

From fluttering to drifting: Inertialess sedimentation of an achiral particle

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There has been much recent interest in the chiral motion of achiral particles that sediment in a viscous fluid in a regime where inertial effects can be neglected. This occurs in a broad range of applications such as those involving biological objects like algae, ultrathin graphene flakes, or colloidal suspensions. It is known that particles with two planes of symmetry can be categorized as “settlers,” “drifters,” or “flutterers,” where the latter sediment along chiral trajectories despite their achiral shapes. Previous work investigated the sedimentation of circular disks bent into a U-shape and identified them as “flutterers.” In this work we analyze the change in behavior of such particles when we break one of their symmetries by pinching the disks isometrically along their axis, a shape change that can arise during the sedimentation of thin elastic sheets. The “fluttering” behavior is found to be robust to such shape changes, with the trajectories now evolving toward helical paths. However, the behavior changes when the degree of pinching becomes too strong, at which point the particles become “drifters” which sediment steadily without rotation. We establish criteria for the transition between the two types of behavior and confirm our predictions in experiments. Finally, we discuss the implications of our observations for the dispersion of dilute suspensions made of such particles.

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I. INTRODUCTION

The motion of a rigid body through a viscous fluid under the influence of an internal or external driving force is a canonical problem which is both fundamental in nature and widely relevant to a range of applications. The study of such problems dates back to seminal work of Stokes [1] who derived the drag on a sphere moving in the limit of vanishing fluid inertia, a limit which is relevant for a host of biological and soft-matter problems, e.g., the size-separation of biological molecules such as DNA [2] and 2D materials [3], the provision of pathways to designing optimal swimmers at the microscale [4,5], separating microplastics and other colloidal suspensions [6,7], and for improving blood tests [8].

At high Reynolds number, simple particles such as rods, disks, spheres or ellipsoids moving in an unbounded and otherwise quiescent fluid readily exhibit fluttering, tumbling and chaotic modes,

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as observed for sedimenting coins [9], say. However, in the limit of zero Reynolds number, simple particles translate steadily without reorientation [1,10,11]; and more complicated dynamics only arise because of the presence of bounding walls [12], background flows such as shear flows [13,14], or when multiple particles interact hydrodynamically [15–17].

In an unbounded and quiescent Stokesian fluid, the sedimentation of a particle is entirely determined by its shape [11], and so nontrivial dynamics arise through translation-rotation coupling, whereby translational motion induces a torque which rotates the body, which, in turn, affect its translational trajectory. It is this limit we are concerned with in the present study. The dynamics are described by Stokes’ theory which links the rate of change of the particle’s position and orientation to the forces and torques acting on it via the so-called resistance matrix [11]. Coupling between translation and rotation are encoded in the off-diagonal terms in this matrix.

The possible types of translation-rotation coupling depend on the particle’s symmetry [11]; for example, it vanishes for particles with three planes of symmetry, leading to simple translational motion. For the case of chiral particles which do not have any planes of symmetry, such as helices [4,18,19], propellers [20], Möbius strips [21], and knots [2], their centers of mass tend to sediment along chiral trajectories, whose handedness is determined by the particle shape.

For achiral particles with fewer than three planes of symmetry, the types of possible sedimentation trajectories can be determined by analysis of their resistance matrices [19,22]. However, there is no known general analytical approach to determine the resistance matrix for an arbitrary particle shape, and so this must be determined numerically or through experiments. The possible behavior of an arbitrary particle shape cannot, therefore, be determined *a priori*. Furthermore, even when a particle exhibits nontrivial sedimentation dynamics, the translation-rotation coupling can be so weak (as in the case of Kelvin’s isotropic helicoid [23]), that it would be difficult to observe the behavior in experiments. This is because the rotation can occur over timescales during which the particles sediment over much larger distances than can be accommodated in typical lab experiments.

