

Dominant-mode closure for transient Taylor dispersion of reduced Brownian-rod transport in plane power-law channels

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Taylor-Aris and generalized Taylor dispersion theories characterize the long-time spreading reached after transverse homogenization in bounded shear. The present work considers the preasymptotic approach to that regime for anisotropic microstructures transported through finite channels with non-Newtonian carriers. We use an established single-exponential memory closure to connect the exact short-time sampled-velocity variance with the long-time Taylor enhancement, and carry this reduction into bounded shear with varying transverse geometry, non-Newtonian rheology, and Brownian orientational dynamics. For scalar unit-reference problems with invariant release, the ratio of these two anchors defines the integral correlation time whenever sampled-velocity memory is governed by one slow mode. For the reduced rod problem, the same construction is used as a Green-Kubo-motivated two-anchor approximation to the dominant shear-generated contribution under a uniform but generally nonequilibrium transverse injection. Exact spectral solutions for isotropic dispersion in plane and circular power-law Poiseuille geometries and Newtonian Brownian-rod calculations with independently supplied rod generalized Taylor dispersion coefficients provide controlled benchmarks for this extension. Applied to a reduced Brownian-rod model in power-law channels, the carrier rheology enters through the imposed velocity and shear fields while the reference particle-scale diffusivities remain fixed. Lattice Boltzmann simulations of the corresponding anisotropic advection-diffusion equation retain the full tensorial direct-diffusion contribution and resolve the transient moment evolution. Across flow indices and rotational Péclet numbers, alignment and rheology-dependent shear sampling shape the raw transients, and the reduced variables follow common master curves to the reported accuracy. The resulting characteristic and tolerance-based development measures connect the approach to Taylor transport with physical residence times and downstream lengths in finite channels.

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I. INTRODUCTION

Taylor-Aris theory provides the canonical long-time description of dispersion in bounded shear, in which adequate transverse sampling of the velocity field reduces streamwise transport to one-dimensional advection-diffusion with an enhanced effective diffusivity [1,2]. Classical analyses clarified how this regime is approached, with Gill formulating transient-dispersion solution methods, Chatwin analyzing the approach to normality, Smith examining the early stage

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and proposing a delay-diffusion description, and Barton refining the moment method for solute dispersion in bounded shear [3–7]. Later work broadened the finite-time picture through multiple-timescale reductions in nonparallel geometries, anomalous early-time analysis in Poiseuille flow and preasymptotic theories that explicitly track relaxation from the initial condition [8–10]. Recent studies have extended Taylor-type and all-time dispersion analyses to passive-solute transport, isotropic colloidal dispersion driven by diffusioosmosis, arbitrarily shaped axisymmetric channels and coupled electroosmotic/pressure-driven flows, thereby broadening the range of geometries, driving mechanisms and time regimes for which finite-time dispersion can be quantified [11–14]. Moment-based extensions of Taylor-Aris dispersion have incorporated wall exchange through Robin boundary conditions and reversible sorption, revealing how surface kinetics reshape transient moments and long-time transport [15–18]. In finite channels, the quantities of interest therefore include the asymptotic Taylor coefficient and the time and downstream distance required to approach it within a prescribed tolerance, because these scales guide channel-length selection and determine whether measurements can be interpreted with a long-time Taylor fit or require the all-time description.

These finite-channel questions connect with a broader dispersion literature on how carrier rheology and spatial heterogeneity reshape longitudinal spreading. Non-Newtonian constitutive behavior modifies Taylor-type spreading in Herschel-Bulkley, Casson, and Carreau-Yasuda tube flows, as well as in viscoelastic transport through arbitrarily shaped axisymmetric channels and pulsatile viscoelastic transport through eccentric annuli [19–23]. Studies of hydrodynamic dispersion in heterogeneous porous media further show that structural compaction and pore-scale velocity variability can change both the scaling and the controlling mechanism of longitudinal spreading, with intra- and interpore contributions encoded in tracer transition-time statistics [24–26]. Lamellar theory, coupled continuous-time-random-walk descriptions, and pore-resolved simulations of transient advection-dominated mixing extend this sampling perspective by connecting velocity heterogeneity and pore-scale shear with the stretching and deformation histories that shape transport and mixing [27–31].

When dispersed entities also carry internal configuration, this sampling process extends into the full position-configuration space described by generalized Taylor dispersion (GTD), which coarse-grains the coupled dynamics into effective drift and dispersivity [32–34], with applications to gyrotactic swimmers and active Brownian particles in pipe, channel and porous-media flows [35–39]. Brownian rods furnish a canonical passive example that represents a broad class of elongated microstructures and connects naturally to experimental systems such as rigid rod-like duplex deoxyribonucleic acid (DNA), Brownian microrods in microchannel flow and rod-shaped carriers in mucus-rich environments [40–42]. In channel flows, Nitsche and Hinch [43] showed that Brownian rigid rods undergo shear-induced lateral migration in parabolic flow, while Nagel *et al.* [44] described confinement-induced oscillatory transport of elongated fibres in narrow channels. For Taylor dispersion itself, Kumar *et al.* [45] analyzed the long-time dispersion of elongated rods in Newtonian Poiseuille flow and derived an effective one-dimensional theory, while Khair [46] established the small- and large-rotational-Péclet-number limits, together providing the asymptotic benchmark for passive rods in Newtonian channels.

Building on this Newtonian benchmark, the present problem brings the non-Newtonian and rod-dispersion literatures together in settings motivated by rod-like tracers and carriers in mucus-rich delivery, while blood and bronchial mucus more generally exemplify biological carrier fluids with pronounced shear-thinning and viscoelasticity [42,47,48]. Related microfluidic applications also employ strongly shear-thinning or viscoelastic polymer solutions, in which anisotropic particles, cells and droplets exhibit rheology-dependent migration, focusing and separation [49–53]. Plane power-law Poiseuille flow is therefore a useful minimal model for pressure-driven transport in narrow slit-like channels with shear-dependent viscosity [54,55]. The flow index n changes both the axial velocity field sampled immediately after release and the wall-normal shear distribution that enters the reduced orientational forcing and cross-stream transport. For $n < 1$, a broad plug-like core carries weak rotational forcing and stronger orienting shear is concentrated near the walls, whereas

for $n > 1$, shear is distributed across a larger fraction of the section and the associated transport heterogeneity extends across a broader cross-stream region. Together, these changes make plane power-law channels a natural setting for carrying the transient closure into a reduced Brownian-rod transport model and identify the bulk rheological inputs that must be separated from unresolved particle-scale rod physics.

At the particle scale, studies of non-Newtonian resistance and deterministic orientation supply complementary parts of the physical picture needed for this separation. The deterministic drag on a sphere changes with the power-law index, and weakly shear-thinning calculations for non-Brownian spheres likewise show rheology-dependent resistance, mobility coupling, and suspension stress [56,57]. For rods and spheroids, the close dependence of generalized-Newtonian results on the adopted constitutive and particle-scale approximations is illustrated by the cell model of Férec *et al.* [58], which retains Newtonian-like rod orientation dynamics while changing the stress contribution, and by the resolved calculations of Domurath *et al.* [59], which find small changes in the angular velocity of long non-Brownian rods. Perturbative calculations for finite spheroids predict rheology-dependent angular velocities and tumbling periods in simple and pressure-driven shear-thinning flows [60,61], with particle-resolved studies of deterministic non-Brownian spheroids and ellipsoids further demonstrating the model dependence of orientational motion [62,63]. The available generalized-Newtonian calculations concentrate on shear-thinning carriers, a focus consistent with their importance in polymer processing and biological transport that leaves the shear-thickening regime substantially less developed at particle scale.

Brownian transport supplies the complementary mobility and relaxation perspective through formulations of anisotropic translational and orientational transport in Newtonian viscous carriers [64], analyses of continuous and oscillatory shear within the classical Jeffery drift and a prescribed rotational diffusivity [65], and second-order-fluid studies of a particular weakly viscoelastic limit [66]. Recent equilibrium and linear-response studies have quantified anisotropic rod diffusion in polymer melts, polymeric gels, and quiescent complex liquids [67–69]. Together, these advances reveal how confinement, microstructure, and scale-dependent viscosity reshape translational and rotational motion and provide descriptions of rheology-dependent resistance, deterministic orientation, and Brownian diffusion. The resulting evidence has yet to yield a generally accepted Jeffery–Smoluchowski closure that couples shear-rate-dependent translational and rotational mobilities to the non-Newtonian disturbance torque for a passive Brownian rod under steady generalized-Newtonian shear across shear-thinning and shear-thickening regimes.

Because this combined particle-scale closure remains unavailable, bulk power-law rheology and rod-scale Brownian transport enter at different levels of description in the present formulation. The Einstein relation connects diffusion to particle mobility, and its familiar inverse-viscosity form follows once an appropriate Newtonian Stokes mobility has been specified. Microrheological studies show that this connection rests on additional assumptions and that the response sampled at molecular or particle scales can depart substantially from the macroscopic viscosity [70,71]. In the comparisons reported below, the reference translational diffusivities parallel and perpendicular to the rod axis, together with the reference rotational diffusivity, are held fixed as the flow index varies. The carrier law sets the velocity and local shear fields, which reshape the orientation distribution and thereby the effective translational diffusivity, migration drift, and long-time dispersion coefficient, so these coarse-grained transport quantities retain an indirect dependence on the flow index. Across these calculations, uniform transverse release in bounded steady shear is combined with the dilute, noninteracting treatment of Kumar *et al.* [45] and Khair [46], under which rod–rod interactions and feedback of the dispersed phase on the carrier flow are neglected. A material-specific extension would require a regularized carrier law together with particle-scale translational and rotational mobilities and the corresponding non-Newtonian disturbance torque. The present work consequently applies the transient closure within this reduced and explicitly delimited formulation.

At this reduced-model level, the build-up to the Taylor regime admits an exact memory-kernel representation for the projected axial moments. Projection-operator formalisms show how eliminating unresolved variables yields memory kernels that retain the lag between transverse equilibration

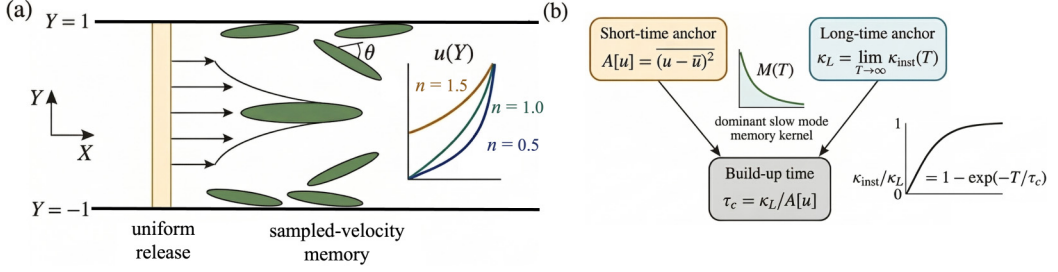


FIG. 1. Schematic of the transient-dispersion problem and the adopted two-anchor closure. Panel (a) shows how a uniform release samples different axial velocities in a plane power-law channel; cross-stream transport progressively erases this sampled-velocity memory, while n changes the velocity and shear landscapes. Panel (b) shows the dominant-mode closure constrained by the short-time anchor $A[u]$, defined in Eq. (14), and the long-time anchor κ_L , defined in Eq. (18). For the invariant scalar unit-reference problem, $\tau_c = \kappa_L / A[u]$ is the integral correlation time obtained from Eq. (25); in the reduced rod application, the same ratio is used as the one-mode build-up timescale of the modeled shear contribution.

and axial spreading [72–74]. The unresolved transverse correlations may contain several relaxation modes and retain information about the streamlines sampled at release. In the reduced rod calculations, orientation has already been eliminated through a local-equilibrium closure and enters this memory through the flow-selected effective transport coefficients.

For an invariant release, this remaining memory reduces to the Green-Kubo relation for the sampled-velocity autocovariance [75,76]. Approximating the correlation by a single exponential gives the established one-timescale closure, closely related to classical delay-diffusion descriptions [6]. The resulting closed correlation obeys a local one-mode relaxation law and admits an equivalent Markov representation at the covariance level without specifying the trajectory-level stochastic process. The present work first uses this stationary construction for scalar unit-reference benchmarks, where the invariant transverse density is uniform and the exact short-time sampled-velocity variance and long-time Taylor enhancement provide the two anchors. In the reduced rod problem, spatially varying effective transverse diffusivity makes a uniform transverse injection generally nonstationary; the same form is therefore used as a Green-Kubo-motivated two-anchor approximation to the dominant shear-generated contribution, while the full tensorial direct-diffusion correction is retained by the advection-diffusion equation (ADE) simulation. The construction is restricted to regimes that admit a long-time Taylor coefficient and whose sampled memory is governed effectively by one slow relaxation timescale.

This two-anchor reduction connects finite channel dimensions with the validity of a long-time Taylor interpretation by predicting the residence time and downstream distance required to approach a prescribed fraction of the long-time enhancement. The resulting development measures support channel-length selection and the interpretation of measurements acquired before transverse homogenization is complete, while Fig. 1 summarizes the physical sampling process and the two-anchor construction that produces them.

The remainder of the manuscript develops this progression through Sec. II, which establishes the exact short- and long-time identities, connects the stationary memory representation to Green-Kubo theory, and introduces the adopted one-mode closure. Section III carries the same two-anchor construction from exact isotropic spectral solutions in plane channels and circular tubes to Brownian rods in Newtonian plane Poiseuille flow, for which independent rod-GTD theory supplies the long-time coefficient [45,46]. Section IV then brings non-Newtonian rheology into the reduced Brownian-rod problem, combining semianalytic transport preprocessing with lattice Boltzmann method (LBM) simulations to obtain the transient moments, finite-channel development scales, and a dimensional Brownian-rod illustration. Sections V and VI discuss the scope of the reduction and summarize the main conclusions.

II. GENERAL FRAMEWORK

A. Setup and observables

Consider a plane channel of half-width a , with cross-stream coordinate $Y = y/a \in [-1, 1]$ and streamwise coordinate $X = x/a$. The dimensionless imposed streamwise velocity $u(Y)$ is normalized by a characteristic speed U_c and is assumed bounded and steady. Time is scaled with the transverse diffusion time t_{cross} according to

$$T = \frac{t}{t_{\text{cross}}}, \quad t_{\text{cross}} = \frac{a^2}{\bar{D}}, \quad (1)$$

where t_{cross} is the cross-stream diffusion time over the channel half-width and $\bar{D}$ is the reference diffusivity used in the nondimensionalization. The streamwise Péclet number is defined by

$$\text{Pe} = \frac{U_c a}{\bar{D}}. \quad (2)$$

Here U_c is the velocity scale, taken below as the centerline speed for the power-law profiles. The quantity of interest is the streamwise variance

$$\Sigma(T) = \text{Var}[X(T)]. \quad (3)$$

To connect the variance growth directly with classical Taylor dispersion, let

$$D_{x,\text{inst}}(t) = \frac{1}{2} \frac{d}{dt} \text{Var}[x(t)] \quad (4)$$

denote the dimensional instantaneous axial dispersion coefficient. The scalings in Eqs. (1) and (3) give

$$\frac{1}{2} \frac{d\Sigma}{dT} = \frac{D_{x,\text{inst}}}{\bar{D}}, \quad (5)$$

so we define the instantaneous Taylor enhancement as

$$\kappa_{\text{inst}}(T) = \frac{1}{2\text{Pe}^2} \left(\frac{d\Sigma}{dT} - 2 \right) = \frac{1}{\text{Pe}^2} \left(\frac{D_{x,\text{inst}}}{\bar{D}} - 1 \right). \quad (6)$$

For unit reference diffusion in the absence of shear, $\Sigma(T) = \Sigma(0) + 2T$, and Eq. (6) gives $\kappa_{\text{inst}} = 0$. Subtracting 2 from the variance growth rate is therefore equivalent, after integration, to removing the reference contribution $2T$ from the variance. We record this relation through the excess variance

$$\Sigma_{\text{ex}}(T) = \Sigma(T) - \Sigma(0) - 2T, \quad (7)$$

for which $\kappa_{\text{inst}} = (2\text{Pe}^2)^{-1} d\Sigma_{\text{ex}}/dT$. The factor Pe^2 follows from the classical Taylor-Aris scaling because a shear-induced displacement of order $\text{Pe} T$ produces a variance of order Pe^2 , and the corresponding dimensional relation is

$$D_{x,\text{inst}} - \bar{D} = \bar{D} \text{Pe}^2 \kappa_{\text{inst}} = \frac{U_c^2 a^2}{\bar{D}} \kappa_{\text{inst}}. \quad (8)$$

Because the short-time anchor depends on which streamlines are occupied at release, let $C(X, Y, T)$ denote the dimensionless concentration density, normalized to unit total mass, and pose the separable initial-value problem

$$C(X, Y, 0) = F_0(X) \rho_0(Y), \quad \int_{-\infty}^{\infty} F_0(X) dX = 1, \quad \int_{-1}^1 \rho_0(Y) dY = 1, \quad (9)$$

where F_0 has finite streamwise variance and ρ_0 specifies the transverse release. Unless stated otherwise, the analysis below takes $\rho_0 = 1/2$, while Sec. IIE returns to a general ρ_0 to determine how the short-time anchor and memory depend on the release and Appendix A supplies the corresponding

finite-width and point-source spectral test. The independently specified initial data for the reduced rod calculations consist of F_0 and ρ_0 . Local angular relaxation is assigned to the unresolved entrance layer before $T = 0$, and the resulting stationary conditional moments are embedded in the effective translational-diffusion tensor $D_{ij}(Y)$, $i, j \in \{x, y\}$, and the wall-normal migration drift $v_d(Y)$ of the coarse-grained rod model. Here $D_{xx}(Y)$ and $D_{yy}(Y)$ denote the axial and transverse components, respectively, and the reduced closure used below has $v_d(Y) = \partial_Y D_{yy}(Y)$. Section III C states the supporting time-scale condition after defining the rotational Péclet number.

