p'T'} c_{\rho'T'}}{(c_{p'\rho'} - c_{p'T'} c_{\rho'T'}) c_{p'\rho'T'} - (1 - c_{p'T'}^2) c_{\rho'\rho'T'} - (c_{p'\rho'} c_{p'T'} - c_{\rho'T'}) c_{\rho'T'T'}} \quad (B3a)$$

and (21) for $\text{CV}_{\rho'}$,

$$0 < \text{CV}_{\rho'} \stackrel{(21)}{=} \frac{1 - c_{p'\rho'}^2 - c_{p'T'}^2 - c_{\rho'T'}^2 + 2 c_{p'\rho'} c_{p'T'} c_{\rho'T'}}{(c_{p'T'} - c_{p'\rho'} c_{\rho'T'}) c_{p'\rho'T'} - (c_{p'T'} c_{p'\rho'} - c_{\rho'T'}) c_{\rho'\rho'T'} - (1 - c_{p'\rho'}^2) c_{\rho'T'T'}} \quad (B3b)$$

clearly showing that the relations between 2CCs and 3CCs change with the fluctuation levels. Notice that identity (B2) is also obtained by straightforward division of (B3a) and (B3b). Because of the positivity of the fluctuation intensities, the numerators and denominators in (B2) and (B3) are of the same sign, independently of the particular flow or fluctuation intensity.

Notice that the linear part of the compatibility relation (19) and (21) is symmetric in the correlation coefficients $\{c_{p'\rho'}, c_{p'T'}, c_{\rho'T'}\}$. Adding and subtracting $\{c_{p'\rho'}^2 c_{p'T'}^2, c_{p'T'}^2 c_{\rho'T'}^2, c_{p'\rho'}^2 c_{\rho'T'}^2\}$

to the compatibility relation (19) and (21) yields three equivalent forms, respectively {(B4a), (B4b), (B4c)},

$$(1 - c_{\rho'\rho'}^2)(1 - c_{\rho'T'}^2) - (c_{\rho'\rho'} c_{\rho'T'} - c_{\rho'T'}^2)^2 \stackrel{(19,21,B1)}{=} \text{NLNT}_{\text{CR}}, \quad (\text{B4a})$$

$$(1 - c_{\rho'T'}^2)(1 - c_{\rho'\rho'}^2) - (c_{\rho'T'} c_{\rho'\rho'} - c_{\rho'\rho'}^2)^2 \stackrel{(19,21,B1)}{=} \text{NLNT}_{\text{CR}}, \quad (\text{B4b})$$

$$(1 - c_{\rho'\rho'}^2)(1 - c_{\rho'T'}^2) - (c_{\rho'\rho'} c_{\rho'T'} - c_{\rho'T'}^2)^2 \stackrel{(19,21,B1)}{=} \text{NLNT}_{\text{CR}}, \quad (\text{B4c})$$

where the equalities between the left-hand sides in (B4) can also be verified directly, and NLNT_{CR} is defined in (B1).

APPENDIX C: FAVRE- AND REYNOLDS-AVERAGED TEMPERATURE FLUCTUATIONS IN COMPRESSIBLE CHANNELS

Many compressible databases contain data for $\{\tilde{T}, \sqrt{\tilde{T}'^2}\}$ and not for $\{\bar{T}, T'_{\text{rms}}\}$. It is therefore necessary to compare the ratio (pseudo-CV) $\mathcal{CV}_{T''}$ (27) with $\text{CV}_{T'}$ that appears in the exact nonlinear equations (Sec. II). Straightforward calculations, using the definition of Favre-averaging [[20], (2.6), pp. 710–711] and (6), yield the exact relation

$$\frac{\overbrace{\text{CV}_{T'}}^{\sqrt{\tilde{T}'^2}}}{\tilde{T}} = \frac{\overbrace{\mathcal{CV}_{T''}}^{\sqrt{\tilde{T}'^2}}}{\tilde{T}} \frac{1 + c_{\rho'T'} \text{CV}_{\rho'} \text{CV}_{T'}}{\sqrt{1 + c_{\rho'T'T'} \text{CV}_{\rho'} - c_{\rho'T'}^2 \text{CV}_{\rho'}^2}}. \quad (\text{C1})$$

Both this relation (C1) and the relation for the ratio $\tilde{T}/\bar{T}$ (6) are direct consequences of the definition of Favre-averaging, completely independent of the particular equation-of-state. Taylor-expanding the denominator in (C1) at $\text{CV}_{\rho'} = 0$ better illustrates the relation between $\text{CV}_{T'}$ (A1a) and $\mathcal{CV}_{T''}$ (27), at least for moderately compressible turbulence, to wit,

$$\frac{\overbrace{\text{CV}_{T'}}^{\sqrt{\tilde{T}'^2}}}{\tilde{T}} \sim \frac{\overbrace{\mathcal{CV}_{T''}}^{\sqrt{\tilde{T}'^2}}}{\tilde{T}} \left(1 - \frac{1}{2} \overbrace{c_{\rho'T'T'} \text{CV}_{\rho'}}^{O(\text{CV}_{\rho'})} + \overbrace{\left(c_{\rho'T'} \frac{\text{CV}_{T'}}{\text{CV}_{\rho'}} + \frac{3}{8} c_{\rho'T'T'}^2 + \frac{1}{2} c_{\rho'T'}^2 \right) \text{CV}_{\rho'}^2}^{O(\text{CV}_{\rho'}^2)} + \dots \right) \quad (\text{C2a})$$