We have recently performed experimental [24] and numerical [25] studies of U-shaped disks that were formed by isometrically warping a circular disk into a cylindrical shape, resulting in a body with two planes of symmetry. This choice of particle was motivated by our interest in the sedimentation of thin elastic sheets which tend to deform into such shapes while undergoing very small in-plane stretching/compression. We showed that these achiral disks typically sediment in a chiral manner, with the chirality of their trajectories determined by the disks’ initial orientation rather than their shape. Such particles are therefore examples of “flutterers” (particles which show quasiperiodic (but never chaotic) reorientational motion), using the terminology of Ref. [26] whose recent theoretical study established the possible types of inertialess sedimentation for particles with two planes of symmetry. (In addition to “flutterers,” this class of particles includes “drifters” whose motion is purely translational at a fixed nonzero angle with respect to gravity, and “settlers” whose motion is also purely translational but parallel to gravity.)

Here we combine numerical simulations and experiments to analyze how breaking one plane of the disk’s symmetry affects this behavior. For this purpose we consider the sedimentation of “pinched” U-shaped disks, obtained by warping plane circular disks around a cone rather than a cylinder. This results in an achiral particle with a single plane of symmetry; see Fig. 1(a). A conical deformation is chosen as it is the canonical isometric deformation that produces a surface with one plane of symmetry. We show that variations in the degree of pinching allow this particle to exhibit the entire range of possible behaviors, from quasiperiodic spiralling (a periodic spiral motion of the center of mass accompanied with a periodic rotational motion of incommensurate period, which is found to occur robustly over a wide range of shapes) to sedimentation along straight trajectories without any accompanying reorientation.

II. MATHEMATICAL FRAMEWORK

We consider a thin disk of area $A = \pi\mathcal{R}^2$, nominal uniform thickness $h \ll \mathcal{R}$, and density ρ_d , sedimenting under the action of gravity, $-\mathbf{g}\mathbf{e}_3$, in a Newtonian fluid of viscosity μ and density

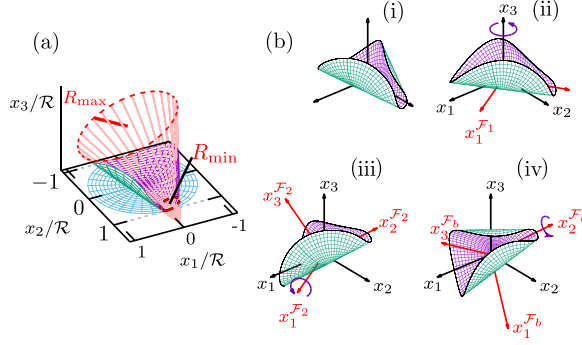


FIG. 1. (a) “Pinched” disk constructed by the isometric deformation of a flat, circular disk (shown in blue) onto the surface of a cone (red lines); (b) the intrinsic sequence of rotations ii–iv from the reference state i, characterized by χ_{yaw} , χ_{pitch} , and χ_{roll} (in that order) that reorientate the disk from laboratory frame $\mathcal{F}_{\text{lab}}$ to its general orientation in a body-fitted frame $\mathcal{F}_b$.

$\rho_f < \rho_d$ at a typical velocity $\mathcal{U} = (\rho_d - \rho_f)g\mathcal{R}^2/\mu$, so that the corresponding Reynolds number is small, i.e., $\text{Re} = \rho_f\mathcal{U}\mathcal{R}/\mu \ll 1$, allowing us to neglect inertial effects in the fluid. For modest density ratios, such that $\rho_d/\rho_f \text{ Re} \ll 1$, we can neglect the inertia of the disk too, implying that the disk sediments quasisteadily. In the experiments discussed in Sec. III B with nylon sheets ($\mathcal{R} \sim 12$ mm) in silicone oil ($\mu = 1.02$ Pa s), the typical maximum velocity $\mathcal{U} \sim \mathcal{O}(1)$ mm s^{−1}, which gives $\text{Re} \sim \mathcal{O}(10^{-2})$, justifies the quasisteady assumption of our mathematical modeling.

Our disks are obtained by the isometric deformation of planar circular disks of radius $\mathcal{R}$ onto the surface of a cone, characterized by the two radii of curvature, $(\mathcal{R}_{\text{min}}, \mathcal{R}_{\text{max}}) = \mathcal{R}(R_{\text{min}}, R_{\text{max}})$. If $R_{\text{min}} = R_{\text{max}}$, then we obtain U-shaped disks with two planes of symmetry; otherwise, the disk is pinched, with its curvature varying with x_2 , leaving only a single plane of symmetry, as shown in Fig. 1(a). We refer to the most tightly curved point of a pinched disk as its “nose.”