To isolate this release-dependent anchor from direct axial diffusion, the short-time derivation below uses the unit-reference form. In the reduced rod application, the coarse-grained ADE retains the complete spatially varying tensor $D_{ij}(Y)$, including its direct axial contribution through $D_{xx}(Y)$. The term $2T$ continues to provide the common reference baseline, while departures of the direct axial diffusivity from $\bar{D}$ remain in $D_{x,\text{inst}}$ and enter the reference-normalized κ_{inst} after division by Pe^2 . With $\langle f \rangle_{\rho_0} \equiv \int_{-1}^1 \rho_0(Y) f(Y) dY$ denoting an average over the initial transverse distribution, it follows that

$$\kappa_{\text{inst}}(0) = \text{Pe}^{-2} \left[\left\langle \frac{D_{xx}}{\bar{D}} \right\rangle_{\rho_0} - 1 \right] \quad (10)$$

need not vanish for the tensorial problem.

B. Exact short-time anchor

Immediately after release, transverse diffusion produces a displacement of order $T^{1/2}$, so the axial velocity sampled over this short interval differs from its initial value only at higher order [1, 5, 10]. The axial displacement contains the direct Brownian contribution from the outset. Let $Y_0 \equiv Y(0)$, and let $W_x(T)$ be a standard Wiener process independent of the separable initial variables $X(0)$ and Y_0 . For unit reference diffusion, the short-time displacement is

$$X(T) = X(0) + \text{Pe} u(Y_0)T + \sqrt{2} W_x(T) + R_T, \quad R_T = O_{L^2}(T^{3/2}), \quad (11)$$

where R_T contains the change in sampled velocity caused by cross-stream motion. For independent axial and transverse Brownian increments, its leading part has zero conditional mean given Y_0 and is orthogonal to W_x , so it first contributes to the variance at $O(T^3)$. Since $\mathbb{E}[W_x(T)] = 0$, $\text{Var}[W_x(T)] = T$, and $W_x(T)$ is independent of $u(Y_0)$, taking the variance of Eq. (11) gives

$$\begin{aligned} \Sigma(T) - \Sigma(0) &= \text{Var}[\text{Pe} u(Y_0)T + \sqrt{2} W_x(T)] + O(T^3) \\ &= 2T + \text{Pe}^2 \{ \mathbb{E}[u(Y_0)^2] - \mathbb{E}[u(Y_0)]^2 \} T^2 + O(T^3). \end{aligned} \quad (12)$$

The first term is the reference axial-diffusion contribution identified in Eq. (6), while the coefficient of the shear-generated $\text{Pe}^2 T^2$ term is the variance of the initially sampled velocity. For the general transverse distribution in Eq. (9), define

$$A_{\rho_0}[u] = \int_{-1}^1 \rho_0(Y) u(Y)^2 dY - \left[\int_{-1}^1 \rho_0(Y) u(Y) dY \right]^2. \quad (13)$$

Thus, $A_{\rho_0}[u] = \text{Var}_{\rho_0}[u(Y_0)]$ is the exact coefficient of the shear-generated $\text{Pe}^2 T^2$ contribution under the unit-reference normalization, and its dependence on the distribution of sampled velocities means that it cannot in general be inferred from the spatial variance $\text{Var}_{\rho_0}(Y_0)$ alone. For a uniform release, (12) becomes

$$\Sigma(T) = \Sigma(0) + 2T + A[u] \text{Pe}^2 T^2 + O(T^3), \quad A[u] = \overline{(u - \bar{u})^2}, \quad (14)$$

where $\bar{u} = \frac{1}{2} \int_{-1}^1 u(Y) dY$ is the cross-sectional mean and $\overline{(\cdot)}$ denotes uniform cross-sectional averaging over $Y \in [-1, 1]$. The coefficient of the shear-generated $\text{Pe}^2 T^2$ term in Eq. (14) is exact. Physically, $A[u]$ is the initial sampled-velocity variance and measures how strongly particles

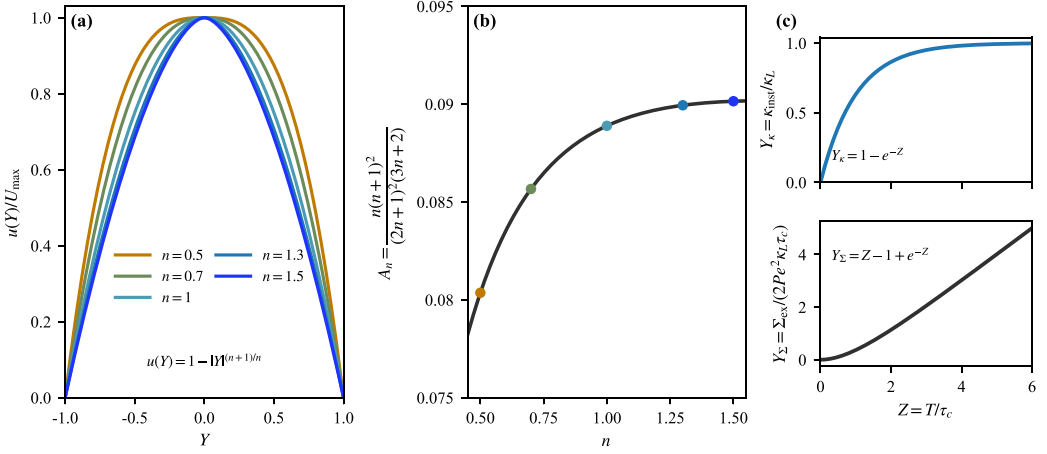


FIG. 2. Power-law channel kinematics and one-mode master curves. (a) Representative plane power-law velocity profiles $u(Y) = 1 - |Y|^{(n+1)/n}$, showing the broader plug-like core at smaller n and the more distributed velocity deficit at larger n . (b) Exact short-time kinematic anchor $A_n = n(n+1)^2/[(2n+1)^2(3n+2)]$ for uniform release, which fixes the curvature of the shear-generated $\text{Pe}^2 T^2$ contribution under the unit-reference normalization. (c) Scalar unit-reference specialization of the one-mode master laws, for which $\kappa_0 = 0$, $M_0 = A[u]$, $Y_\kappa = \kappa_{\text{inst}}/\kappa_L$, and $Y_\Sigma = \Sigma_{\text{ex}}/(2\text{Pe}^2\kappa_L\tau_c)$, as functions of $Z = T/\tau_c$.

released on faster streamlines outrun those released on slower ones before transverse diffusion exchanges them between streamlines; because it is fixed entirely by the release kinematics, it remains independent of the subsequent transverse relaxation mechanism.

For the plane power-law profile

$$u(Y) = 1 - |Y|^m, \quad m = \frac{n+1}{n}, \quad (15)$$

where n is the power-law flow index and m is the corresponding profile exponent. Figure 2(a) gives the visual counterpart of Eq. (15): smaller n broadens the plug-like core, and larger n spreads the velocity deficit over more of the channel. The fluid cross-sectional mean velocity is

$$\bar{u}_f = \bar{u} = \frac{n+1}{2n+1}, \quad (16)$$

and the short-time coefficient reduces to

$$A_n = A[u] = \frac{m^2}{(m+1)^2(2m+1)} = \frac{n(n+1)^2}{(2n+1)^2(3n+2)}. \quad (17)$$

The coefficient of the shear-generated $\text{Pe}^2 T^2$ contribution in a power-law channel is therefore known exactly from n alone. Figure 2(b) plots this analytic anchor and shows its monotone increase over the range used below. For Brownian rods, this kinematic coefficient depends only on n , while the direct axial-diffusion contribution retained by the reduced ADE follows the reference treatment described in Sec. II A.

C. Long-time anchor and memory-kernel representation

Assume that the long-time Taylor enhancement exists as

$$\kappa_L = \lim_{T \rightarrow \infty} \kappa_{\text{inst}}(T) = \lim_{T \rightarrow \infty} \frac{1}{2\text{Pe}^2} \left(\frac{d\Sigma}{dT} - 2 \right). \quad (18)$$

Then the variance has the long-time form

$$\Sigma(T) = \Sigma(0) + 2(1 + \text{Pe}^2 \kappa_L)T + O(1), \quad T \rightarrow \infty. \quad (19)$$

To connect the short- and long-time limits through an all-time representation, consider the excess variance defined in Eq. (7) and set

$$\kappa_0 = \kappa_{\text{inst}}(0). \quad (20)$$

It satisfies $\Sigma_{\text{ex}}(0) = 0$ and $\Sigma'_{\text{ex}}(0) = 2\text{Pe}^2 \kappa_0$. Taylor's formula with integral remainder therefore gives

$$\Sigma_{\text{ex}}(T) = 2\text{Pe}^2 \kappa_0 T + \int_0^T (T-s) \Sigma''_{\text{ex}}(s) ds, \quad (21)$$

which has the structure of a history integral and motivates the memory-kernel definition

$$M(T) = \frac{1}{2\text{Pe}^2} \Sigma''_{\text{ex}}(T) = \frac{d\kappa_{\text{inst}}}{dT}. \quad (22)$$

Equation (22) converts the identity (21) into

$$\Sigma_{\text{ex}}(T) = 2\text{Pe}^2 \kappa_0 T + 2\text{Pe}^2 \int_0^T (T-s) M(s) ds. \quad (23)$$

The long-time limit (18) gives

$$M_0 \equiv M(0), \quad \int_0^\infty M(s) ds = \kappa_L - \kappa_0 \equiv \Delta\kappa. \quad (24)$$

For the scalar unit-reference problem with uniform invariant release, $\kappa_0 = 0$ and the short-time expansion (14) gives $M_0 = A[u]$. For a full tensorial problem, however, M_0 may also contain the initial relaxation of the direct axial-diffusion contribution and need not equal $A[u]$. The excess variance at time T therefore contains both its initial linear contribution and the accumulated relaxation represented by $M(T)$. In the reduced rod application, locally equilibrated orientation statistics affect this redistribution through the precomputed effective coefficients rather than through a separately evolved angular state.

For an invariant release, $M(T)$ is the stationary autocovariance of the sampled axial velocity, and Eq. (23) is the corresponding Green-Kubo representation [75,76]. The two quantities M_0 and $\Delta\kappa$ in Eq. (24) are then the zero-lag autocovariance and its time integral, respectively. For a nonequilibrium release, M remains the exact curvature kernel defined by Eq. (22), but it is not a stationary autocovariance; a one-mode representation is then Green-Kubo-motivated rather than an exact stationary reduction.

D. Dominant-mode closure

The exact kernel is generally multiscale, reflecting a superposition of cross-sectional relaxation contributions with different decay rates. Classical analyses of the approach to normality, together with later multi-mode and center-manifold reductions, show that faster contributions decay first and the final approach to the Taylor-Aris regime is controlled by the slowest surviving contribution [4,77–79]. The two exact constraints in Eq. (24) first define the normalized kernel-area timescale

$$\tau_c \equiv \frac{1}{M_0} \int_0^\infty M(s) ds = \frac{\Delta\kappa}{M_0} = \frac{\kappa_L - \kappa_0}{M_0}. \quad (25)$$

For an invariant release, $M(T)$ is the stationary sampled-velocity autocovariance, so τ_c is its integral correlation time. For a nonequilibrium release, it is the corresponding normalized area timescale of the exact curvature kernel. We compress the relaxation hierarchy into one effective mode by

adopting the single-exponential closure

$$M(T) \approx M_0 e^{-T/\tau_c}. \quad (26)$$

This construction preserves both the exact initial amplitude and the exact kernel area. At the level of the closed correlation, it satisfies $\tau_c dM/dT = -M$ and therefore admits an equivalent one-mode Markov representation. The covariance leaves the trajectory-level transition law open and does not prescribe a particular velocity-renewal process. Under this approximation, the integral correlation time is also the e-folding time of the effective mode and represents its collective transverse-relaxation scale. Substitution of Eq. (26) into Eq. (23) gives

$$\Sigma_{\text{th}}(T) = \Sigma(0) + 2T + 2\text{Pe}^2 \{ \kappa_0 T + \Delta\kappa [T - \tau_c(1 - e^{-T/\tau_c})] \}, \quad (27)$$

and differentiation of Eq. (27) yields

$$\kappa_{\text{inst}}^{\text{th}}(T) = \kappa_0 + \Delta\kappa(1 - e^{-T/\tau_c}). \quad (28)$$

The general closure is fully determined once κ_0 , M_0 , and κ_L are supplied. The scalar unit-reference specialization has $\kappa_0 = 0$, $M_0 = A[u]$, and $\Delta\kappa = \kappa_L$, recovering $\tau_c = \kappa_L/A[u]$ and the formulas used for the invariant scalar benchmarks.

The reduced time

$$Z = \frac{T}{\tau_c} = \frac{M_0 T}{\Delta\kappa} \quad (29)$$

defined in Eq. (29) then produces the master laws

$$\frac{\kappa_{\text{inst}}^{\text{th}} - \kappa_0}{\Delta\kappa} = 1 - e^{-Z}, \quad \frac{\Sigma_{\text{th}}(T) - \Sigma(0) - 2T - 2\text{Pe}^2 \kappa_0 T}{2\text{Pe}^2 \Delta\kappa \tau_c} = Z - 1 + e^{-Z}. \quad (30)$$

For the scalar unit-reference specialization, we denote the corresponding reduced observables by $Y_\kappa \equiv \kappa_{\text{inst}}/\kappa_L$ and $Y_\Sigma \equiv \Sigma_{\text{ex}}/(2\text{Pe}^2 \kappa_L \tau_c)$. Figure 2(c) displays the scalar unit-reference specialization of the two dimensionless master curves from Eq. (30): the shifted instantaneous enhancement approaches its plateau as $1 - e^{-Z}$, and the shifted reduced excess variance follows $Z - 1 + e^{-Z}$. Both curves follow from assuming that one slow transverse mode controls the decay of the memory kernel.

E. Initial-distribution dependence and the point-source limit

Equation (13) extends the exact short-time anchor to any normalized transverse release, although the complete memory also depends on how that release relaxes after injection. This additional dependence is exposed most directly in the isotropic advection-diffusion problem by letting Y_T denote the reflected transverse diffusion process and defining

$$C_{\rho_0}(s, r) = \text{Cov}_{\rho_0}[u(Y_s), u(Y_r)]. \quad (31)$$

After bare streamwise diffusion has been removed, the shear-induced variance has the exact representation

$$\Sigma_{\text{ex}}(T) = \text{Pe}^2 \int_0^T \int_0^T C_{\rho_0}(s, r) ds dr. \quad (32)$$

In this representation, $A_{\rho_0}[u] = C_{\rho_0}(0, 0)$ fixes the initial curvature, while the temporal structure of C_{ρ_0} carries the subsequent history. If ρ_0 is the invariant transverse distribution, then the covariance depends only on $|s - r|$ and Eq. (32) reduces to the stationary autocovariance form of Eq. (23); for a nonequilibrium release, its dependence on both times retains the relaxation of the source profile, which cannot be reconstructed from either the spatial variance of the release or the single value $A_{\rho_0}[u]$ [10,80].