implying that to leading-order

$$\text{CV}_{\rho'}^2 \ll 1 \stackrel{(\text{C2a})}{\Rightarrow} \frac{\text{CV}_{T'}}{\mathcal{CV}_{T''}} \sim 1 - \frac{1}{2} c_{\rho'T'T'} \text{CV}_{\rho'} + O(\text{CV}_{\rho'}^2), \quad (\text{C2b})$$

the sign of $c_{\rho'T'T'}$ [Fig. 2(a)] determining the order relation between $\text{CV}_{T'}$ and $\mathcal{CV}_{T''}$, whereas the ratio

$$(6,7) \Rightarrow \frac{\mathcal{CV}_{T''}}{\text{CV}_{\rho'}} := \frac{\tilde{T}}{\bar{T}} = 1 + c_{\rho'T'} \text{CV}_{\rho'} \text{CV}_{T'} \Rightarrow \frac{\tilde{T}}{\bar{T}} \sim 1 + O(\text{CV}_{\rho'} \text{CV}_{T'}). \quad (\text{C2c})$$

By (C2), contrary to the difference between Favre- and Reynolds-averages of mean-flow quantities which is of second-order in fluctuation intensities (C2c), the difference between Favre and Reynolds fluctuation intensities is of first-order in the fluctuation intensities themselves (C2b). The observed small differences between Favre- and Reynolds-averages of mean-flow quantities ([30], p. 4299) imply large differences for the fluctuation intensities (Fig. 10).

In turbulent plane channel flow, the ratio $\text{CV}_{T'}/\mathcal{CV}_{T''}$ presents two extrema across the channel [Fig. 10(b)], a maximum $\max_y(\text{CV}_{T'}/\mathcal{CV}_{T''}) > 1$ very close to the wall ($y^* \lesssim 4$) and a minimum $\min_y(\text{CV}_{T'}/\mathcal{CV}_{T''}) < 1$ in the buffer region ($15 \lesssim y^* \lesssim 25$). This minimum corresponds to the maximum departure from unity of $\text{CV}_{T'}/\mathcal{CV}_{T''}$ [Fig. 10(b)]. Further away from the wall $\text{CV}_{T'}/\mathcal{CV}_{T''}$

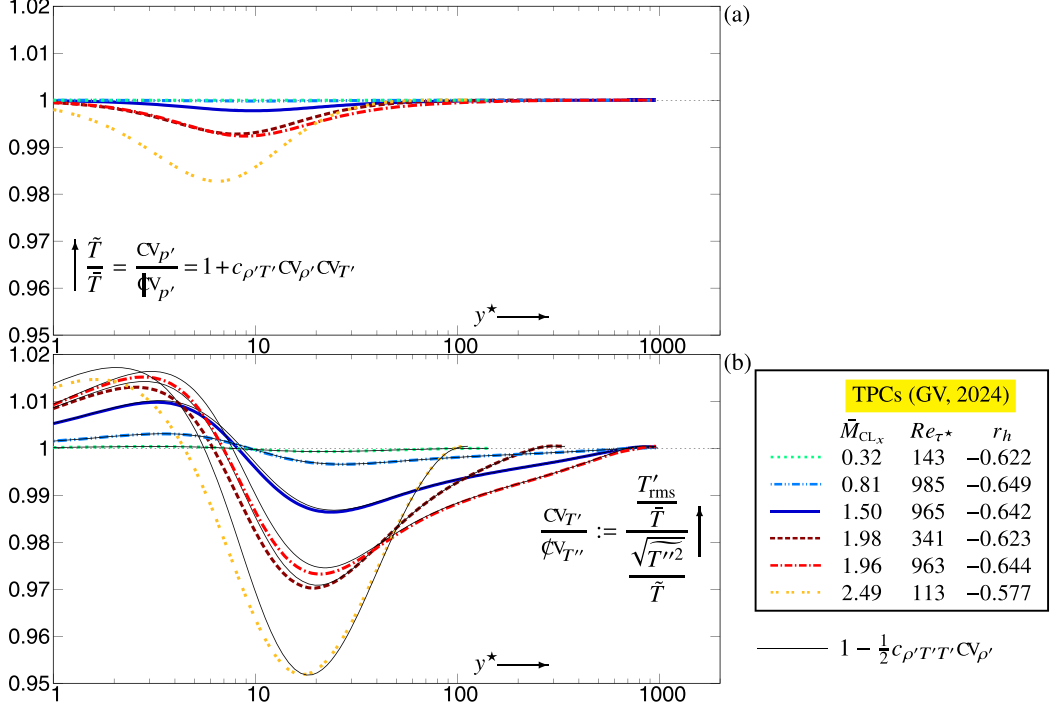


FIG. 10. Ratios of (a) Favre-to-Reynolds-averaged temperatures $\tilde{T}/\bar{T}$ (6) and of (b) $CV_{T'} := T'_{rms}/\bar{T}$ (A1a) on $\Phi_{V_{T''}} := \sqrt{T''^2}/\bar{T}$ (27), and comparison with the leading-order approximation $CV_{T'} \sim \Phi_{V_{T''}}(1 - \frac{1}{2} c_{\rho'T'T'} CV_{\rho'})$ (C2b), plotted against the HCB-scaled [25] wall-distance y^* (log scale), for selected flows from the compressible turbulent plane channel (TPC) flows DNS database [24] (GV, 2024), representative of the available range $97 \leq Re_{\tau^*} \leq 985$ and $0.32 \leq \bar{M}_{CLx} \leq 2.49$.