We describe the disk’s kinematics in terms of the position vector $\mathbf{r}_M(t)$ to its center of mass (generally located outside the disk), and characterise its orientation using an intrinsic sequence of rotations (i.e., a chained sequence of rotations about the body-fitted axes) defined by the Tait-Bryan angles $\chi_{\text{yaw}}(t)$, $\chi_{\text{pitch}}(t)$, and $\chi_{\text{roll}}(t)$, with the rotations applied in precisely that order (because rotations do not commute), as shown in Fig. 1(b). Here $\chi_{\text{yaw}} = \chi_{\text{pitch}} = \chi_{\text{roll}} = 0$ define the reference orientation in which the disk’s principal axes are aligned with the laboratory frame and the disk is in an upright-U orientation. Given the values of $(\chi_{\text{yaw}}, \chi_{\text{pitch}}, \chi_{\text{roll}})$ the disk is reoriented by first rotating it by an angle χ_{yaw} about the x_3 axis [Figs. 1(b.i) and 1(b.ii)]; the subsequent pitching rotation by χ_{pitch} about the body-fitted $x_1^{\mathcal{F}_b}$ axis then rotates the disk out of the horizontal plane [Figs. 1(b.ii) and 1(b.iii)]; finally we “roll” the disk about its body-fitted $x_2^{\mathcal{F}_b}$ axis [Figs. 1(b.iii) and 1(b.iv)].

A. Resistance-matrix formalism

Assuming that wall effects can be neglected, the disk’s quasisteady motion is completely determined by the resistance matrix $\mathbf{R}$ which links the disk’s instantaneous translational and rotational velocities, $d\mathbf{r}_M/dt$ and $\boldsymbol{\omega}$, to the net external force $\mathbf{F}$ and the torque $\mathbf{T}_M$ acting on its center of mass,

$$\begin{bmatrix} \mathbf{F} \\ \mathbf{T}_M \end{bmatrix} = \mathbf{R} \begin{bmatrix} d\mathbf{r}_M/dt \\ \boldsymbol{\omega} \end{bmatrix}; \quad (1)$$

see Ref. [11]. If the vectors are decomposed into the body-fitted coordinate system, $\mathcal{F}_b$, shown in Fig. 1(b), then the entries of the resistance matrix are constant and depend only on the shape of the disk. For a freely sedimenting disk, the net force, $\mathbf{F}$, is due to buoyancy in the direction

of gravity which, in the body-fitted frame $\mathcal{F}_b$, depends on the pitch and roll angles, so $\mathbf{F}^{\mathcal{F}_b} = \mathbf{F}^{\mathcal{F}_b}(\chi_{\text{pitch}}(t), \chi_{\text{roll}}(t))$. The external torque is zero, $\mathbf{T}_M = \mathbf{0}$.

Equation (1) represents a system of six autonomous ODEs for the evolution of the three components of $\mathbf{r}_M(t)$ and the three Tait-Bryan angles $\chi_{\text{yaw}}(t)$, $\chi_{\text{pitch}}(t)$, $\chi_{\text{roll}}(t)$. In Appendix A we show that the ODEs for χ_{pitch} and χ_{roll} decouple from the ODEs for χ_{yaw} and $\mathbf{r}_M$ and have the form

$$\begin{aligned}\frac{d\chi_{\text{pitch}}}{dt} &= f_{\text{pitch}}(\chi_{\text{pitch}}, \chi_{\text{roll}}), \\ \frac{d\chi_{\text{roll}}}{dt} &= f_{\text{roll}}(\chi_{\text{pitch}}, \chi_{\text{roll}}).\end{aligned}\quad (2)$$

Once $\chi_{\text{pitch}}(t)$ and $\chi_{\text{roll}}(t)$ have been determined from Eq. (2), the other variables can be obtained successively by integrating the remaining ODEs,

$$\frac{d\chi_{\text{yaw}}}{dt} = f_{\text{yaw}}(\chi_{\text{pitch}}, \chi_{\text{roll}}), \quad (3)$$

$$\frac{d\mathbf{r}_M}{dt} = \mathbf{f}_M(\chi_{\text{yaw}}, \chi_{\text{pitch}}, \chi_{\text{roll}}). \quad (4)$$