Replacing $A[u]$ by $A_{\rho_0}[u]$ in Eq. (26) therefore preserves the exact initial curvature whenever $A_{\rho_0}[u] > 0$, but its all-time accuracy is governed by the modal content of the release. Within the centered Gaussian family examined in Appendix A, increasing the Gaussian width parameter $\sigma_Y > 0$ broadens ρ_0 toward the invariant uniform distribution, drives $A_{\rho_0}[u]/A[u]$ toward unity, and progressively recovers the uniform-release one-mode transient. This improvement accompanies the diminishing nonstationary relaxation of the source profile and the stronger relative dominance of the slow transverse contribution that also controls κ_L , whereas localized distributions first acquire sampled-velocity variability as diffusion carries particles away from the injection streamline, adding an initial build-up that is absent from the single-amplitude description.

For a point release $\rho_0(Y) = \delta(Y - Y_0)$, the resulting value $A_{\rho_0}[u] = 0$ removes the T^2 ballistic contribution, and a local expansion of an interior source with unit transverse diffusivity gives, when $u'(Y_0) \neq 0$,

$$\Sigma_{\text{ex}}(T) = \frac{2}{3}\text{Pe}^2[u'(Y_0)]^2 T^3 + o(T^3), \quad \kappa_{\text{inst}}(T) = [u'(Y_0)]^2 T^2 + o(T^2). \quad (33)$$

At a smooth stationary point of the velocity profile, $u'(Y_0) = 0$, the next contribution is

$$\Sigma_{\text{ex}}(T) = \frac{1}{3}\text{Pe}^2[u''(Y_0)]^2 T^4 + o(T^4), \quad \kappa_{\text{inst}}(T) = \frac{2}{3}[u''(Y_0)]^2 T^3 + o(T^3). \quad (34)$$

Equations (33) and (34) show how diffusion makes the curvature kernel rise from $M(0) = 0$ before its time integral approaches the initial-condition-independent coefficient κ_L . Since a single exponential with zero initial amplitude also has zero area, the point-source transient is propagated as an initial-value modal problem, whose plane-Poiseuille spectral evaluation is given in Appendix A. The invariant uniform-release closure (26) provides the exact stationary reference for the scalar benchmarks; the rod calculations below instead use the same two-anchor form as a Green-Kubo-motivated approximation for a uniform but generally nonequilibrium transverse injection.

F. Development scales

For the general one-mode closure, the approach from κ_0 to κ_L is characterized by the build-up time

$$T_d = \tau_c = \frac{\Delta\kappa}{M_0}. \quad (35)$$

For the scalar unit-reference case, $\Delta\kappa = \kappa_L$ and $M_0 = A[u]$, so Eq. (35) reduces to $\tau_c = \kappa_L/A[u]$. This characteristic e-folding time measures the decay of sampled-velocity memory and the accompanying build-up of the one-dimensional Taylor-Aris description. Classical and later analyses emphasise that this transition is controlled by cross-sectional relaxation on diffusion times of order $a^2/\bar{D}$, and that true transverse uniformity can remain slower than the onset of longitudinal normality [1,4,79]. Let u_{adv} denote the mean dimensionless convective speed appropriate to the long-time transverse measure. The associated characteristic convective-distance estimate is

$$X_d = u_{\text{adv}} \text{Pe} T_d. \quad (36)$$

The definitions of T , X , and Pe recover the corresponding dimensional scales as

$$t_d = \frac{a^2}{\bar{D}} T_d, \quad x_d = a X_d = u_{\text{adv}} U_c t_d. \quad (37)$$

At $T = T_d$, the one-mode law in Eq. (28) gives $(\kappa_{\text{inst}} - \kappa_0)/\Delta\kappa = 1 - e^{-1} \simeq 0.632$, so T_d and X_d provide characteristic measures of the build-up. A prescribed remaining relative gap $0 < \varepsilon < 1$ defines a corresponding operational time and characteristic distance through

$$T_\varepsilon = T_d \ln\left(\frac{1}{\varepsilon}\right), \quad X_\varepsilon = X_d \ln\left(\frac{1}{\varepsilon}\right), \quad (38)$$

for which $1 - (\kappa_{\text{inst}} - \kappa_0)/\Delta\kappa = \varepsilon$ within the closure. The choice $\varepsilon = 0.1$ gives the 90-percent measures $T_{90} = T_d \ln 10$ and $X_{90} = X_d \ln 10$. Because X_d and X_ε use a fixed asymptotic convective speed, they provide characteristic estimates. Let $X_c(T) \equiv \mathbb{E}[X(T)]$ and $x_c(t) \equiv aX_c(T)$ denote the dimensionless and dimensional streamwise centroids, respectively. The corresponding exact transient centroid displacements are $X_c(T_d) - X_c(0)$ and $X_c(T_\varepsilon) - X_c(0)$. Together, the characteristic and tolerance-based scales convert sampled-memory decay into residence-time and channel-length estimates for long-time fits, one-dimensional reduced models, and finite-window dispersion measurements [11,12].

III. EXTENSION ACROSS GEOMETRY AND MICROSTRUCTURE

This section carries the closure from isotropic scalar transport through a change of transverse geometry and then to Brownian rods with orientational microstructure. The plane-channel and circular-tube problems provide exact spectral references, while the Newtonian rod problem supplies the long-time coefficient independently through rod-GTD theory. Together these benchmark calculations show how the same two-anchor construction operates as geometry and internal dynamics become more complex. Here and below, the reported normalized errors are per-case relative root-mean-square error (RMSE) values computed over the sampled transient; excess-variance errors are normalized by the dynamic range of the corresponding reference curve, instantaneous-enhancement errors by κ_L , and the quoted means and maxima are then taken over the relevant case set.

A. Exact isotropic spectral benchmark in plane power-law Poiseuille flow

Isotropic diffusion isolates the transverse relaxation mechanism from any particle-scale microstructure. For this passive-scalar benchmark, the transient problem reduces to the classical Taylor-dispersion initial-value problem in a channel. Its long-time limit is governed by Taylor-Aris theory, while the all-time moment evolution can be constructed by the discrete-spectrum method of moments for bounded shear dispersion [1,2,7,10]. In plane power-law Poiseuille flow, this framework yields a complete spectral transient and provides a direct reference for the extension.

For isotropic diffusion the transverse operator is the Neumann Laplacian on $[-1, 1]$, and specializing the classical Taylor-Aris long-time formula to the power-law profile (15) gives the isotropic long-time Taylor enhancement $\kappa_\infty^{\text{iso}}(n)$ as [1,2]

$$\kappa_\infty^{\text{iso}}(n) = \frac{2n^2(n+1)^2}{3(2n+1)^2(4n+1)(5n+2)}. \quad (39)$$

Expanding the centered velocity field in the cosine eigenfunctions of the Neumann Laplacian, following the bounded-channel moment formulation of Barton [7], define the modal coefficients by

$$b_k(n) = -\sqrt{2} \int_0^1 Y^{(n+1)/n} \cos(k\pi Y) dY, \quad k = 1, 2, \dots \quad (40)$$

The exact transient can then be written as

$$\kappa_{\text{inst}}^{\text{exact}}(T; n) = \sum_{k=1}^{\infty} \frac{b_k(n)^2}{k^2\pi^2} (1 - e^{-k^2\pi^2 T}), \quad (41)$$

$$\Sigma^{\text{exact}}(T; n) = \Sigma(0) + 2T + 2\text{Pe}^2 \sum_{k=1}^{\infty} \frac{b_k(n)^2}{k^2\pi^2} \left[T - \frac{1 - e^{-k^2\pi^2 T}}{k^2\pi^2} \right]. \quad (42)$$

TABLE I. Distribution of modal weight in the exact isotropic benchmark. The first three numerical columns give the partition of the exact short-time anchor $A_n = \sum_{k \geq 1} b_k(n)^2$ into the shares carried by the first mode, the second mode, and the remaining modes. The final column gives the share of the long-time enhancement $\kappa_\infty^{\text{iso}} = \sum_{k \geq 1} b_k(n)^2 / (k^2 \pi^2)$ carried by the first mode.

n	Mode 1 (E_1)	Mode 2 (E_2)	Modes $k \geq 3$ ($E_{\geq 3}$)	Mode 1 (K_1)
0.5	0.813	0.144	0.043	0.954
0.7	0.877	0.096	0.027	0.971
1.0	0.924	0.058	0.018	0.983
1.3	0.947	0.038	0.015	0.989
1.5	0.956	0.030	0.014	0.991

Equations (39)–(42) therefore define the exact isotropic benchmark. Recalling from Sec. II that the memory kernel satisfies $M(T) = d\kappa_{\text{inst}}/dT$, straightforward differentiation of Eq. (41) yields

$$M^{\text{exact}}(T; n) = \sum_{k=1}^{\infty} b_k(n)^2 e^{-k^2 \pi^2 T}. \quad (43)$$

Evaluation of Eq. (43) at $T = 0$ recovers the exact short-time anchor, $M^{\text{exact}}(0; n) = \sum_{k=1}^{\infty} b_k(n)^2 = A_n$, so the ratio $b_k(n)^2/A_n$ gives the fraction of the initial kinematic variance carried by mode k . Taking $T \rightarrow \infty$ in Eq. (41) gives $\kappa_\infty^{\text{iso}}(n) = \sum_{k=1}^{\infty} b_k(n)^2 / (k^2 \pi^2)$, so $b_k(n)^2 / (k^2 \pi^2 \kappa_\infty^{\text{iso}})$ gives the modal contribution to the long-time plateau. Table I lists the initial-energy shares $E_1 = b_1^2/A_n$, $E_2 = b_2^2/A_n$, $E_{\geq 3} = \sum_{k \geq 3} b_k(n)^2/A_n$, and the fundamental-mode plateau share $K_1 = (b_1^2/\pi^2)/\kappa_\infty^{\text{iso}}$.

The spectral content is already sharply concentrated in the fundamental mode. Across the sampled range, mode $k = 1$ carries 81.3% to 95.6% of the initial kernel energy, while the first two modes together carry 95.7% to 98.6%. The concentration is even stronger in the long-time coefficient because every mode is weighted by an additional factor $1/k^2$: the fundamental mode alone accounts for 95.4% to 99.1% of $\kappa_\infty^{\text{iso}}$. The one-mode closure therefore draws support from the modal amplitudes as well as from the modal decay rates.

The flow-index dependence follows directly from the projection integral (40). Smaller n makes the underlying velocity profile u more plug-like. After subtraction of the cross-sectional mean, the centered field $u - \bar{u}$ therefore has a flatter core and a velocity deficit concentrated more strongly in the near-wall layers. Its cosine expansion retains a broader spectral tail, so higher transverse modes carry a larger share of both the initial kernel energy and the transient response. As n increases, the centered profile is distributed more smoothly across the channel and the higher-mode projections weaken. Table I shows this trend quantitatively: E_2 falls from 0.144 at $n = 0.5$ to 0.030 at $n = 1.5$, while E_1 rises from 0.813 to 0.956.

The hierarchy of decay rates encoded in Eq. (43) is therefore explicit: mode k relaxes on the timescale $(k^2 \pi^2)^{-1}$, so every mode with $k \geq 2$ decays at least four times faster than the fundamental mode. To validate this exact benchmark, direct numerical simulations were performed for all 5×3 isotropic cases, spanning the power-law indices $n \in \{0.5, 0.7, 1.0, 1.3, 1.5\}$ and Péclet numbers $\text{Pe} \in \{10, 100, 1000\}$, using an entropic LBM solver for the ADE described in Appendix B [31]. Figure 3(a) gives the raw transient comparison at $\text{Pe} = 100$: the entropic-LBM direct simulation reproduces the exact spectral solution with mean normalized excess-variance error 3.75×10^{-3} and maximum 5.52×10^{-3} . On the same unscaled axes, the one-mode reconstruction follows the exact transient with mean errors 7.52×10^{-3} in excess variance and 7.82×10^{-3} in instantaneous enhancement, with maxima 1.27×10^{-2} and 1.50×10^{-2} , respectively. Figure 3(b) then removes the case-dependent scales by plotting all (n, Pe) simulation cases in Y_Σ and Z , where the results collapse onto the exact normalized solution and the one-mode master curve. The mean instantaneous-enhancement error decreases monotonically from 1.50×10^{-2} at $n = 0.5$ to 3.70×10^{-3} at $n = 1.5$, consistent with the increasing concentration of modal weight in the fundamental mode. The exact

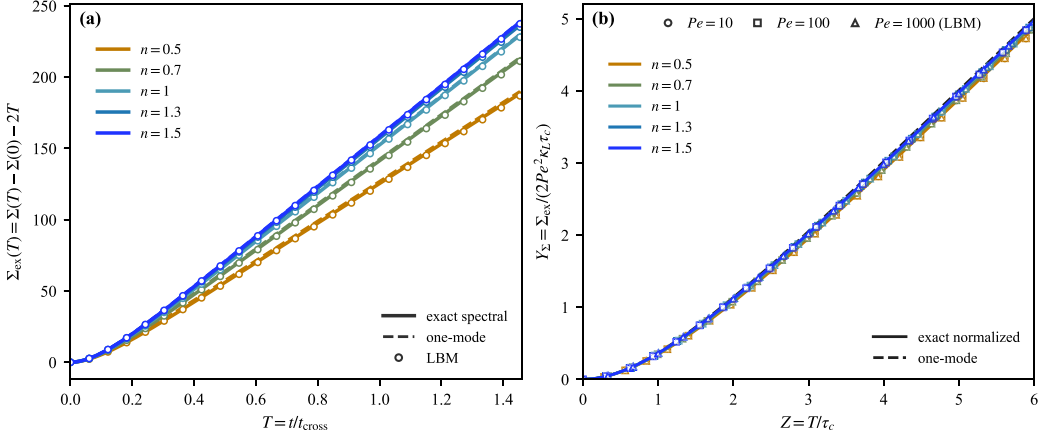


FIG. 3. Exact isotropic reference in plane power-law Poiseuille flow. (a) Excess variance $\Sigma_{\text{ex}}(T) = \Sigma(T) - \Sigma(0) - 2T$ at $Pe = 100$, comparing the exact spectral solution, the one-mode closure, and the entropic-LBM direct simulation for representative power-law indices n . (b) Reduced excess-variance collapse over all isotropic simulation cases ($n = 0.5$ – 1.5 , $Pe = 10$ – 1000), plotted with the exact normalized solution and the one-mode master curve in the variables Y_{Σ} and $Z = T/\tau_c$.

spectral solution therefore shows why the one-mode closure remains accurate across the whole transient: the first mode carries most of the initial kinematic variance, dominates the long-time plateau, and survives for the longest time.