increases, and near the centerline it becomes again very slightly higher than 1 [Fig. 10(b)]. The leading-order expansion $CV_{T'}/\Phi_{V_{T''}} \sim 1 - \frac{1}{2} c_{\rho'T'T'} CV_{\rho'}$ (C2b) approximates quite well the DNS data [Fig. 10(b)], the slight overestimation of the very-near-wall $\max_y(CV_{T'}/\Phi_{V_{T''}})$ at the higher $\bar{M}_{CLx} \gtrsim 2$ notwithstanding. Consistently with this approximation, the two extrema of $CV_{T'}/\Phi_{V_{T''}}$ [Fig. 10(b)] occur very close to the negative and positive extrema of $c_{\rho'T'T'}$ [Fig. 2(a)], and the changes of the sign of $c_{\rho'T'T'}$ across the channel are consistent with the ratio $CV_{T'}/\Phi_{V_{T''}}$ being greater or less than unity. Notice that, in TPC flow, both of these extrema occur in a region where $CV_{T'}/CV_{\rho'}$ is nearly constant and close to 0.9 [Fig. 2(d)].

The exact relation (C1) and its expansions (C2) suggest that the increasing departure from unity, with increasing Mach number, of the ratio $CV_{T'}/\Phi_{V_{T''}}$ can be correlated [Fig. 11(a)] with the near-wall peak $CV_{\rho'_{PEAK}} := \max_y(CV_{\rho'})$ ([20], Fig. 5, p. 720), with the idea that the value of this peak also represents the general intensity of density fluctuations. For the TPC data [24], the value of $\min_y(CV_{T'}/\Phi_{V_{T''}})$ [Fig. 11(a)] varies linearly with $CV_{\rho'_{PEAK}}$, with a slightly sharper slope at low $Re_{\tau^*} \approx 110$ than at higher $Re_{\tau^*} \approx 1000$ [Fig. 11(a)]. The very-near-wall $\max_y(CV_{T'}/\Phi_{V_{T''}})$ also correlates well with $CV_{\rho'_{PEAK}}$, but its departure from unity grows sublinearly, and in contrast to $\min_y(CV_{T'}/\Phi_{V_{T''}})$ the departure is slightly greater with increasing Re_{τ^*} [Fig. 11(a)]. The same type of correlation for the extrema of $CV_{T'}/\Phi_{V_{T''}}$ is observed with respect to $CV_{T'_{PEAK}} := \max_y(CV_{T'})$ [Fig. 11(b)]. Indeed, in TPC flow, the ratio $CV_{T'}/CV_{\rho'}$ is nearly constant close to 0.9 for $5 \lesssim y^* \lesssim 25$ [Fig. 2(f)] so that plots against $CV_{T'_{PEAK}}$ [Fig. 11(b)] or $CV_{\rho'_{PEAK}}$ [Fig. 11(a)] are quite similar. These results are valid for the $c_{\rho'T'T'}$ values observed in these TPC very-cold-wall [21] flows [Fig. 2(a)]. Data for the profiles of all the quantities appearing in (C1), $\{CV_{\rho'}, CV_{T'}, c_{\rho'T'}, c_{\rho'T'T'}\}$, are needed in