B. Geometry transfer to circular power-law tubes

The plane benchmark above varies the carrier profile while retaining a slit-like cross-section, so we next preserve isotropic diffusion, uniform release and the same five power-law indices while replacing the transverse interval by a circular tube of radius a . At each fixed n , this pairing changes the transverse geometry while retaining the same constitutive index and centerline-speed normalization, and the sweep over n determines whether the geometric effect persists across plug-like and more broadly sheared velocity profiles. With $R = r/a$, the fully developed profile and the area average are

$$u_{\text{tube}}(R) = 1 - R^m, \quad m = \frac{n+1}{n}, \quad \langle f \rangle_{\text{tube}} = 2 \int_0^1 R f(R) dR, \quad (44)$$

which give the mean sampled speed

$$\bar{u}_{\text{tube}} = \langle u_{\text{tube}} \rangle_{\text{tube}} = \frac{m}{m+2} = \frac{n+1}{3n+1}. \quad (45)$$

The initial sampled-velocity variance and the radial Taylor cell problem can then be evaluated in closed form as

$$A_n^{\text{tube}} = \langle (u_{\text{tube}} - \bar{u}_{\text{tube}})^2 \rangle_{\text{tube}} = \frac{m^2}{(m+1)(m+2)^2} = \frac{n(n+1)^2}{(2n+1)(3n+1)^2}, \quad (46)$$

$$\kappa_{\infty,n}^{\text{tube}} = \frac{m^2}{2(m+4)(m+2)^3} = \frac{n^2(n+1)^2}{2(5n+1)(3n+1)^3}, \quad \tau_n^{\text{tube}} = \frac{\kappa_{\infty,n}^{\text{tube}}}{A_n^{\text{tube}}} = \frac{n(2n+1)}{2(5n+1)(3n+1)}. \quad (47)$$

The complete transient follows from the radial diffusion generator

$$\mathcal{L}_{\text{tube}} f = \frac{1}{R} \frac{d}{dR} \left(R \frac{df}{dR} \right), \quad (48)$$

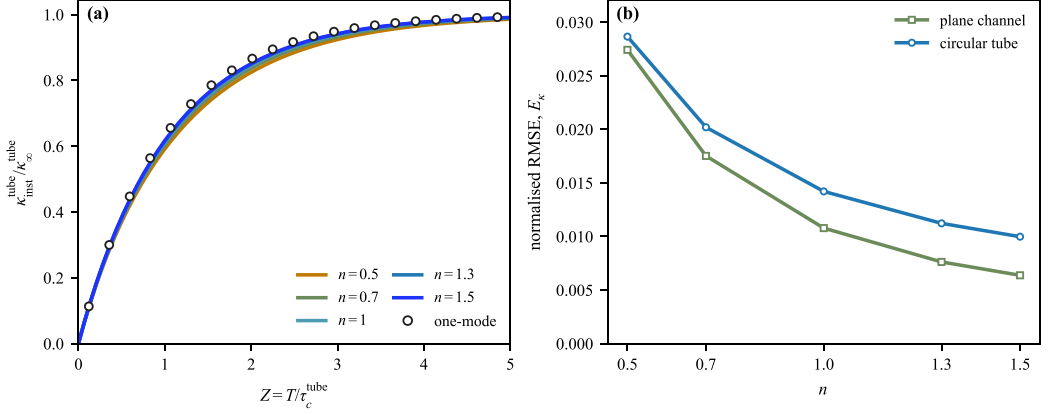


FIG. 4. Geometry dependence of the isotropic spectral benchmark. (a) Exact circular-tube instantaneous enhancement for the five power-law indices, shown by solid curves, together with the anchored one-mode law shown by open circles. (b) Normalized instantaneous-enhancement RMSE of the one-mode law for the plane-channel and circular-tube spectra over $0 \leq Z \leq 5$. At each n , the comparison retains isotropic diffusion, uniform release and the same power-law exponent while changing the transverse geometry.

with regularity at the axis and a reflecting wall. Its nonconstant orthonormal eigenfunctions and decay rates are

$$\phi_j(R) = \frac{J_0(\alpha_j R)}{|J_0(\alpha_j)|}, \quad \lambda_j = \alpha_j^2, \quad J_1(\alpha_j) = 0, \quad j \geq 1, \quad (49)$$

where J_ν is the Bessel function of the first kind of order ν , α_j is the j th positive zero of J_1 , and the normalization is taken with respect to the area average in Eq. (44). Defining the velocity projections by

$$b_j^{\text{tube}}(n) = \langle (u_{\text{tube}} - \bar{u}_{\text{tube}}) \phi_j \rangle_{\text{tube}}, \quad (50)$$

the exact instantaneous enhancement becomes

$$\kappa_{\text{inst}}^{\text{tube}}(T; n) = \sum_{j=1}^{\infty} \frac{[b_j^{\text{tube}}(n)]^2}{\alpha_j^2} (1 - e^{-\alpha_j^2 T}), \quad (51)$$

and its endpoint sums recover $A_n^{\text{tube}} = \sum_{j \geq 1} [b_j^{\text{tube}}]^2$ and $\kappa_{\infty, n}^{\text{tube}} = \sum_{j \geq 1} [b_j^{\text{tube}}]^2 / \alpha_j^2$. The cosine spectrum of the plane channel has therefore been replaced by an area-weighted Bessel spectrum while the two anchors and the closure construction retain the same roles.

Figure 4 compares the exact Bessel transients with the anchored one-mode law and places their normalized instantaneous-enhancement errors beside those of the plane-channel spectrum over the same reduced-time interval. In both geometries the error decreases as n increases and the centered velocity field becomes more strongly concentrated in the slowest velocity-coupled mode, while the circular-tube error is modestly larger because its area weighting and radial eigenfunctions distribute a greater share of the low- n profile over faster modes. The normalized RMSE remains below 3 % throughout the tube family, and the slowest-mode contribution to $\kappa_{\infty, n}^{\text{tube}}$ rises from 0.938 at $n = 0.5$ to 0.981 at $n = 1.5$. The paired spectra consequently show that transverse geometry changes the modal partition quantitatively while preserving the asymptotically anchored structure of the reduction.

The isotropic calculation provides a direct geometric test in which particle configuration dynamics remain fixed. For Brownian rods in three-dimensional ducts, local Jeffery-Brownian orientation moments generate a tensorial transverse operator, as developed by Feng and Chu for circular tubes and regular-polygonal ducts [81,82]. Both formulations retain an invariant transverse mode,

a cell-problem Taylor coefficient and a discrete transient spectrum, so the same asymptotically anchored reduced closure may be expected to extend to three-dimensional Brownian-rod dispersion whenever one effective velocity-coupled mode governs the relaxation. A systematic assessment of that tensorial extension is reserved for a separate study.

C. Brownian-rod benchmark in Newtonian Poiseuille flow

Brownian rods add orientational microstructure to the dispersion problem. The cross-stream sampling of streamlines is then affected by translational diffusion, rotational advection-diffusion, and the anisotropy of translational transport relative to the rod axis. The Newtonian channel $n = 1$ isolates that microstructural physics in a fixed parabolic profile:

$$u(Y) = 1 - Y^2, \quad \bar{u} = \frac{1}{2} \int_{-1}^1 u(Y) dY = \frac{2}{3}, \quad (52)$$

where the profile is normalized by the centerline speed U_c . The short-time anchor then follows directly from the general definition $A[u] = (\overline{u - \bar{u}})^2$:

$$A_1 = A[u] = \frac{1}{2} \int_{-1}^1 \left(1 - Y^2 - \frac{2}{3}\right)^2 dY = \frac{4}{45}. \quad (53)$$

The Newtonian benchmark therefore carries no fitted short-time parameter: both $u(Y)$ and A_1 are fixed analytically by the parabolic plane-Poiseuille profile.

The rod microphysics enters through the rotational and anisotropic-translational dynamics. Following Kumar *et al.* [45], the rotational Péclet number is

$$\text{Pe}_r = \frac{U_c}{aD_\theta}, \quad (54)$$

where D_θ is the rotational diffusivity. This number measures deterministic shear rotation relative to rotational diffusion. For a planar prolate rod of aspect ratio p , with orientation angle θ measured from the streamwise direction and $u_Y = du/dY$, the deterministic part of the Jeffery rotation is

$$\omega_J(Y, \theta) = u_Y(Y) \frac{p^2 \sin^2 \theta + \cos^2 \theta}{p^2 + 1}, \quad (55)$$

so the angular velocity is nonuniform in θ . Let $\mathbf{e}(\theta) = (\cos \theta, \sin \theta)$ be the unit vector along the rod axis. The orientation-resolved translational diffusivity tensor is

$$D_{ij}(\theta) = D_\perp \delta_{ij} + (D_\parallel - D_\perp) e_i(\theta) e_j(\theta), \quad i, j \in \{x, y\}, \quad (56)$$

where δ_{ij} is the Kronecker δ . Its wall-normal component is

$$D_{yy}(\theta) = D_\parallel \sin^2 \theta + D_\perp \cos^2 \theta, \quad D_\parallel > D_\perp \quad (p > 1). \quad (57)$$

Here $D_\parallel$ and $D_\perp$ are the translational diffusivities parallel and perpendicular to the rod axis, respectively. Equation (57) is the key microscopic ingredient behind the dispersion enhancement: when the rod is nearly streamwise aligned ($\theta \approx 0$), its cross-stream diffusion is reduced toward $D_\perp$.

The reduced spatial description uses stationary angular averages of the orientation-dependent transport in Eq. (57). Its local-equilibrium interpretation therefore requires orientational relaxation to precede the cross-stream and Taylor-dispersion dynamics. The relevant separation is governed by the ratio

$$\varepsilon_\theta = \frac{\text{Pe}_r}{\text{Pe}} = \frac{\bar{D}}{a^2 D_\theta}. \quad (58)$$

For rods small compared with the channel width, Kumar *et al.* [45] estimate $\varepsilon_\theta \sim a_p^2/a^2 \ll 1$, where a_p denotes the particle length scale and a is the channel half-width. Their macrotransport

calculation and that of Khair [46] both adopt the corresponding ordering $\text{Pe}/\text{Pe}_r \gg 1$. On the transverse-diffusion time $T = t\bar{D}/a^2$, the slowest nonuniform, π -periodic mode at zero shear decays as $\exp[-4(\text{Pe}/\text{Pe}_r)T]$. This gives the e-folding time

$$T_\theta = \frac{\text{Pe}_r}{4\text{Pe}} = \frac{\varepsilon_\theta}{4} \ll 1. \quad (59)$$

Away from zero shear, the local Jeffery-Brownian operator modifies the order-one relaxation factor while retaining the separation $T_\theta = O(\text{Pe}_r/\text{Pe})$. A preparation-induced angular bias therefore relaxes within a thin initial or entrance layer before the cross-stream and Taylor-dispersion dynamics resolved here.

Within this scale-separated regime, Equation (55) already shows that Jeffery rotation is strongly nonuniform in θ , so a prolate rod spends a disproportionate fraction of each cycle close to streamwise alignment [45,83]. Rotational diffusion competes with that deterministic bias through Pe_r . When $\text{Pe}_r \ll 1$, the orientation distribution remains nearly isotropic and the rods behave, to leading order, like tracers with an orientationally averaged diffusivity. When $\text{Pe}_r \gg 1$, the shear-driven dynamics dominate and the rods become strongly aligned with the flow, which is precisely the regime in which the largest departures from classical spherical-particle Taylor dispersion are observed [45,46]. For the Newtonian profile (52), the local shear is $u_Y = -2Y$, so the rotational forcing grows linearly away from the centerline; equivalently, in the notation used below for power-law channels, $q(Y) = -(\text{Pe}_r/2)u_Y(Y) = \text{Pe}_r Y$.

The long-time enhancement changes because flow alignment directly modifies how efficiently rods cross streamlines. In the rod-GTD theory of Kumar *et al.* [45], the orientationally averaged lateral diffusivity

$$D_Y(Y) = \langle D_{yy}(\theta) \rangle_\theta, \quad (60)$$

where $\langle \cdot \rangle_\theta$ denotes averaging over the local orientation distribution. This lateral diffusivity decreases in the high-shear regions near the walls as the rods align with the streamwise direction, while the same shear-orientation coupling also produces a weak wallward migration [43,45]. Since Taylor dispersion is controlled by the competition between axial shear and transverse homogenization, weaker lateral diffusion slows the transverse cell-problem relaxation and increases the shear-generated contribution to the long-time axial dispersion coefficient. Under fixed U_c , a , and $\bar{D}$, this contribution is represented by the dimensionless rod coefficient $\kappa_L^{\text{rod}}(p, \text{Pe}_r)$. Its enhancement is therefore the macroscopic signature of flow alignment suppressing cross-stream mixing [45,46].

The stationary local Jeffery-Brownian distribution therefore supplies all angular averages in the reduced description. Under $\text{Pe}/\text{Pe}_r \gg 1$, adiabatic elimination places the resolved time origin after the thin entrance layer in which the rods adjust to the imposed local shear. The conditional distribution approaches isotropy where rotational forcing is weak and carries a flow-induced bias elsewhere.

This Newtonian benchmark also separates the two shear-dispersion anchors cleanly. The short-time anchor remains the exact kinematic coefficient $A_1 = 4/45$. After the fast angular layer has been eliminated, the initially sampled velocity field alone determines the leading shear-generated $\text{Pe}^2 T^2$ contribution, which is independent of p and Pe_r . The stationary orientation moments enter through the direct axial-diffusion contribution and the effective transport governing subsequent cross-stream exchange. The long-time shear-dispersion anchor is $\kappa_L^{\text{rod}}(p, \text{Pe}_r)$, which depends on the orientation statistics generated by the competition between Jeffery rotation, rotational diffusion, and anisotropic translational diffusion. The short-time anchor records the release kinematics, while the long-time anchor carries the coarse-grained effect of rod microphysics.

A uniform rod injection is not, in general, the invariant transverse measure: when the effective $D_Y(Y)$ varies, the no-flux stationary density is proportional to $D_Y^{-1}(Y)$, as derived in Eq. (C4). The Newtonian-rod reconstruction is therefore interpreted as a Green-Kubo-motivated one-pole approximation for the dominant shear-generated contribution. For this contribution, the scalar specialization of Eqs. (25)–(28) uses $M_0 = A_1$, $\kappa_0 = 0$, and $\tau_c = \kappa_L^{\text{rod}}/A_1$. The direct simulations retain

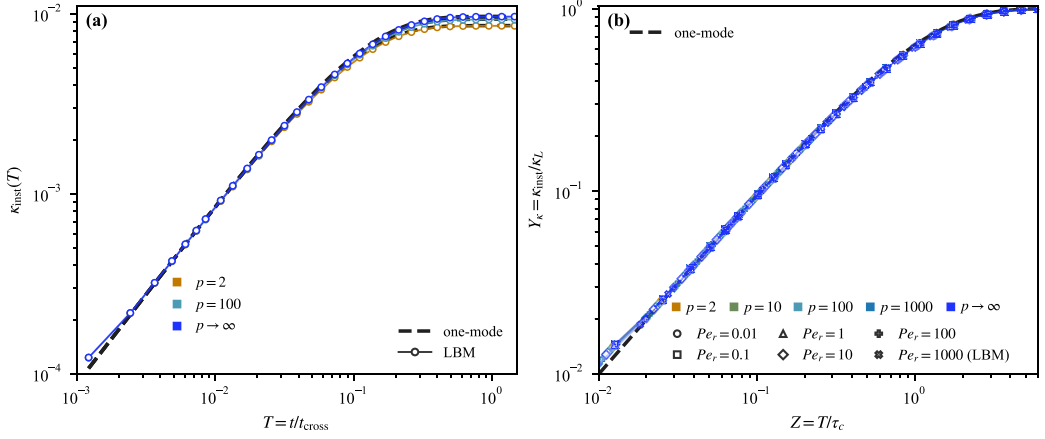


FIG. 5. Brownian rods in Newtonian Poiseuille flow. (a) Representative instantaneous enhancements $\kappa_{\text{inst}}(T)$ for selected rod aspect ratios p , shown on log-log axes and comparing direct simulation with the Green-Kubo-motivated one-mode reconstruction of the dominant shear-generated contribution, anchored by A_1 and the independently supplied rod-GTD value of κ_L^{rod} . (b) Reduced collapse of the full Newtonian-rod direct-simulation ensemble over p and Pe_r , shown on log-log axes using the unshifted rod normalization $Y_\kappa = \kappa_{\text{inst}}/\kappa_L^{\text{rod}}$ versus $Z = T/\tau_c$. The direct simulations retain the small tensorial direct-diffusion contribution that is not separately parametrized by the reconstruction.

the tensorial direct-diffusion contribution, which is not separately parametrized by this two-anchor curve.

With these shear anchors fixed, the transient closure uses the rod microphysics through the independently supplied long-time coefficient κ_L^{rod} , with no additional transient fit parameter. The Newtonian rod problem therefore tests how accurately this approximation reconstructs the full directly simulated transient.

At $Pe = 1000$, direct simulations were performed for 30 Newtonian-rod cases, covering $p = 2, 10, 100, 1000, \infty$ and $Pe_r = 0.01, 0.1, 1, 10, 100, 1000$. The long-time coefficients used to anchor the reconstruction are taken independently from the rod-GTD benchmark of Kumar *et al.* [45], while Khair [46] provides the asymptotic interpretation of the low- and high- Pe_r limits; the coefficients range from 8.47×10^{-3} to 1.18×10^{-2} , giving the shear-specialized time $\tau_c = \kappa_L^{\text{rod}}/A_1$ between 9.52×10^{-2} and 1.32×10^{-1} . Figure 5(a) shows representative instantaneous-enhancement curves on log-log axes, comparing direct simulations with the one-mode reconstruction based on A_1 and the independently supplied rod-GTD plateau. Figure 5(b) then places the full 30-case direct-simulation ensemble in the unshifted rod normalization $Y_\kappa = \kappa_{\text{inst}}/\kappa_L^{\text{rod}}$ and Z , also on log-log axes, where the reduced transients follow the scalar specialization of Eq. (30) to the reported accuracy. This benchmark therefore tests how accurately the shear-anchored one-mode approximation reconstructs directly simulated transients once the long-time rod physics has been supplied independently. The endpoint $Pe_r = Pe = 1000$ also appears in the extreme theory-simulation comparison discussed by Kumar *et al.* [45] and Khair [46]. Its role here is to stress-test the reduced one-mode closure after the stationary orientation coefficients have been supplied. The physical validity of local orientational equilibrium follows separately from the scale-separated regime.

Across these Newtonian-rod direct simulations, the mean normalized errors are 4.49×10^{-3} in excess variance and 1.30×10^{-2} in instantaneous enhancement, with maxima 4.78×10^{-3} and 1.31×10^{-2} . The same reduction remains accurate when the particles carry orientational microstructure, provided that the long-time coefficient is supplied independently.

IV. REDUCED BROWNIAN-ROD TRANSPORT IN POWER-LAW CHANNELS

A. Problem formulation and spatial heterogeneity of local shear

Section IV studies a semianalytically defined reduced Brownian-rod transport model in a plane power-law channel. The imposed profile (15) fixes the release kinematics and the shear map used in the adopted Jeffery-type orientational forcing. Semianalytic preprocessing supplies the effective transport tensor $D_{ij}(Y)$, the wall-normal migration drift $v_d(Y)$, the asymptotic rod speed u_m^{rod} , and the long-time shear-dispersion anchor κ_{ref} . The local shear entering the adopted Jeffery-type orientational forcing is

$$q(Y) = -\frac{\text{Pe}_r}{2} u_Y(Y) = \frac{(n+1)\text{Pe}_r}{2n} |Y|^{1/n} \text{sgn}(Y), \quad (61)$$

where $u_Y = du/dY$, sgn is the sign function, and Pe_r is the rotational Péclet number defined in Eq. (54). Equation (61) shows that the flow index changes both the axial velocity profile seen at release and the wall-normal distribution of orientational forcing. Writing $q_w = (n+1)\text{Pe}_r/(2n)$ for the wall value, the normalized shear map is $|q(Y)|/|q_w| = |Y|^{1/n}$. For a prescribed threshold fraction $0 < \chi < 1$, the low-shear set $|q| < \chi |q_w|$ therefore occupies $|Y| < \chi^n$, giving a dimensionless core width $2\chi^n$ and hence the full-channel-width fraction χ^n . For the representative threshold $\chi = 0.1$, this fraction is 31.6% at $n = 0.5$, 10.0% at $n = 1$, and only 3.16% at $n = 1.5$. The carrier rheology thus changes the wall-normal landscape entering the reduced orientational dynamics very strongly.

To show how this shear landscape enters the coarse-grained transport inputs, let $\Psi_n(\theta; Y)$ denote the stationary local orientation distribution generated by the adopted angular drift and rotational diffusion, normalized by $\int_0^\pi \Psi_n(\theta; Y) d\theta = 1$. At fixed particle-frame diffusivities, the local translational tensor follows from the orientation average

$$D_{ij}^{\text{eff}}(Y; n) = \int_0^\pi D_{ij}(\theta) \Psi_n(\theta; Y) d\theta.$$

For notational economy, we henceforth write $D_{ij}(Y) \equiv D_{ij}^{\text{eff}}(Y; n)$ when n is fixed. The dependence of $q(Y; n)$ on the flow index therefore passes through Ψ_n into $D_{ij}(Y)$, the migration drift $v_d = \partial_Y D_{yy}$, and the long-time anchor κ_{ref} . The adopted Jeffery-type drift fixes the modeling level of this construction by evaluating the classical Jeffery angular velocity with the imposed local shear rate. A particle-resolved treatment of the nonlinear disturbance flow would supply further rheology-dependent corrections to the torque and rotational mobility, which lie beyond the reduced orientational operator used here [58–61].

This preprocessing implements the separation of scales in Eq. (58). When $\text{Pe}/\text{Pe}_r \gg 1$, the angular statistics remain slaved to the local shear on the timescale resolved by the coarse-grained ADE. The resolved time origin follows the omitted $O(\text{Pe}_r/\text{Pe})$ entrance relaxation and uses $\Psi_n(\theta; Y)$ from the outset. Angular initial data have consequently been eliminated from the reduced model, while the underlying physical reorientation retains its finite duration. The formal endpoint $\text{Pe}_r = \text{Pe} = 100$ lies outside this scale-separated regime and extends the numerical test of the reduced transient closure and its LBM implementation after the stationary coefficients have been imposed. The physical justification for local orientational equilibrium rests on the time-scale ordering in Eq. (58).

Within this reduced model, the spatial redistribution has an immediate transport consequence. In shear-thinning channels ($n < 1$), the velocity profile remains close to the centerline value over a broad core, while the velocity deficit and the strongest gradients are compressed into thin near-wall layers. Rods released in that core encounter weak deterministic rotation, so the local orientation distribution remains closer to isotropy and rotational diffusion retains a larger role. By contrast, rods entering the thin near-wall layers encounter stronger Jeffery-type forcing and spend longer intervals near streamwise alignment. For $n > 1$, appreciable local shear extends over a much larger fraction of the cross-section, so the transition from Brownian to shear-dominated behavior is distributed more smoothly across Y . A trajectory migrating across streamlines therefore samples sharply

different effective orientational environments in a shear-thinning channel and a more broadly active alignment field in a shear-thickening one.

The transient closure tracks two rheological inputs. The exact coefficient A_n sets the short-time spread of release velocities, while the shear map (61) enters the semianalytic reduced-rod preprocessing that supplies the effective transport profiles $D_{xx}(Y)$, $D_{xy}(Y)$, $D_{yy}(Y)$, $v_d(Y)$, the asymptotic rod speed u_m^{rod} , and the long-time shear-dispersion anchor κ_{ref} . Those quantities define the anisotropic ADE solved here by the entropic LBM [84]. Full details of the preprocessing and the coupled ADE-LBM pipeline are given in Appendix C.

The uniform transverse injection used by the LBM is generally not the invariant density of this reduced rod ADE. Because $v_d = \partial_Y D_{yy}$, the no-flux stationary density is $\rho_\infty(Y) \propto D_{yy}^{-1}(Y)$, as shown in Eq. (C4); it becomes uniform only when D_{yy} is constant. Accordingly, the rod reconstruction is a Green-Kubo-motivated two-anchor one-pole approximation for a nonequilibrium uniform injection. The pair $(A_n, \kappa_{\text{ref}})$ closes the dominant shear-generated contribution through the specialization $\kappa_0 = 0$, $M_0 = A_n$, and $\tau_c = \kappa_{\text{ref}}/A_n$. The LBM itself retains the spatially varying D_{xx} and hence the tensorial direct-diffusion correction in Eq. (10); that correction is not separately parametrized by the simplified rod curve.

The direct-simulation campaign considered below is at $\text{Pe} = 100$ and $p = 1000$, with $n \in \{0.5, 0.7, 1.0, 1.3, 1.5\}$ and $\text{Pe}_r \in \{0.01, 0.1, 1, 10, 100\}$. Across this parameter grid, the semianalytic long-time anchors span $6.94 \times 10^{-3} \leq \kappa_{\text{ref}} \leq 1.09 \times 10^{-2}$, giving $0.0864 \leq \tau_c = \kappa_{\text{ref}}/A_n \leq 0.1204$. For the 25 archived reduced power-law rod-model cases, the late-time fitted coefficient from the full tensorial LBM runs agrees closely with the semianalytic shear anchor, with mean relative difference 1.41×10^{-3} and maximum 3.71×10^{-3} . These small discrepancies quantify the net size of the omitted direct-diffusion and residual multimode effects over the present parameter range.

B. Alignment dynamics and delayed transverse homogenization: The Pe_r effect

At fixed n , the exact short-time anchor A_n is unchanged, so the dependence on Pe_r enters through the semianalytically supplied transport inputs. Within the adopted reduced model, the deterministic orientational drift is represented by the classical Jeffery angular velocity evaluated with $q(Y; n)$, while rotational diffusion continuously broadens the orientation distribution [45,46,65,83]. The large- Pe_r regime is therefore characterized by prolonged residence near $\theta = 0$, namely near streamwise alignment, whereas the small- Pe_r regime remains close to orientational isotropy. This distinction matters directly for dispersion because the effective wall-normal diffusivity sampled by aligned configurations decreases toward $D_\perp$ in Eq. (57). Streamwise alignment therefore weakens the transverse exchange needed to homogenise the sampled velocity field.

This suppression is also reflected in the reduced-model long-time shear anchor: throughout the power-law application grid, $\kappa_{\text{ref}}/\kappa_\infty^{\text{iso}}(n)$ remains within about 0.05% of unity for $\text{Pe}_r \leq 0.1$, whereas it reaches 1.22–1.23 at $\text{Pe}_r = 100$.

This mechanism organizes the delayed transverse homogenization seen in the Pe_r -sweep panels of Figs. 6(a) and 6(c). When $\text{Pe}_r \ll 1$, rotational diffusion repeatedly randomizes the rods, the orientationally averaged transverse diffusivity remains close to its isotropic value, and streamline sampling is rapidly renewed. When $\text{Pe}_r \gg 1$, rods linger near streamwise alignment for long intervals, so transverse exchange proceeds through the much smaller coefficient $D_\perp$. The packet then preserves the velocity of its initially sampled streamlines for longer, which increases both the long-time shear anchor κ_{ref} and the build-up time $\tau_c = \kappa_{\text{ref}}/A_n$. Averaged over n , increasing Pe_r from 0.01 to 100 raises $T_d = \tau_c$ from 9.36×10^{-2} to 1.15×10^{-1} and the characteristic distance X_d from 6.32 to 7.64. At fixed $n = 1$, the same sweep raises κ_{ref} by 22.4%. The late-time fitted LBM values preserve the same ordering and remain close to the semianalytic anchors throughout the grid. These trends are consistent with the reduced rod-GTD transport picture adopted here and with the lateral migration mechanism identified for Brownian rods in parabolic channels by Nitsche and Hinch [43].

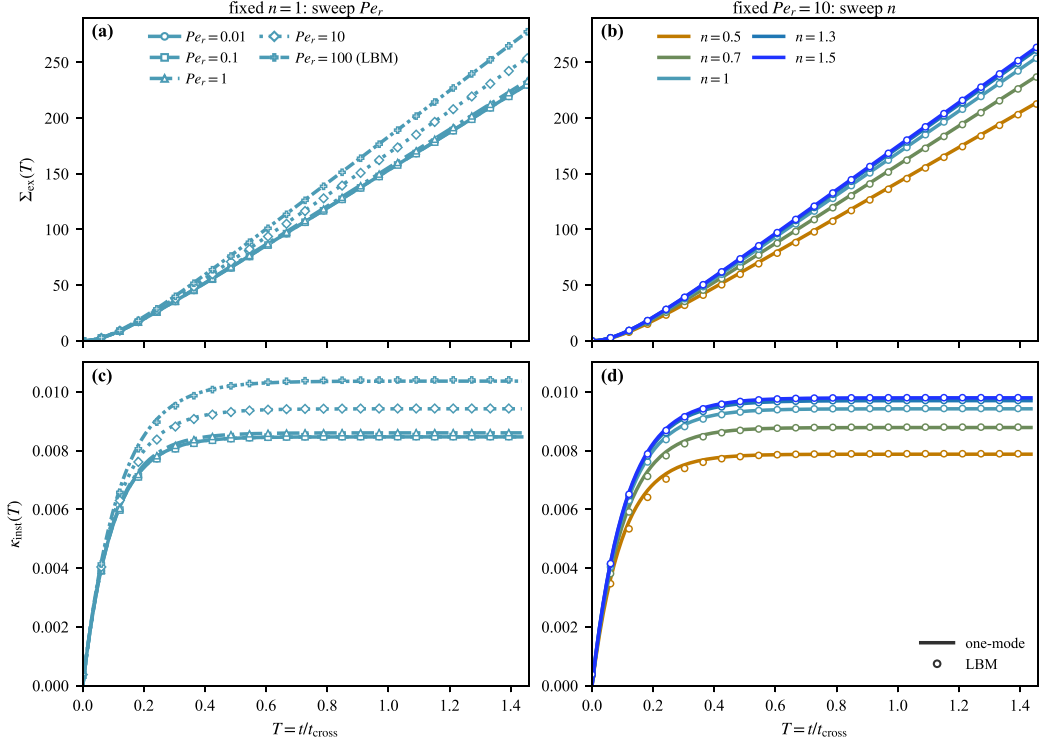


FIG. 6. Reduced Brownian-rod transport in plane power-law channels: representative raw transients. (a) Excess-variance transients for a rotational-Péclet-number sweep at fixed $n = 1$. (b) Excess-variance transients for a flow-index sweep at fixed $\text{Pe}_r = 10$. (c) Instantaneous-enhancement transients for the same Pe_r sweep at fixed $n = 1$. (d) Instantaneous-enhancement transients for the same n sweep at fixed $\text{Pe}_r = 10$. In all panels, the solid curves are the Green-Kubo-motivated one-mode reconstruction of the dominant shear-generated contribution using A_n and κ_{ref} ; the LBM data retain the full tensorial direct-diffusion correction.

C. Dual role of rheology in the reduced model: Kinematic anchoring and dynamic relaxation

The flow index n acts on two separated parts of the transient. The short-time contribution is purely kinematic. At $T = 0$, the fast local-orientation layer has already been eliminated and its stationary moments have been absorbed into fixed effective coefficients. For $T \ll 1$, the concentration has hardly diffused across streamlines, so the shear-generated excess variance is determined entirely by the spread of the release velocities. That spread is measured exactly by A_n . Across the present parameter range, A_n increases from 0.0804 at $n = 0.5$ to 0.0901 at $n = 1.5$, a 12.2% increase fixed analytically by Eq. (17). The origin of this monotonicity is geometric: smaller n produces a broader plug-like region with velocities clustered near the centerline value, whereas larger n distributes the velocity deficit over more of the section and broadens the sampled-velocity distribution. The early-time coefficient therefore records the release kinematics before cross-stream transport has had time to reorganize the packet.

Within the adopted reduced model, the long-time effect of n enters through the semianalytically precomputed orientation statistics and effective transport profiles. Increasing n spreads appreciable shear over a larger fraction of the channel, so the alignment field ceases to be confined to narrow near-wall layers and instead extends more broadly across Y . The consequence is a more pervasive suppression of effective transverse mixing once the rods begin to align with the local streamlines. This trend is reflected directly in the semianalytic long-time anchors: at every sampled Pe_r , κ_{ref} increases monotonically with n . For example, at $\text{Pe}_r = 10$ κ_{ref} rises from 7.88×10^{-3} at $n = 0.5$

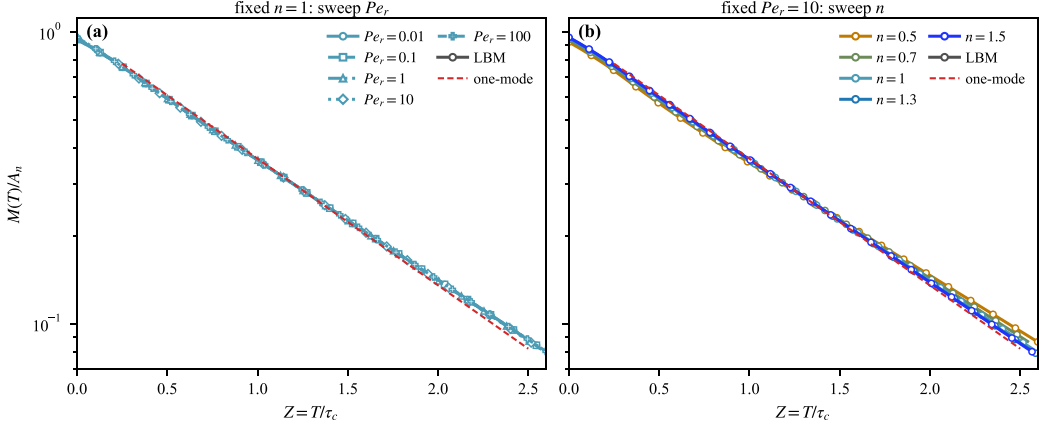


FIG. 7. Semilog diagnostic of the reduced-model memory-kernel decay in plane power-law channels. (a) Normalized kernel $M(T)/A_n$ for a rotational-Péclet-number sweep at fixed $n = 1$. (b) Normalized kernel $M(T)/A_n$ for a flow-index sweep at fixed $Pe_r = 10$. In both panels, $M(T)$ is extracted from the full tensorial LBM transient as $M = (2Pe^2)^{-1}d^2\Sigma_{ex}/dT^2 = d\kappa_{inst}/dT$ and plotted against the reduced time $Z = T/\tau_c$. The dashed red line shows the Green-Kubo-motivated shear reference $M(T)/A_n = e^{-Z}$, corresponding to the limit $\alpha = 1$. The extended linear regime shows that one effective slow mode dominates to the reported accuracy, while the fitted α values include the residual multimode and direct-diffusion effects.

to 9.80×10^{-3} at $n = 1.5$; at $Pe_r = 100$ it rises from 8.44×10^{-3} to 1.09×10^{-2} . The fitted LBM tails preserve the same ordering and remain close to these values. Within this reduced-model setting, carrier rheology shifts the long-time dispersion coefficient through the shear map and the associated effective transport profiles.