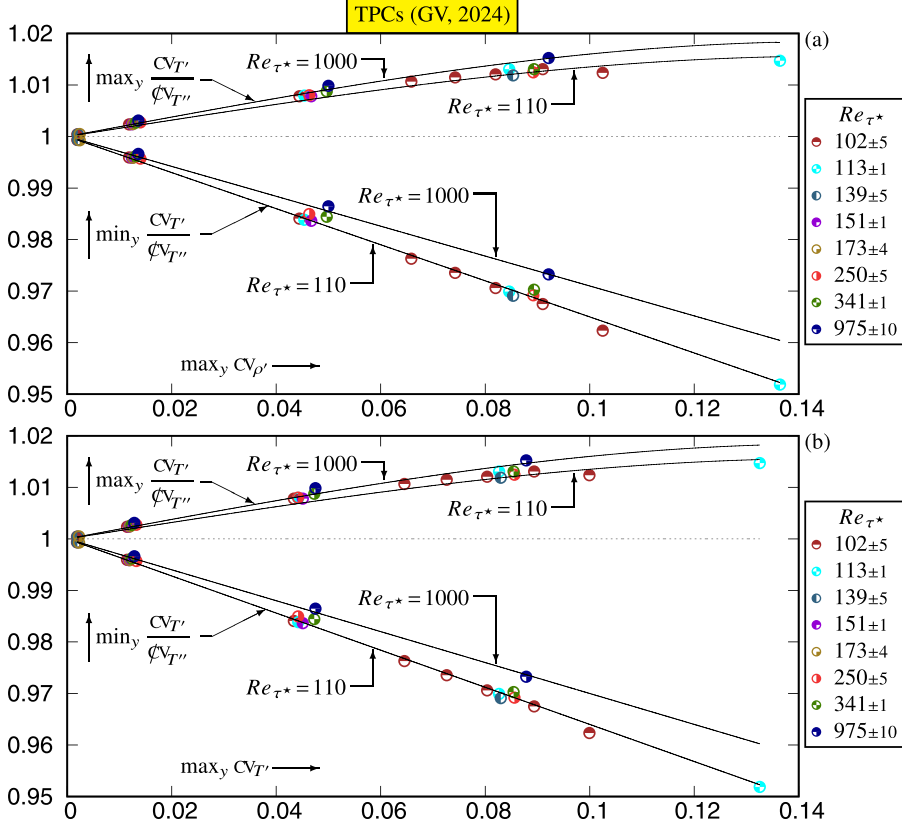


FIG. 11. Maximal and minimal values across the channel of the ratio of $CV_{T'} := T'_{\text{rms}}/\bar{T}$ (A1a) on $CV_{T''} := \sqrt{T''^2}/\bar{T}$ (27), plotted against $CV_{p'}$ (a) and against $CV_{T'}$ (b), for compressible turbulent plane channel (TPC) flows (DNS database [24]; GV, 2024; $97 \leq Re_{\tau^*} \leq 985$ and $0.32 \leq \bar{M}_{CL_x} \leq 2.49$).

boundary-layer flows with different thermal wall conditions to assess the difference between $CV_{T'}$ and $CV_{T''}$ in such flows.

Contrary to the nonlinear mapping onto the $(CV_{T'}/CV_{p'}, \sqrt{\Phi_{p'}/CV_{p'}^2 - c_{NLN} CV_{T'}})$ -plane (Fig. 1), which is based on the exact inequalities (15), the approximate linearized mapping onto the $(CV_{T'}/CV_{p'}, CV_{p'}/CV_{p'})$ -plane is limited to moderately compressible turbulence conditions (29). The interest of the above analysis, with respect to the main focus of the paper, is the approximation error of the linearized realizability (33).

Regarding channels [Fig. 7(a)] where $CV_{T'}$ data are available, the approximation error consists of replacing the ordinates by $CV_{p'}$. In the range $0.32 \leq \bar{M}_{CL_x} \leq 2.49$ of available TPC data [24], there are no qualitative differences from the exact nonlinear realizability plane (Fig. 1). Direct comparison of the approximation error in the vicinity of the $CV_{T'}$ peak around $y^* \approx 10$ [Figs. 12(b) and 12(c)] shows very small differences for $\bar{M}_{CL_x} \in \{1.51, 1.96\}$. At $\bar{M}_{CL_x} = 2.49$ [Fig. 12(a)], the linearization error is more noticeable but remains quite small.

Regarding boundary-layers, the approximation error is increased by the use of DNS data for $CV_{T''}$ in lieu of $CV_{T'}$. By (C2) this source of error is higher than the approximation of $\Phi_{p'}$ by $CV_{p'}$. In the absence of TBL data for $CV_{T'}$ or for $c_{\rho'T'T'}$ (C2b) we cannot precisely quantify this effect. In channels (Fig. 12) where all correlations are available [24] the effect of this approximation seems small, especially regarding the distance from the lower realizability boundary, although using $CV_{T''}$ moves the curve closer to this boundary [Fig. 12(a)].