These two effects explain why the development time is only weakly sensitive to n even though both of its ingredients vary appreciably. In the power-law specialization of Eq. (35), T_d depends on the ratio κ_{ref}/A_n , so the increase of κ_{ref} with n is partly offset by the simultaneous increase of A_n . The representative sweep at $Pe_r = 10$ makes the cancellation explicit: raising n from 0.5 to 1.5 increases A_n by 12.2% and κ_{ref} by 24.4%, yet T_d increases by only 10.9%. Both quantities grow with n , but their ratio remains confined to a much narrower range. This competition is one of the distinctive features of reduced rod transport in power-law channels.

Figure 6 displays the raw transients associated with this split behavior. Comparing Fig. 6(a) with Fig. 6(c) isolates the fixed- $n = 1$ Pe_r sweep: the excess variance and instantaneous enhancement both shift to slower relaxation as Pe_r increases, reflecting stronger flow alignment and weaker cross-stream exchange. In the complementary fixed- $Pe_r = 10$ sweep, Figs. 6(b) and 6(d) show the rheological response: the short-time curvature of Σ_{ex} and the long-time level of κ_{inst} move together across n , because n changes both the sampled-velocity variance and the effective relaxation time. The shear-anchored one-mode approximation captures both sweeps to the reported accuracy, despite the small direct-diffusion correction retained in the LBM data.

D. Semilog memory-kernel diagnostic

A direct mechanism diagnostic follows from the reduced-model kernel extracted from the transient moments. Section II gives $M(T) = d\kappa_{inst}/dT = (2Pe^2)^{-1}d^2\Sigma_{ex}/dT^2$, while the shear-contribution approximation predicts $M(T)/A_n = e^{-Z}$ with $Z = T/\tau_c$. Figure 7 evaluates this prediction for the same representative sweeps used in Fig. 6. In both panels, the numerical kernel is extracted from the full tensorial LBM variance histories and then normalized by the shear anchor A_n . Thus, the numerical kernel also contains any relaxation of the direct axial-diffusion contribution,

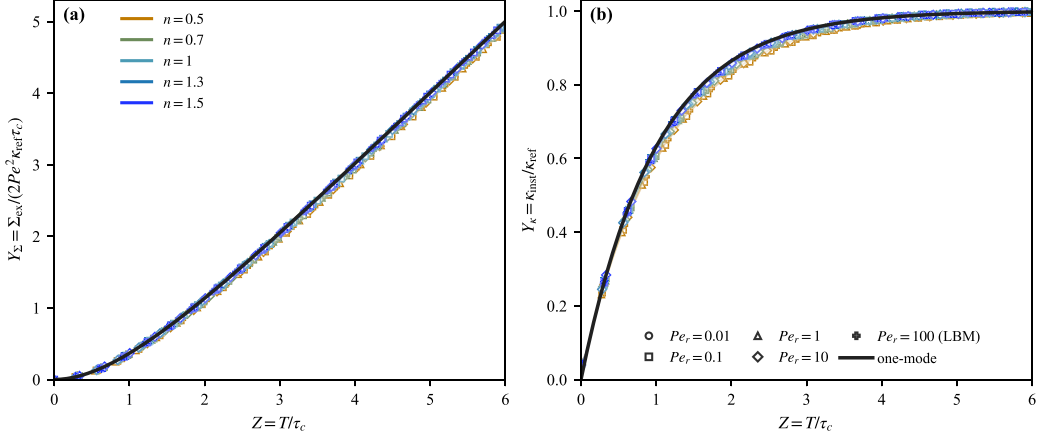


FIG. 8. Reduced collapse for the semianalytically closed power-law rod-transport cases. (a) Unshifted rod normalization $Y_{\Sigma}^{\text{rod}} = \Sigma_{\text{ex}} / (2\text{Pe}^2 \kappa_{\text{ref}} \tau_c)$ for all 25 (n, Pe_r) cases, plotted against $Z = T/\tau_c$. (b) Corresponding $Y_{\kappa}^{\text{rod}} = \kappa_{\text{inst}} / \kappa_{\text{ref}}$. The black curves are the scalar one-mode master laws used as the Green-Kubo-motivated shear reference; the plotted LBM moments retain the small tensorial direct-diffusion correction.

whereas the dashed exponential represents only the adopted shear closure. On semilog axes, a linear segment corresponds to $M(T)/A_n \sim \exp(-\alpha Z)$, with the one-mode limit given by $\alpha = 1$.

The extracted kernels form long semilog straight segments over the principal build-up interval. Least-squares fits over $0.25 \leq Z \leq 2.5$ give decay rates $0.888 \leq \alpha \leq 0.975$ across the 25 archived cases, with mean $R^2 = 0.99984$. The rheological trend is especially clear in Fig. 7(b): at fixed $\text{Pe}_r = 10$, α increases from 0.900 at $n = 0.5$ to 0.973 at $n = 1.5$. The more plug-like profile at $n = 0.5$ leaves a broader tail of faster transverse modes, so its kernel departs most visibly from the ideal one-mode slope $\alpha = 1$. As n increases, the profile projects more strongly onto the slowest mode and the semilog trend tightens. The slow-mode picture therefore appears directly in the decay of sampled-velocity memory. The reduced collapse in Fig. 8 is the moment-level consequence of the same kernel structure.

E. Reduced collapse in the one-mode variables

The same timescale organizes the transient moments compactly. To preserve the normalization used in the archived rod figures, define the unshifted plotting variables

$$Y_{\Sigma}^{\text{rod}} = \frac{\Sigma_{\text{ex}}}{2\text{Pe}^2 \kappa_{\text{ref}} \tau_c}, \quad Y_{\kappa}^{\text{rod}} = \frac{\kappa_{\text{inst}}}{\kappa_{\text{ref}}}, \quad Z = \frac{T}{\tau_c}. \quad (62)$$

These are not the exact shifted tensorial variables in Eq. (30): the full LBM observables in their numerators retain the small direct-diffusion contribution, while their denominators use the shear anchor κ_{ref} . The transients from all 25 archived reduced power-law rod-model LBM cases are compared with the scalar specialization of the one-mode master curves in Fig. 8. Figure 8(a) shows Y_{Σ}^{rod} , and Fig. 8(b) shows Y_{κ}^{rod} . In both observables, the case set follows the corresponding single-mode curve to the reported accuracy when plotted against Z . The collapse shows that, over the sampled (n, Pe_r) range, the transient moments are organized chiefly by the release-velocity variance and the semianalytic long-time shear anchor. Once those two quantities are fixed, the reduced dynamics are well described by a single relaxation time τ_c , with the remaining tensorial and multimode effects measured by the reported errors. Appendix C reports the corresponding κ_{ref} and late-time fitted coefficient κ_{num} for all archived cases.

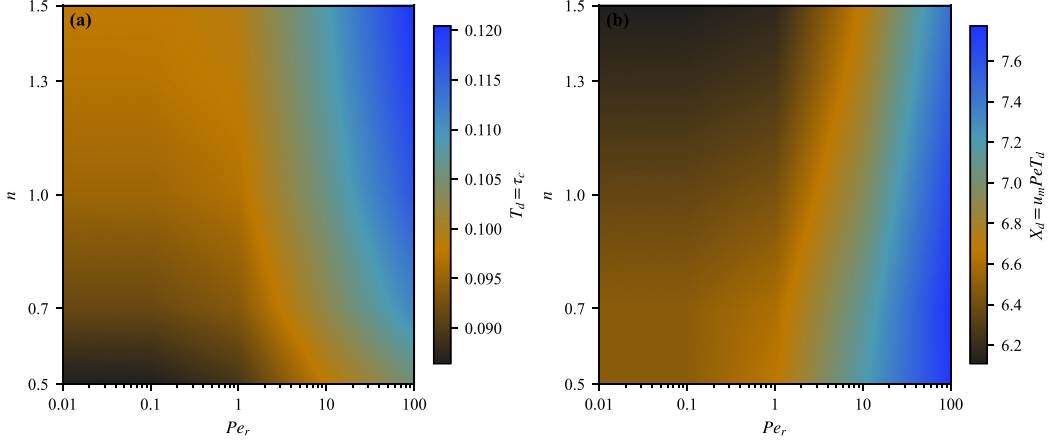


FIG. 9. Development maps for reduced Brownian-rod transport in plane power-law channels. Panel (a) shows the shear-closure e-folding time $T_d = \kappa_{\text{ref}}/A_n$ over the sampled (n, Pe_r) grid, and panel (b) shows the characteristic convective-distance estimate $X_d = u_m^{\text{rod}} \text{Pe} T_d$, based on the asymptotic rod speed. The tolerance-based estimates follow from Eq. (38); exact transient locations would use centroid displacements.

The accuracy of the collapse confirms that this coarse-graining loses very little information at the level of the transient moments. The mean normalized errors over the full application grid are 2.64×10^{-3} in excess variance and 2.50×10^{-2} in instantaneous enhancement, with maxima 6.18×10^{-3} and 2.75×10^{-2} . These values are already small enough for development-scale extraction, but the trend with n is equally revealing. The mean excess-variance error decreases monotonically from 5.75×10^{-3} at $n = 0.5$ to 8.50×10^{-4} at $n = 1.5$, while the mean instantaneous-enhancement error decreases from 2.72×10^{-2} to 2.40×10^{-2} . The same rheological ordering was already visible in the exact isotropic spectral benchmark, where increasing n shifts more modal weight into the fundamental transverse mode. The rod problem retains that ordering after the orientational degrees of freedom have been integrated out into the long-time anchor κ_{ref} . Larger n therefore strengthens effective slow-mode dominance in the reduced dynamics.

This behavior clarifies the reduction. Local shear and semianalytically closed orientation dynamics primarily shift the long-time transport anchor, while the transient shape in the reduced variables remains close to the same master law as in the simpler benchmarks. Over the sampled range, the transport inputs change the anchor location more strongly than the functional form of the approach to it.

F. Operational development scales

Having established the reduced collapse, we now translate its relaxation time into quantities that can be compared directly with the residence time and downstream extent of a finite channel. The main transport outputs of the reduction are

$$T_d = \tau_c = \frac{\kappa_{\text{ref}}}{A_n}, \quad X_d = u_m^{\text{rod}} \text{Pe} T_d. \quad (63)$$

These are the shear-contribution specializations of Eqs. (35) and (36). In the present application, the semianalytic rod-GTD calculation supplies both κ_{ref} and the asymptotic rod speed u_m^{rod} . Figure 9(a) maps the characteristic build-up time T_d over the (n, Pe_r) grid, and Fig. 9(b) maps the characteristic convective-distance estimate X_d . At $T = T_d$, the modeled shear-generated enhancement has reached the fraction $1 - e^{-1}$ of its plateau. Equation (38) gives the analogous estimates for any selected proximity to that plateau.

The dominant trend is the increase with Pe_r , which is visible in the color gradient of Fig. 9(a). After averaging over n , the cases $\text{Pe}_r = 0.01$ and 0.1 both give $T_d \approx 9.36 \times 10^{-2}$, with $X_d \approx 6.32$

and 6.33, respectively, while $\text{Pe}_r = 100$ gives $T_d \approx 1.15 \times 10^{-1}$ and $X_d \approx 7.64$. Across the full simulation grid the ranges are

$$0.0864 \leq T_d \leq 0.1204, \quad 6.11 \leq X_d \leq 7.77. \quad (64)$$

The bounds in Eq. (64) translate the microstructural argument of Sec. IV B into a finite-channel result because stronger alignment delays streamline exchange and carries the approach to the Taylor plateau farther downstream.

The dependence on n is weaker and separates the temporal and spatial constraints. At fixed Pe_r , T_d increases mildly with n because the growth of κ_{ref} is partly offset by the simultaneous growth of A_n . The streamwise estimate contains the additional, case-dependent factor u_m^{rod} from Eq. (C7). This rod speed is weighted by the invariant density D_y^{-1} and is distinct from the fluid cross-sectional mean $\bar{u}_f$ in Eq. (16); the reported X_d values already use the semianalytic rod speed. The tilted contours in Fig. 9(b) make this separation visible. Over the representative $\text{Pe}_r = 10$ sweep, T_d increases from 9.80×10^{-2} to 1.09×10^{-1} , while X_d decreases from 7.19 to 6.69. The residence time and characteristic downstream estimate therefore encode distinct operational constraints in a non-Newtonian channel.

For a dimensional channel length L_{ch} measured downstream from the release plane, the available dimensionless distance is $\Lambda = L_{\text{ch}}/a$. Comparison of Λ with X_ε gives a characteristic residence-length criterion based on the asymptotic rod speed. When $\Lambda < X_\varepsilon$, the estimate indicates that the packet retains a larger fraction of sampled-velocity memory and that a long-time Taylor-Aris fit may carry a finite-development bias, so the all-time closure supplies the evolving enhancement over the available observation window [1,2]. Because a nonequilibrium injection can have a time-dependent mean speed, the exact threshold location would instead be the transient centroid displacement $X_c(T_\varepsilon) - X_c(0)$. The ratio Λ/X_ε is therefore a design estimate rather than an exact position criterion.

G. Dimensional Brownian-rod illustration

To show how the reduced measures translate into a physical residence time and observation length, we use the tobacco-mosaic-virus-scale example discussed by Kumar *et al.* [45]. Its Newtonian carrier corresponds to the $n = 1$ member of the same power-law family and provides a dimensional anchor based on a complete set of reported particle and channel parameters. The rod has major- and minor-axis scales $a_p = 300$ nm and $b_p = 20$ nm, giving $p = a_p/b_p = 15$, and it is transported in water through a plane channel of half-width $a = 2$ μm with centerline speed $U_c = 1$ mm s $^{-1}$. The reported values $\text{Pe} \simeq 750$ and $\text{Pe}_r \simeq 10$ imply $\bar{D} \simeq 2.67$ μm^2 s $^{-1}$ and $D_\theta \simeq 50.0$ s $^{-1}$ through Eqs. (2) and (54). These parameters produce the time-scale hierarchy

$$t_{\text{cross}} = \frac{a^2}{\bar{D}} = 1.50 \text{ s}, \quad \frac{\text{Pe}}{\text{Pe}_r} = 75, \quad t_\theta = \frac{\text{Pe}_r}{4\text{Pe}} t_{\text{cross}} = 5.00 \text{ ms}, \quad (65)$$

which places local orientational relaxation well before the streamwise dispersion development considered by the closure.