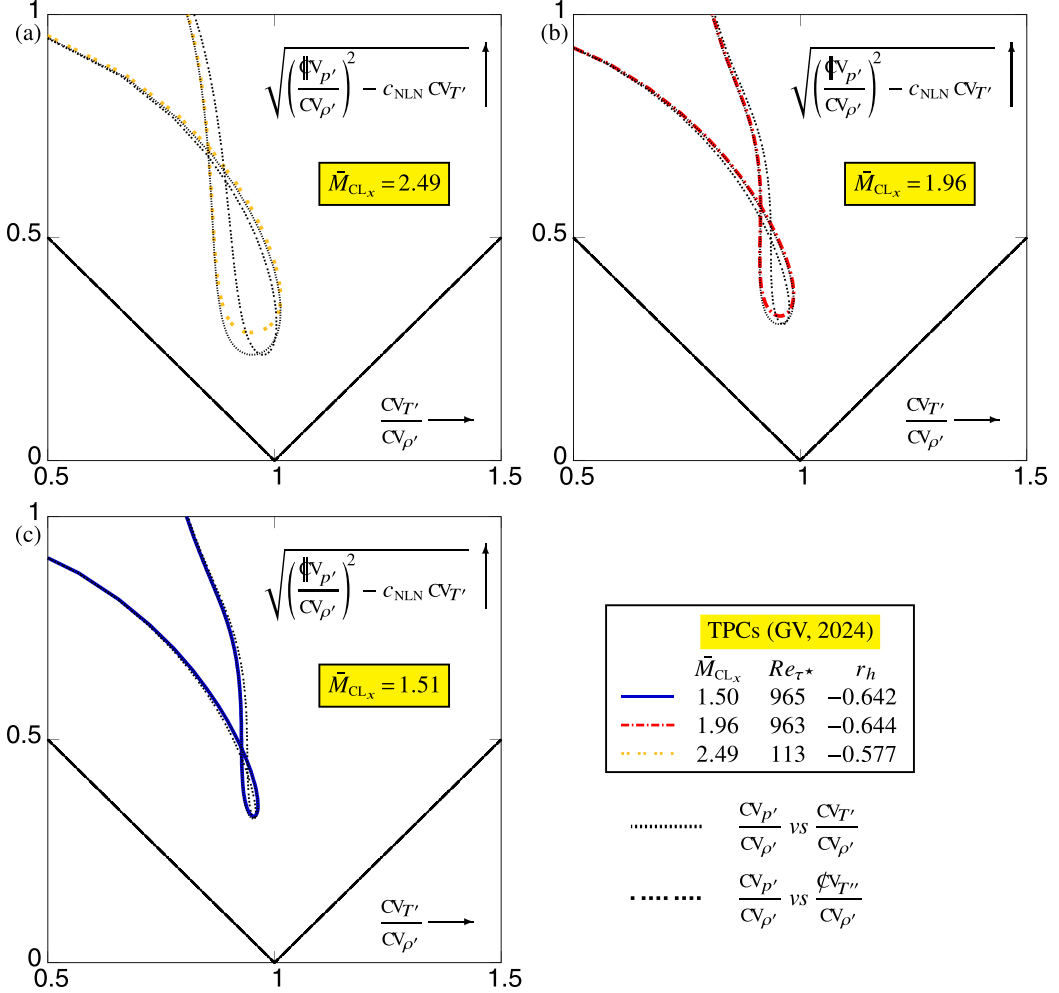


FIG. 12. Zoom near the peaks of $CV_{T'}$ and $CV_{\rho'}$ of the loci of selected DNS data ($\tilde{M}_{CL,x} \in \{1.5, 2.0, 2.5\}$) for compressible turbulent plane channel flow [24] on the nonlinear realizability $(CV_{T'}/CV_{\rho'}, \sqrt{(\Phi_{V_{p'}}^2 / CV_{\rho'}^2) - c_{NLN} CV_{T'}})$ -plane (Fig. 1), compared with their linearized approximations using Reynolds-averaging $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ or Favre-averaging $(\overline{CV_{T''}}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ for the temperature fluctuations [(27) and (C1)].

APPENDIX D: APPROXIMATE RELATIONS IN THE LINEAR RÉGIME

The linearized approximations of the exact equations (9a)–(9c) read

$$\frac{CV_{p'}}{CV_{\rho'}} \stackrel{(9a,29)}{\approx} c_{p'\rho'} + c_{p'T'} \frac{CV_{T'}}{CV_{\rho'}}, \quad (D1a)$$

$$c_{p'\rho'} \frac{CV_{p'}}{CV_{\rho'}} \stackrel{(9b,29)}{\approx} 1 + c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}}, \quad (D1b)$$

$$c_{p'T'} \frac{CV_{p'}}{CV_{\rho'}} \stackrel{(9c,29)}{\approx} c_{\rho'T'} + \frac{CV_{T'}}{CV_{\rho'}}. \quad (D1c)$$

The relations (9d) and (9e) between variances are approximated by

$$CV_{p'}^2 \stackrel{(9d, 29)}{\approx} CV_{\rho'}^2 + CV_{T'}^2 + 2 c_{\rho'T'} CV_{T'} CV_{\rho'} \iff \frac{CV_{p'}^2}{CV_{\rho'}^2} \approx 1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2 c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} \quad (D2)$$

and the expressions (16) for the correlation coefficients become

$$(16,29) \Rightarrow c_{p'\rho'} \approx \frac{\left(\frac{CV_{p'}}{CV_{\rho'}}\right)^2 - \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 + 1}{2 \frac{CV_{p'}}{CV_{\rho'}}}, \quad (D3a)$$

$$c_{p'T'} \approx \frac{\left(\frac{CV_{p'}}{CV_{\rho'}}\right)^2 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 - 1}{2 \frac{CV_{p'}}{CV_{\rho'}} \frac{CV_{T'}}{CV_{\rho'}}}, \quad (D3b)$$

$$c_{\rho'T'} \approx \frac{\left(\frac{CV_{p'}}{CV_{\rho'}}\right)^2 - \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 - 1}{2 \frac{CV_{T'}}{CV_{\rho'}}}. \quad (D3c)$$

The correlation coefficients on the three straight lines which constitute the realizability boundary (33) can be easily calculated by (D3) and are equal to ± 1 ,

$$\frac{CV_{p'}}{CV_{\rho'}} = 1 - \frac{CV_{T'}}{CV_{\rho'}} > 0 \stackrel{(D3)}{\Rightarrow} c_{p'\rho'} \approx +1 ; c_{p'T'} \approx -1 ; c_{\rho'T'} \approx -1, \quad (D4a)$$

$$\frac{CV_{p'}}{CV_{\rho'}} = \frac{CV_{T'}}{CV_{\rho'}} - 1 > 0 \stackrel{(D3)}{\Rightarrow} c_{p'\rho'} \approx -1 ; c_{p'T'} \approx +1 ; c_{\rho'T'} \approx -1, \quad (D4b)$$

$$\frac{CV_{p'}}{CV_{\rho'}} = 1 + \frac{CV_{T'}}{CV_{\rho'}} > 1 \stackrel{(D3)}{\Rightarrow} c_{p'\rho'} \approx +1 ; c_{p'T'} \approx +1 ; c_{\rho'T'} \approx +1. \quad (D4c)$$

Approximate linearized compatibility. In the linear régime of moderately compressible turbulence, the expressions (18) for the ratios $\{CV_{p'}/CV_{\rho'}, CV_{T'}/CV_{\rho'}\}$ are approximated by

$$0 < \frac{CV_{p'}}{CV_{\rho'}} \stackrel{(18a,29)}{\approx} \frac{c_{p'T'} - c_{p'\rho'} c_{\rho'T'}}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}}, \quad (D5a)$$

$$\stackrel{(18b,29)}{\approx} \frac{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}}{1 - c_{p'T'}^2}, \quad (D5b)$$

$$\stackrel{(18c,29)}{\approx} \frac{1 - c_{\rho'T'}^2}{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}}, \quad (D5c)$$

$$0 < \frac{CV_{T'}}{CV_{\rho'}} \stackrel{(18d,29)}{\approx} \frac{1 - c_{p'\rho'}^2}{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}}, \quad (D5d)$$

$$\stackrel{(18e,29)}{\approx} \frac{c_{p'\rho'} c_{p'T'} - c_{\rho'T'}}{1 - c_{p'T'}^2}, \quad (D5e)$$

$$\stackrel{(18f,29)}{\approx} \frac{c_{p'T'} - c_{p'\rho'} c_{\rho'T'}}{c_{p'\rho'} - c_{p'T'} c_{\rho'T'}}. \quad (D5f)$$

The approximate compatibility relation (31) can be obtained by dropping the nonlinear term in (19) and (21) or directly from (D5). The only difference between (18) and (20) is in the expression of the nonlinear terms, so that the linearization of (20) yields the same expressions as (D5).

Inequalities (30a)–(30c) are linearized approximations of the nonlinear inequalities (22), demonstrated from identities (18b), (18e), (20d), and (20a), which were obtained for the proof of the nonlinear compatibility equation (19) and (21). Notice that inequalities (30a)–(30c) can also be obtained directly by applying the covariance inequalities (A1c), $\{1 - c_{p'\rho'}^2 \geq 0, 1 - c_{p'T'}^2 \geq 0, 1 - c_{\rho'T'}^2 \geq 0\}$, to the linearized relations (D5).

Regions. Dropping the nonlinear terms from the exact inequalities (17) defines approximately [(30d)–(30g)] regions on the $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane with the respect to the signs of the correlation coefficients. After straightforward calculations, using (30d)–(30g) combined with the realizability inequality (33) and taking account the limiting behavior (D4), these are

$$\textcircled{A} : \frac{CV_{T'}}{CV_{\rho'}} \leq 1; 0 < 1 - \frac{CV_{T'}}{CV_{\rho'}} \leq \frac{CV_{p'}}{CV_{\rho'}} \leq \sqrt{1 - \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2} \xrightarrow{(29,30,33)} c_{p'T'} \lesssim 0 \lesssim c_{p'\rho'}; c_{\rho'T'} \lesssim 0, \quad (\text{D6a})$$

$$\textcircled{B} : \frac{CV_{T'}}{CV_{\rho'}} \geq 1; 0 < \frac{CV_{T'}}{CV_{\rho'}} - 1 \leq \frac{CV_{p'}}{CV_{\rho'}} \leq \sqrt{\left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2 - 1} \xrightarrow{(29,30,33)} c_{p'\rho'} \lesssim 0 \lesssim c_{p'T'}; c_{\rho'T'} \lesssim 0, \quad (\text{D6b})$$

$$\textcircled{C} : \sqrt{\left|1 - \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2\right|} \leq \frac{CV_{p'}}{CV_{\rho'}} \leq \sqrt{1 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2} \xrightarrow{(29,30,33)} c_{p'\rho'} \gtrsim 0; c_{p'T'} \gtrsim 0; c_{\rho'T'} \lesssim 0, \quad (\text{D6c})$$

$$\textcircled{D} : \sqrt{1 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right)^2} \leq \frac{CV_{p'}}{CV_{\rho'}} \leq 1 + \left(\frac{CV_{T'}}{CV_{\rho'}}\right) \xrightarrow{(29,30,33)} c_{p'\rho'} \gtrsim 0; c_{p'T'} \gtrsim 0; c_{\rho'T'} \gtrsim 0. \quad (\text{D6d})$$

The order relations (D6) between the correlation coefficients will be further sharpened (35) using additional inequalities (30a) and (30c) obtained from the compatibility relation (D5). Regions $\textcircled{A}$ (D6a) and $\textcircled{B}$ (D6b), near the lower realizability boundary [Fig. 6(a)], correspond to $c_{p'T'} \lesssim 0 \lesssim c_{p'\rho'}$ (D6a) or $c_{p'\rho'} \lesssim 0 \lesssim c_{p'T'}$ (D6b), respectively. Both remaining regions $\textcircled{C}$ (D6c) and $\textcircled{D}$ (D6d) have $c_{p'T'} \gtrsim 0$ and $c_{p'\rho'} \gtrsim 0$. By (D6), turbulence where both $c_{p'T'} < 0$ and $c_{p'\rho'} < 0$ simultaneously is not realizable in the linear regime.