For this $n = 1$ case, the semianalytic rod-GTD preprocessing at $p = 15$, $\text{Pe}_r = 10$, and $\text{Pe} = 750$ gives $u_m^{\text{rod}} = 0.659$ and $\kappa_{\text{ref}} = 9.13 \times 10^{-3}$. The corresponding shear-generated long-time coefficient exceeds the equal- $\bar{D}$ spherical value $\kappa_s = 8/945$ by 7.8 %, reflecting the weaker cross-stream exchange produced by streamwise rod alignment. Restoring dimensions through Eqs. (37) and (38) gives

$$T_d = 0.1027, \quad X_d = 50.7, \quad t_d = 0.154 \text{ s}, \quad x_d = 0.101 \text{ mm}. \quad (66)$$

$$\begin{aligned} t_{90} &= t_d \ln 10 = 0.355 \text{ s}, \quad x_{90} = x_d \ln 10 = 0.234 \text{ mm}, \quad \Delta D_{x,\infty}^{(\text{shear})} \equiv \bar{D} \text{Pe}^2 \kappa_{\text{ref}} = \frac{U_c^2 a^2}{\bar{D}} \kappa_{\text{ref}} \\ &= 1.37 \times 10^4 \mu\text{m}^2 \text{ s}^{-1}. \end{aligned} \quad (67)$$

Here $\Delta D_{x,\infty}^{(\text{shear})}$ is the dimensional shear-induced increment associated with κ_{ref} ; the full tensorial coefficient additionally retains the direct axial-diffusion correction described in Sec. II A. At $T = T_d$, the modeled shear-generated enhancement equals $(1 - e^{-1})\kappa_{\text{ref}} \simeq 0.632\kappa_{\text{ref}}$, while x_{90} is the analogous estimate for a ten-percent remaining gap. A measurement window ending near 0.10 mm after release therefore samples a substantial transient contribution, and 0.234 mm is the estimated downstream length for the stated 90-percent criterion under the one-mode closure. Every non-Newtonian entry in Fig. 9 can be converted in the same way by inserting u_m^{rod} , κ_{ref} , and the chosen physical scales. Because these distances use the asymptotic rod speed, the exact locations would be $x_c(t_d) - x_c(0)$ and $x_c(t_{90}) - x_c(0)$; the reported values are characteristic convective-distance estimates.

V. DISCUSSION

The present reduction applies to bounded steady shear provided that a long-time Taylor coefficient exists and that the decay of the relevant memory is effectively dominated by one slow transverse mode. For an invariant scalar release, the stationary Green-Kubo identity and the general one-pole formulas use the three quantities κ_0 , M_0 , and κ_L ; the unit-reference scalar benchmarks reduce to $\kappa_0 = 0$ and $M_0 = A[u]$. For the uniform but nonequilibrium rod injection, A_n and κ_{ref} instead define a Green-Kubo-motivated approximation to the dominant shear-generated contribution. For the isotropic reference and the Newtonian rod problem, the long-time anchor is supplied independently. For the power-law application, the reduced model is defined by the imposed carrier profile $u(Y)$, the shear map $q(Y)$, and semianalytically supplied transport inputs $D_{ij}(Y)$, $v_d(Y)$, u_m^{rod} , and κ_{ref} . The LBM runs then resolve the full tensorial transient moment evolution of the corresponding coarse-grained ADE. The close agreement reported here shows that the omitted direct-diffusion correction and residual multimode content remain small over the sampled range.

To interpret the flow-index trends within this modeling scope, note that the comparisons across n retain the same reference particle-scale diffusivities at fixed Pe_r and particle geometry. Rheology therefore reaches the dispersion process through the two coupled pathways resolved by the model. Changes in $u(Y; n)$ set the release-velocity variance and the short-time anchor, while changes in $q(Y; n)$ reshape the orientation distribution and pass into the effective tensor, migration drift, and long-time anchor. The resulting $D_{ij}(Y; n)$ are consequently flow-index dependent even though the particle-frame mobility parameters remain fixed. The reported trends make the background-flow contribution explicit and identify the additional particle-scale rheological coupling required for a material-specific prediction.

A direct insertion of the apparent power-law viscosity into a local Stokes–Einstein expression would leave that particle-scale coupling ill posed near the channel centerline. Denoting the apparent viscosity, local shear rate, and hypothetical local diffusivity by η_{app} , $\dot{\gamma}$, and D_{local} , respectively, the power-law relations $\eta_{\text{app}} \propto |\dot{\gamma}|^{n-1}$ and $|\dot{\gamma}(Y)| \propto |Y|^{1/n}$ would give $D_{\text{local}}(Y) \propto |Y|^{(1-n)/n}$. This quantity approaches zero for $n < 1$ and diverges for $n > 1$ as Y approaches zero. These limits reflect the zero-shear singularity of the unregularized power-law constitutive relation. Recent equilibrium and linear-response studies of rods in polymer melts, gels, and quiescent complex liquids provide useful information on medium-dependent mobility [67–69]; extending those results to steady generalized-Newtonian shear would still require the coupled translational, rotational, and disturbance-torque response. A material-specific Brownian mobility would therefore need to draw on a regularized constitutive model and particle-scale microrheology, while systems displaying memory-dependent or anomalous transport would also require a stochastic description beyond the Fickian ADE considered here [70,71].

Within the one-mode framework, these particle-scale uncertainties affect how material-specific anchors would be obtained, whereas the subsequent conversion of supplied anchors into development scales remains unchanged. Operationally, once A_n , κ_{ref} , and u_m^{rod} are available from analysis, simulation, or experiment, the rod model yields characteristic residence times and convective-

distance estimates. For a nonequilibrium injection, an exact downstream threshold location requires the transient centroid displacement rather than multiplication by a fixed asymptotic speed.

The present demonstration covers uniform release in steady plane-channel shear on the sampled range $Pe = 100$, $p = 1000$, $n \in \{0.5, 0.7, 1.0, 1.3, 1.5\}$, and $Pe_r \in \{0.01, 0.1, 1, 10, 100\}$. The local-orientational-equilibrium interpretation applies in the scale-separated regime $Pe/Pe_r \gg 1$. The endpoint $Pe_r = Pe$ provides a formal numerical stress test of the closed model, consistent with the extreme comparison discussed by Kumar *et al.* [45] and Khair [46]. In the power-law rod application, the transient solver evolves the reduced ADE with precomputed $D_{ij}(Y)$ and $v_d(Y)$, while semianalytic preprocessing supplies the stationary local orientation statistics. The flow-selected, generally anisotropic conditional distribution $\Psi_n(\theta; Y)$ thereby supplies the angular state at the resolved time origin.

An injected population with a preparation-induced angular density $f_0(\theta; Y) \neq \Psi_n(\theta; Y)$ would generate a finite initial relaxation layer in the orientation-resolved joint density $P(X, Y, \theta, T)$. A streamwise bias would initially sample D_{yy} closer to $D_\perp$, suppress transverse exchange, and prolong sampled-velocity memory. A cross-stream bias would sample D_{yy} closer to $D_\parallel$ and promote transverse exchange. Under $Pe/Pe_r \gg 1$, this dependence decays over $T = O(Pe_r/Pe)$ and remains inside the omitted entrance layer, leaving the resolved reduced transient and the unique long-time Taylor coefficient unchanged. When the scale separation weakens, or when observations focus on the entrance layer, the early transient retains its dependence on f_0 and requires the full configuration-space equation. Such dynamics lie beyond the present local-equilibrium closure.

This angular-preparation dependence is distinct from the spatial initial-condition dependence associated with a nonuniform transverse release. For such a release, $A_{\rho_0}[u]$ supplies the exact coefficient of the shear-generated short-time contribution, while the two-time covariance in Eq. (32) carries the remaining spatial initial-condition memory. The corresponding point-source transient therefore approaches the same Taylor plateau through a finite multimode relaxation.

VI. CONCLUSION

Transient Taylor dispersion in a bounded channel has the general initial value $\kappa_0 = \kappa_{\text{inst}}(0)$, initial kernel amplitude M_0 , and long-time enhancement κ_L . For an invariant release, the exact memory representation has Green-Kubo form, and the adopted single-exponential approximation gives $\tau_c = (\kappa_L - \kappa_0)/M_0$ and the shifted laws (28) and (30). In the scalar unit-reference case, $\kappa_0 = 0$ and $M_0 = A[u] = \overline{(u - \bar{u})^2}$, recovering the familiar two-anchor formulas. For uniform release, A_n in Eq. (17) remains the exact coefficient of the shear-generated $Pe^2 T^2$ contribution in plane power-law Poiseuille flow.

For a general transverse release, the exact coefficient becomes the initial sampled-velocity variance $A_{\rho_0}[u]$, while a localized source also carries a nonstationary two-time memory through which a point release, despite having zero ballistic coefficient, develops shear-induced spreading at order T^3 or higher and converges to the common long-time Taylor coefficient through the transverse modes.

The isotropic power-law benchmarks show directly through their cosine and Bessel spectra why a one-mode reduction remains plausible across transverse geometries, since the faster modes lose their influence first and the slowest velocity-coupled contribution controls the late build-up of axial dispersion. The circular-tube comparison retains errors below 3 % across the sampled flow indices and confirms that geometry changes the modal partition while leaving the scalar two-anchor construction intact. Brownian rods in Newtonian Poiseuille flow show that a Green-Kubo-motivated shear-anchored approximation remains accurate when orientational dynamics and anisotropic diffusion are present, even though uniform rod injection is generally not the invariant transverse measure.

The same approximation applies numerically to the reduced Brownian-rod transport problem in plane power-law channels over the sampled parameter range. Its physical local-equilibrium interpretation follows from $Pe/Pe_r \gg 1$, and the $Pe_r = Pe$ endpoint evaluates the closed-model numerics. The long-time shear anchor is supplied semianalytically as κ_{ref} within the reduced

rod-GTD preprocessing, while entropic-LBM runs retain the full spatially varying tensor and resolve the transient evolution of the associated coarse-grained ADE. Across the 25 archived cases, the fitted late-time coefficients remain close to the semianalytic anchors, with mean relative difference 1.41×10^{-3} and maximum 3.71×10^{-3} . The resulting build-up remains effectively one-timescale in the unshifted rod plotting variables $Z = T/\tau_c$, Y_Σ^{rod} , and Y_κ^{rod} , with the reported errors containing the small direct-diffusion and residual multimode effects. The finite-channel consequences comprise the characteristic time $T_d = \tau_c$ and the convective-distance estimate $X_d = u_m^{\text{rod}} \text{Pe} T_d$, together with the tolerance-based estimates T_ε and X_ε . Exact transient threshold locations would instead be obtained from centroid displacements. These quantities increase strongly with Pe_r , reflecting the slower cross-stream homogenization associated with stronger flow-induced rod alignment, while their dependence on the flow index is split between the exact kinematic effect of A_n and the long-time effect contained in κ_{ref} . The flow-index dependence therefore represents the combined influence of the imposed velocity profile and shear-rate landscape under fixed reference Brownian mobility parameters, while material-specific corrections to the mobility and particle-scale torque remain an open extension of the present reduced description.

More broadly, the stationary scalar construction carries a two-anchor one-timescale closure into new geometries once the short-time sampled-velocity variance and long-time Taylor state are known. A complete tensorial closure additionally requires κ_0 and the corresponding kernel amplitude M_0 . When only the dominant shear-generated contribution is retained, as in the present rod reconstruction, the same functional form remains a useful Green-Kubo-motivated approximation whose accuracy must be checked against the full transport model. Its dimensional and tolerance-based forms provide characteristic residence-time and convective-distance estimates for a selected degree of approach to the shear-dispersion plateau.

DATA AVAILABILITY

The data that support the findings of this article are openly available [85].

APPENDIX A: SPECTRAL CHECK OF INITIAL-DISTRIBUTION EFFECTS

To examine the initial-distribution dependence identified in Sec. II E, we use the isotropic Newtonian benchmark $u(Y) = 1 - Y^2$ and expand the transverse concentration in the orthonormal Neumann eigenfunctions

$$\phi_0(Y) = \frac{1}{\sqrt{2}}, \quad \lambda_0 = 0, \quad \phi_j(Y) = \cos\left[\frac{j\pi(Y+1)}{2}\right], \quad \lambda_j = \left(\frac{j\pi}{2}\right)^2, \quad j \geq 1. \quad (\text{A1})$$

For an initial transverse distribution ρ_0 , its modal projection and the velocity-coupling matrix are

$$b_j = \int_{-1}^1 \phi_j(Y) \rho_0(Y) dY, \quad U_{ij} = \int_{-1}^1 \phi_i(Y) u(Y) \phi_j(Y) dY. \quad (\text{A2})$$

Let $\mathbf{b} = (b_0, b_1, \dots)^\top$ and $\mathbf{U} = (U_{ij})$ denote the release-projection vector and velocity-coupling matrix, respectively. Writing $\mathbf{m}^{(q)}$ for the modal streamwise moments of order $q = 0, 1, 2$, a finite modal truncation is propagated by

$$\frac{d}{dT} \begin{bmatrix} \mathbf{m}^{(0)} \\ \mathbf{m}^{(1)} \\ \mathbf{m}^{(2)} \end{bmatrix} = \begin{bmatrix} -\Lambda & 0 & 0 \\ \text{Pe}\mathbf{U} & -\Lambda & 0 \\ 0 & 2\text{Pe}\mathbf{U} & -\Lambda \end{bmatrix} \begin{bmatrix} \mathbf{m}^{(0)} \\ \mathbf{m}^{(1)} \\ \mathbf{m}^{(2)} \end{bmatrix}, \quad \Lambda = \text{diag}(\lambda_0, \lambda_1, \dots), \quad (\text{A3})$$

with $\mathbf{m}^{(0)}(0) = \mathbf{b}$ and $\mathbf{m}^{(1)}(0) = \mathbf{m}^{(2)}(0) = 0$. Because bare axial diffusion is added separately as $2T$, this system propagates the shear-induced contribution entering κ_{inst} and includes a point release directly through $b_j = \phi_j(Y_0)$, following the moment treatment of initial point concentrations by Vedel *et al.* [80]. The converged spectral calculation shown in Fig. 10 recovers the analytic plane-Poiseuille

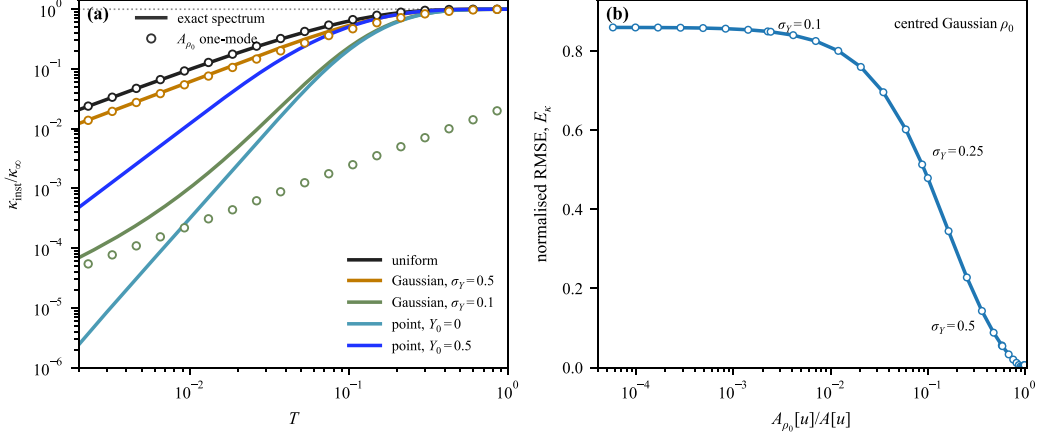


FIG. 10. Effect of the initial transverse distribution in isotropic plane Poiseuille flow. (a) Exact spectral instantaneous enhancement for uniform release, two centered Gaussian releases, a centerline point release and an off-center point release. The three sets of open symbols give the one-mode expression for the uniform and two Gaussian releases, for which $A_{\rho_0}[u]$ is positive, while no one-mode symbols are drawn for either point release because $A_{\rho_0}[u] = 0$; the dotted horizontal line is the common Taylor plateau. (b) Normalized RMSE of that one-mode expression over $0 \leq T \leq 1$ for centered Gaussian releases, plotted against the fraction of the uniform sampled-velocity variance present initially.

plateau $\kappa_\infty = 8/945$, with each Gaussian release obtained by normalizing $\exp[-Y^2/(2\sigma_Y^2)]$ over the channel.