Combining (30) and (D6) readily yields

$$\begin{aligned} \textcircled{A} \quad & \xrightarrow{(D6a)} \quad \xrightarrow{(D6a)} \quad \xrightarrow{(D6a)} \quad \xrightarrow{(A1c)} \quad \xrightarrow{c_{p'\rho'}} \quad \xrightarrow{(30c)} \quad \xrightarrow{(30b)} \quad \xrightarrow{c_{p'T'}} \quad \xrightarrow{(D6a,A1c)} \\ & c_{p'T'} \lesssim 0 \lesssim c_{p'\rho'} \leq 1 \Rightarrow \frac{c_{p'\rho'}}{c_{p'T'}} \lesssim c_{\rho'T'} \lesssim \frac{c_{p'T'}}{c_{p'\rho'}} \lesssim \\ & \xrightarrow{(D6a,A1c)} \quad \xrightarrow{(D6a)} \\ & c_{p'T'} \lesssim c_{p'\rho'} c_{p'T'} \lesssim 0, \end{aligned} \quad (\text{D7a})$$

$$\begin{aligned} \textcircled{B} \quad & \xrightarrow{(D6b)} \quad \xrightarrow{(D6b)} \quad \xrightarrow{(D6b)} \quad \xrightarrow{(A1c)} \quad \xrightarrow{c_{p'T'}} \quad \xrightarrow{(30b)} \quad \xrightarrow{(30c)} \quad \xrightarrow{c_{p'\rho'}} \quad \xrightarrow{(D6b,A1c)} \\ & c_{p'\rho'} \lesssim 0 \lesssim c_{p'T'} \leq 1 \Rightarrow \frac{c_{p'T'}}{c_{p'\rho'}} \lesssim c_{\rho'T'} \lesssim \frac{c_{p'\rho'}}{c_{p'T'}} \lesssim \\ & \xrightarrow{(D6b,A1c)} \quad \xrightarrow{(D6b)} \\ & c_{p'\rho'} \lesssim c_{p'\rho'} c_{p'T'} \lesssim 0. \end{aligned} \quad (\text{D7b})$$

Notice that (D7) also implies an order relation between the absolute values of the correlations coefficients, because taking into account the signs of the correlation coefficients in regions $\textcircled{A}$ (D6a)

and $\textcircled{B}$ (D6b), yields

$$\begin{aligned} \textcircled{A} \quad (D7a) \quad \frac{c_{p'\rho'}}{c_{p'T'}} &\lesssim \frac{c_{p'T'}}{c_{p'\rho'}} \quad (D6a) \quad c_{p'T'}^2 \lesssim c_{p'\rho'}^2 \quad (D6a) \quad -c_{p'T'} \approx |c_{p'T'}| \lesssim |c_{p'\rho'}| \\ &\approx c_{p'\rho'} \quad (D7a,A1c) \quad \frac{c_{p'\rho'}}{c_{p'T'}} \lesssim -1, \end{aligned} \quad (D8a)$$

$$\begin{aligned} \textcircled{B} \quad (D7b) \quad \frac{c_{p'T'}}{c_{p'\rho'}} &\lesssim \frac{c_{p'\rho'}}{c_{p'T'}} \quad (D6b) \quad c_{p'\rho'}^2 \lesssim c_{p'T'}^2 \quad (D6b) \quad -c_{p'\rho'} \approx |c_{p'\rho'}| \lesssim |c_{p'T'}| \\ &\approx c_{p'T'} \quad (D7b,A1c) \quad \frac{c_{p'T'}}{c_{p'\rho'}} \lesssim -1, \end{aligned} \quad (D8b)$$

which significantly sharpen (D6a) and (D6b). In regions $\textcircled{C}$ and $\textcircled{D}$, both $c_{p'\rho'}$ and $c_{p'T'}$ are positive [(D6c) and (D6d)]. Therefore, because of the covariance inequality (A1c), we have

$$\textcircled{C} \text{ and } \textcircled{D} \quad (D6c,D6d,A1c) \quad \begin{aligned} 0 &\lesssim c_{p'\rho'} \lesssim 1 \Rightarrow 0 \lesssim \frac{c_{p'\rho'}c_{p'T'}}{c_{p'T'}c_{p'\rho'}} \lesssim \frac{c_{p'T'}}{c_{p'\rho'}} \Rightarrow c_{p'\rho'}c_{p'T'} \lesssim \min(c_{p'\rho'}, c_{p'T'}). \\ 0 &\lesssim c_{p'T'} \lesssim 1 \Rightarrow 0 \lesssim \frac{c_{p'\rho'}c_{p'T'}}{c_{p'T'}c_{p'\rho'}} \lesssim \frac{c_{p'\rho'}}{c_{p'T'}} \Rightarrow c_{p'\rho'}c_{p'T'} \lesssim \min(c_{p'\rho'}, c_{p'T'}). \end{aligned} \quad (D8c)$$

With hindsight from this analysis [(D6)–(D8)], we remark that the relation (30d), $CV_{T'} > CV_{\rho'} \Rightarrow c_{p'T'} > c_{p'\rho'}$, also holds for the absolute values of the correlation coefficients, as can be easily proven by squaring (D1b) and (D1c), so that

$$CV_{T'} \begin{matrix} \leq \\ \geq \end{matrix} CV_{\rho'} \quad (D1b,D1c) \Rightarrow \begin{cases} c_{p'T'} & \begin{matrix} \approx \\ \geq \\ \leq \end{matrix} & c_{p'\rho'}, \\ |c_{p'T'}| & \begin{matrix} \approx \\ \geq \\ \leq \end{matrix} & |c_{p'\rho'}|. \end{cases} \quad (D9)$$

The order relations in each zone (34) of the $(CV_{T'}/CV_{\rho'}, CV_{p'}/CV_{\rho'})$ -plane [Fig. 6(a)] determined by (D6)–(D8) are summarized by (35) in the main paper.

Approximate regions. Dropping the $\propto CV_{T'}$ nonlinear terms in (11b) and (23) yields the approximate expressions for the correlation coefficients $\{c_{p'\rho'}, c_{p'T'}\}$ in terms of $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$

$$c_{p'\rho'} \stackrel{(23a,29,D2)}{\approx} \frac{1 + c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}}}{\sqrt{1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}}}}, \quad (D10a)$$

$$c_{p'T'} \stackrel{(23b,29,D2)}{\approx} \frac{c_{\rho'T'} + \frac{CV_{T'}}{CV_{\rho'}}}{\sqrt{1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}}}}, \quad (D10b)$$

where the linearized approximation (D2) was used to replace $\mathbb{V}_{p'} \stackrel{(29)}{\approx} CV_{p'}$ in the denominators of (23). The positivity of the radical and hence of the pressure-variance approximation (D2) follows from the covariance inequality $|c_{\rho'T'}| \leq 1$ (A1c), which also ensures that the approximations (D10) automatically satisfy the covariance inequalities $|c_{p'\rho'}| \leq 1$ and $|c_{p'T'}| \leq 1$,

$$|c_{\rho'T'}| \stackrel{(A1c)}{\leq} 1 \Rightarrow \begin{cases} CV_{\rho'}^2 + CV_{T'}^2 + 2c_{\rho'T'} CV_{\rho'} CV_{T'} \geq (CV_{T'} - CV_{\rho'})^2 \geq 0 \\ 0 < \left(1 + c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}}\right)^2 = 1 + c_{\rho'T'}^2 \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} \leq 1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} \\ 0 < \left(c_{\rho'T'} + \frac{CV_{T'}}{CV_{\rho'}}\right)^2 = c_{\rho'T'}^2 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}} \leq 1 + \frac{CV_{T'}^2}{CV_{\rho'}^2} + 2c_{\rho'T'} \frac{CV_{T'}}{CV_{\rho'}}. \end{cases} \quad (D11)$$

It is straightforward to define regions in the $(CV_{T'}/CV_{\rho'}, c_{\rho'T'})$ -plane which satisfy specific inequalities for variances or CCs. Neglecting nonlinear terms in (25) yields

$$\frac{CV_{p'}}{CV_{\rho'}} \underset{>}{\overset{<}{\approx}} 1 \quad \stackrel{(25a, 29)}{\iff} \quad c_{\rho'T'} \underset{>}{\overset{<}{\approx}} -\frac{1}{2} \frac{CV_{T'}}{CV_{\rho'}}, \quad (D12a)$$

$$\frac{CV_{p'}}{CV_{T'}} \underset{>}{\overset{<}{\approx}} 1 \quad \stackrel{(25b, 29)}{\iff} \quad c_{\rho'T'} \underset{>}{\overset{<}{\approx}} -\frac{1}{2} \frac{CV_{\rho'}}{CV_{T'}} \quad (D12b)$$

for the fluctuation intensities. Neglecting nonlinear terms in (24) yields

$$c_{p'\rho'} \underset{>}{\overset{<}{\approx}} 0 \quad \stackrel{(24a, 29)}{\iff} \quad c_{\rho'T'} \underset{>}{\overset{<}{\approx}} -\frac{CV_{\rho'}}{CV_{T'}}, \quad (D13a)$$

$$c_{p'T'} \underset{>}{\overset{<}{\approx}} 0 \quad \stackrel{(24b, 29)}{\iff} \quad c_{\rho'T'} \underset{>}{\overset{<}{\approx}} -\frac{CV_{T'}}{CV_{\rho'}} \quad (D13b)$$

for the sign of the correlation coefficients.

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