Both point releases in Fig. 10(a) approach the common Taylor plateau in finite time, with the centerline and off-center releases first reaching $0.9\kappa_\infty$ at $T = 0.358$ and $T = 0.263$, respectively, and their distinct initial powers following Eqs. (33) and (34). For the centered Gaussian family, Fig. 10(b) quantifies the one-mode error through

$$E_\kappa = \left\{ \frac{1}{T_{\text{err}}} \int_0^{T_{\text{err}}} \left[\frac{\kappa_{\text{inst}}^{\text{sp}}(T) - \kappa_{\text{inst}}^{\text{1m}}(T)}{\kappa_\infty} \right]^2 dT \right\}^{1/2}, \quad T_{\text{err}} = 1, \quad (\text{A4})$$

where superscripts sp and 1m denote the spectral and one-mode results. The errors increase from 5.58×10^{-2} at $\sigma_Y = 0.5$ to 5.13×10^{-1} at $\sigma_Y = 0.25$ and 8.50×10^{-1} at $\sigma_Y = 0.1$, compared with 7.46×10^{-3} for uniform release, so broadening the centered Gaussian release toward the invariant distribution progressively recovers the one-mode transient, while narrowing it exposes the finite build-up generated by transverse diffusion. That history remains present in the modal system (A3), including its point-source limit.

APPENDIX B: ENTROPIC LATTICE BOLTZMANN SOLVER FOR THE ADE

The direct numerical benchmark data reported in Figs. 3 and 5–8 are obtained with an entropic LBM solver for the ADE. On a lattice with spatial spacing Δx and time step Δt , the scalar concentration is represented by standard two-dimensional nine-velocity (D2Q9) populations $g_i(\mathbf{x}, t)$, $i = 0, \dots, 8$, associated with

$$\mathbf{c}_0 = (0, 0), \quad \mathbf{c}_{1,\dots,4} = c(1, 0), c(0, 1), c(-1, 0), c(0, -1), \quad (\text{B1})$$

$$\mathbf{c}_{5,\dots,8} = c(1, 1), c(-1, 1), c(-1, -1), c(1, -1), \quad c = \frac{\Delta x}{\Delta t}. \quad (\text{B2})$$

The corresponding weights are

$$W_0 = \frac{4}{9}, \quad W_{1,\dots,4} = \frac{1}{9}, \quad W_{5,\dots,8} = \frac{1}{36}, \quad c_s^2 = \frac{c^2}{3}. \quad (\text{B3})$$

Here c_s is the lattice sound speed. The populations are updated by the collision-streaming rule

$$g_i(\mathbf{x} + \mathbf{c}_i \Delta t, t + \Delta t) = g_i(\mathbf{x}, t) + \Omega_i(\mathbf{x}, t), \quad i = 0, \dots, 8, \quad (\text{B4})$$

and the scalar concentration is the zeroth moment

$$C(\mathbf{x}, t) = \sum_{i=0}^8 g_i(\mathbf{x}, t). \quad (\text{B5})$$

The imposed advecting velocity $\mathbf{u}(\mathbf{x}, t)$ enters through the second-order low-Mach ADE equilibrium

$$g_i^{eq} = CW_i \left[1 + \frac{\mathbf{c}_i \cdot \mathbf{u}}{c_s^2} + \frac{(\mathbf{c}_i \cdot \mathbf{u})^2}{2c_s^4} - \frac{\mathbf{u} \cdot \mathbf{u}}{2c_s^2} \right], \quad (\text{B6})$$

which satisfies $\sum_i g_i^{eq} = C$ and $\sum_i \mathbf{c}_i g_i^{eq} = C\mathbf{u}$. The nonequilibrium first-order flux is

$$\mathbf{j} = \sum_{i=0}^8 \mathbf{c}_i (g_i - g_i^{eq}). \quad (\text{B7})$$

Following the ADE-adapted Karlin-Bösch-Chikatamarla (KBC) construction, the nonequilibrium population is decomposed into a flux part and a higher-order kinetic part,

$$g_i - g_i^{eq} = \Delta s_i + \Delta h_i, \quad \Delta s_i = \mathcal{P}_i(\mathbf{j}), \quad \Delta h_i = g_i - g_i^{eq} - \Delta s_i, \quad (\text{B8})$$

with the D2Q9 lifting operator

$$\mathcal{P}_i(\mathbf{a}) = \frac{W_i}{c_s^2} \mathbf{c}_i \cdot \mathbf{a}, \quad \mathbf{a} \in \mathbb{R}^2. \quad (\text{B9})$$

This construction gives

$$\sum_i \Delta s_i = \sum_i \Delta h_i = 0, \quad \sum_i \mathbf{c}_i \Delta s_i = \mathbf{j}, \quad \sum_i \mathbf{c}_i \Delta h_i = \mathbf{0}. \quad (\text{B10})$$

The collision operator relaxes the physical flux tensorially and damps the remaining higher-order component by a scalar entropic amplitude:

$$\Omega_i = -\mathcal{P}_i(\mathbf{S}\mathbf{j}) - \lambda \Delta h_i. \quad (\text{B11})$$

Here $\mathbf{S} \in \mathbb{R}^{2 \times 2}$ is the flux-relaxation matrix. For a prescribed symmetric positive-definite diffusivity tensor $\mathbf{D}$, the ADE closure is

$$\mathbf{D} = c_s^2 \Delta t \left(\mathbf{S}^{-1} - \frac{1}{2} \mathbf{I} \right), \quad \mathbf{S} = \left(\frac{1}{2} \mathbf{I} + \frac{\mathbf{D}}{c_s^2 \Delta t} \right)^{-1}. \quad (\text{B12})$$

Here $\mathbf{I}$ is the 2×2 identity matrix. Thus, the principal axes and eigenvalues of the physical diffusion tensor are imposed directly on the first-order nonequilibrium flux. In the isotropic benchmark, $\mathbf{D} = D_0 \mathbf{I}$ and $\mathbf{S} = s \mathbf{I}$, where

$$s = \left(\frac{1}{2} + \frac{D_0}{c_s^2 \Delta t} \right)^{-1}. \quad (\text{B13})$$

The scalar λ controls only the higher-order ghost sector. In the entropic implementation it is chosen along the post-flux relaxation line by the KBC entropy condition

$$\frac{d}{d\lambda} \sum_i (g_i + \Omega_i^{(s)} - \lambda \Delta h_i) \ln \left(\frac{g_i + \Omega_i^{(s)} - \lambda \Delta h_i}{W_i} \right) = 0, \quad \Omega_i^{(s)} = -\mathcal{P}_i(\mathbf{S}\mathbf{j}), \quad (\text{B14})$$

so that the leading-order diffusion tensor in Eq. (B12) is unaffected by the ghost relaxation.

No-flux channel walls are imposed with halfway bounce-back boundary conditions. The numerically evolved concentration field is post-processed to obtain $\Sigma(T)$, $\Sigma_{\text{ex}}(T) = \Sigma(T) - \Sigma(0) - 2T$, and $\kappa_{\text{inst}}(T)$ from Eq. (6). Power-law-rod-specific preprocessing, archived run parameters, and the distinction between the semianalytic anchor and the fitted late-time coefficient are described separately in Appendix C.

APPENDIX C: SEMIANALYTIC REDUCED ROD-TRANSPORT INPUTS AND TRANSIENT ADE VALIDATION IN POWER-LAW CHANNELS

1. Semianalytic long-time transport anchor

For each cross-stream location Y , the local shear strength $q(Y)$ defines a steady orientation problem on $\theta \in [0, \pi)$. In the archived power-law-channel calculations, this problem is solved semianalytically by a truncated Fourier collocation method. The resulting stationary orientation moments are used to assemble the effective transport profiles $D_{xx}(Y)$, $D_{xy}(Y)$, and $D_{yy}(Y) = D_y(Y)$, while the migration drift is obtained from the wall-normal variation of the lateral diffusivity as $v_d(Y) = \partial_Y D_y(Y)$. The same preprocessing then yields the asymptotic rod speed u_m^{rod} and the long-time shear-dispersion anchor. In this Appendix, that anchor is denoted by κ_{ref} . Solving the angular problem before the transient implements the rapid local-orientational-equilibration closure. The LBM receives the resulting fields and coefficients without angular release data.

2. Coarse-grained transient ADE solved by LBM

The transient LBM solver advances a coarse-grained concentration field $C(X, Y, T)$ with the semianalytically supplied transport inputs held fixed in Y :

$$\partial_T C + \nabla \cdot [\mathbf{V}(Y) C] = \nabla \cdot (\mathbf{D}(Y) \nabla C), \quad \mathbf{V}(Y) = (\text{Pe } u(Y), -v_d(Y)), \quad (\text{C1})$$

$$\mathbf{D}(Y) = \begin{pmatrix} D_{xx}(Y) & D_{xy}(Y) \\ D_{xy}(Y) & D_{yy}(Y) \end{pmatrix}. \quad (\text{C2})$$

For an X -independent stationary state $C = \rho_\infty(Y)$, the no-flux condition associated with Eq. (C1) is

$$D_y(Y) \frac{d\rho_\infty}{dY} + v_d(Y) \rho_\infty(Y) = 0. \quad (\text{C3})$$

Since $v_d = dD_y/dY$, Eq. (C3) is $d(D_y \rho_\infty)/dY = 0$. Normalization therefore gives

$$\rho_\infty(Y) = \frac{D_y^{-1}(Y)}{\int_{-1}^1 D_y^{-1}(\eta) d\eta}. \quad (\text{C4})$$

Thus, the uniform transverse release used in the calculations is invariant only when D_y is constant. In the general rod case it is a nonequilibrium injection that relaxes toward ρ_∞ . Here $u(Y)$ is the imposed plane power-law Poiseuille profile, and the tensorial diffusivities and migration drift are the effective transport inputs supplied by the semianalytic preprocessing described above. The release is uniform in Y and Gaussian in X , and no-flux conditions are imposed at $Y = \pm 1$. The transient state consists solely of the reduced concentration field $C(X, Y, T)$. Angular coordinates and angular initial data have already been eliminated, with orientational physics carried by the stationary precomputed coefficients $D_{ij}(Y)$ and $v_d(Y)$.

3. Discretization and archived run parameters

The archived calculations reported in Sec. IV employ the D2Q9 entropic lattice Boltzmann discretization of the anisotropic ADE described in Appendix B. Tensorial diffusion is imposed through the flux-relaxation matrix $\mathbf{S}$ in Eq. (B12), whereas the relaxation amplitude of the higher-order kinetic component is determined by the entropy condition given there. The computational domain consists of a 4000×33 lattice with cell-centered Y nodes and no-flux conditions at the upper and lower walls. All 25 application cases are run at $\text{Pe} = 100$ and $p = 1000$. In lattice units, $u_{\max}^{\text{lbn}} = 0.02$, $\bar{D}_{\text{lbn}} = 0.0033$, and the initial Gaussian has width parameter $\sigma_x = 24$ and is centered at $x = 500$; the release remains uniform across Y . Each run is advanced for 120000 steps, corresponding to a final nondimensional time $T_f = 1.4545$, and diagnostics are recorded every 500 steps. The late-time slope used for coefficient extraction is fitted over the second half of the recorded window, beginning at $T = 0.7273$. These settings are held fixed over the full 5×5 parameter grid in (n, Pe_r) .

4. Post-processing and consistency check

From the evolving concentration field we compute

$$\Sigma(T) = \text{Var}[X(T)], \quad \Sigma_{\text{ex}}(T) = \Sigma(T) - \Sigma(0) - 2T, \quad (\text{C5})$$

$$\kappa_{\text{inst}}(T) = \frac{1}{2\text{Pe}^2} \left(\frac{d\Sigma}{dT} - 2 \right). \quad (\text{C6})$$

Here the subtraction removes the unit reference-diffusion rate used throughout the framework. Because the ADE in Eq. (C1) retains the spatially varying $D_{xx}(Y)$, the measured κ_{inst} and its fitted tail contain the corresponding direct axial-diffusion correction relative to $\bar{D}$, scaled by Pe^{-2} , together with the shear-generated contribution. In particular, the uniform release generally has the nonzero initial value in Eq. (10). The semianalytic κ_{ref} below supplies only the shear-dispersion anchor, and the small Pe^{-2} correction remains part of the numerical consistency comparison rather than a separate anchor of the simplified rod curve. With $D_y(Y) = D_{yy}(Y)$ and $\bar{f} = \frac{1}{2} \int_{-1}^1 f(Y) dY$, the semianalytic long-time coefficient stored in the archived summaries as κ_{ref} is evaluated from the rod-GTD closure as

$$u_m^{\text{rod}} = \frac{\overline{D_y^{-1} u(Y)}}{\overline{D_y^{-1}}}, \quad (\text{C7})$$

$$G(Y) = \int_{-1}^Y d\eta \int_{-1}^{\eta} D_y^{-1}(\xi) [u(\xi) - u_m^{\text{rod}}] d\xi, \quad (\text{C8})$$

$$\kappa_{\text{ref}} = - \frac{\overline{D_y^{-1} G(Y) [u(Y) - u_m^{\text{rod}}]}}{\overline{D_y^{-1}}}. \quad (\text{C9})$$

The weighting in Eq. (C7) is precisely the invariant density (C4); consequently u_m^{rod} is the asymptotic rod speed and is generally different from the fluid mean $\bar{u}_f$ in Eq. (16). These are the quantities evaluated by the semianalytic preprocessing described earlier in this Appendix. An independent numerical late-time value κ_{num} is extracted from the tail slope of $\Sigma(T)$ over the fit window described above, equivalently as

$$\kappa_{\text{num}} = \frac{1}{2\text{Pe}^2} \left[\left(\frac{d\Sigma}{dT} \right)_{\text{tail}} - 2 \right]. \quad (\text{C10})$$

The one-mode reconstructions reported in Sec. IV use κ_{ref} as the long-time anchor; κ_{num} is recorded only as a consistency check on the reduced ADE-LBM transient. Table II lists both quantities for the 25 archived reduced power-law rod-model cases together with the relative difference $|\kappa_{\text{num}} - \kappa_{\text{ref}}|/\kappa_{\text{ref}}$.

TABLE II. Semianalytic long-time coefficients κ_{ref} and late-time fitted coefficients κ_{num} for the 25 archived reduced power-law rod-model cases in Sec. IV. The final column reports the relative difference $|\kappa_{\text{num}} - \kappa_{\text{ref}}|/\kappa_{\text{ref}}$.

n	Pe_r	κ_{ref}	κ_{num}	$ \kappa_{\text{num}} - \kappa_{\text{ref}} /\kappa_{\text{ref}}$
0.5	0.01	0.00694451	0.00695709	1.812×10^{-3}
0.5	0.1	0.00694791	0.00696034	1.790×10^{-3}
0.5	1	0.00717186	0.00718418	1.717×10^{-3}
0.5	10	0.00787619	0.00789081	1.856×10^{-3}
0.5	100	0.00844200	0.00845651	1.719×10^{-3}
0.7	0.01	0.00784217	0.00785182	1.231×10^{-3}
0.7	0.1	0.00784460	0.00785414	1.216×10^{-3}
0.7	1	0.00802225	0.00803166	1.174×10^{-3}
0.7	10	0.00878419	0.00879662	1.415×10^{-3}
0.7	100	0.00955033	0.00956493	1.529×10^{-3}
1	0.01	0.00846563	0.00847355	9.365×10^{-4}
1	0.1	0.00846734	0.00847518	9.258×10^{-4}
1	1	0.00860254	0.00861031	9.031×10^{-4}
1	10	0.00942139	0.00943089	1.008×10^{-3}
1	100	0.01036429	0.01040272	3.707×10^{-3}
1.3	0.01	0.00872641	0.00873404	8.733×10^{-4}
1.3	0.1	0.00872777	0.00873531	8.645×10^{-4}
1.3	1	0.00883909	0.00884650	8.390×10^{-4}
1.3	10	0.00969922	0.00970457	5.516×10^{-4}
1.3	100	0.01072890	0.01076492	3.357×10^{-3}
1.5	0.01	0.00881109	0.00881892	8.883×10^{-4}
1.5	0.1	0.00881230	0.00882005	8.802×10^{-4}
1.5	1	0.00891329	0.00892081	8.436×10^{-4}
1.5	10	0.00979520	0.00979852	3.386×10^{-4}
1.5	100	0.01085569	0.01088713	2.896×10^{-3}

5. Validation level

The Sec. IV validation is carried out for the reduced rod-transport model defined above. The semianalytic closure supplies $D_{ij}(Y)$, $v_d(Y)$, u_m^{rod} , and κ_{ref} , and the LBM then assesses the transient moment evolution of the corresponding coarse-grained ADE. Because the LBM release is uniform rather than ρ_∞ , this is a nonequilibrium-injection test of the Green-Kubo-motivated shear closure, not an exact stationary Green-Kubo reduction. Orientation-resolved transient simulation of the full rod Fokker-Planck problem remains a separate numerical development.

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