The functions on the right-hand side of these ODEs are given in Appendix A; they depend on the coefficients of the resistance matrix which we computed using an augmented finite-element-based solution of the 3D Stokes equations (implemented in `oomph-lib` [27]) to resolve the singularities that develop along the edge of the disk; see Refs. [28,29]. We refer to Refs. [25,30] for full details of the implementation and validation; Appendix B of this paper provides a brief overview of the method.

The structure of Eqs. (2)–(4) reflects the fact that in an unbounded fluid, the rate at which the disk changes its inclination relative to the direction of gravity (via changes to χ_{pitch} and χ_{roll}) is not affected by changes to the yaw angle, χ_{yaw} , or its position in the fluid, $\mathbf{r}_M$.

For later reference we note that

$$f_{\text{yaw}}(\chi_{\text{pitch}}, \chi_{\text{roll}} = 0) = f_{\text{yaw}}(\chi_{\text{pitch}}, \chi_{\text{roll}} = \pi) = 0, \quad (5)$$

because a disk whose inclination is changed only by pitching does not induce any rotation about the direction of gravity. We also have

$$f_{\text{yaw}}(\chi_{\text{pitch}}, \chi_{\text{roll}}) = -f_{\text{yaw}}(\chi_{\text{pitch}}, -\chi_{\text{roll}}). \quad (6)$$

Furthermore, the pinched disk’s remaining plane of symmetry implies that

$$f_{\text{pitch}}(\chi_{\text{pitch}}, \chi_{\text{roll}}) = f_{\text{pitch}}(\chi_{\text{pitch}}, -\chi_{\text{roll}}) \quad (7)$$

and

$$f_{\text{roll}}(\chi_{\text{pitch}}, \chi_{\text{roll}}) = -f_{\text{roll}}(\chi_{\text{pitch}}, -\chi_{\text{roll}}). \quad (8)$$

B. Two-dimensional phase plane

Since χ_{yaw} and $\mathbf{r}_M$ are enslaved to the evolution of χ_{pitch} and χ_{roll} , the system’s evolution can be analyzed in a two-dimensional phase plane formed by these two angles. Figure 2 provides a representative example: it shows paths in the $(\chi_{\text{pitch}}, \chi_{\text{roll}})$ -phase plane for a doubly symmetric U-shaped disk with $R_{\text{min}} = R_{\text{max}} = 1$, the case studied in Ref. [25]. In general, the paths can be seen to be closed loops along which the disk alternates between pitching and rolling-dominated reorientations—the behavior exhibited by “flutterers.” We showed in Ref. [25] that this is due to the fact that the disk is stable to rolling but unstable to pitching when it is in its upright-U orientation; conversely, when the disk is in its upside-down-U orientation, it is stable to pitching but unstable to rolling. In the $(\chi_{\text{pitch}}, \chi_{\text{roll}})$ -phase plane the upright and upside-down-U orientations (in which the disk sediments vertically without undergoing any reorientation, implying that for these isolated,

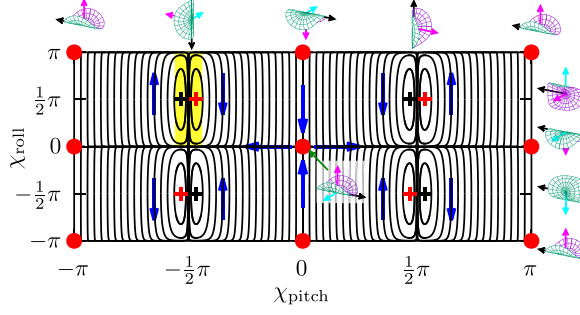


FIG. 2. Phase-space trajectories for a doubly symmetric U-shaped disk with $R_{\min} = R_{\max} = 1$. Filled red circles indicate the position of saddle-points, crosses indicate the location of centers, and blue arrows show the direction of travel along the paths. The color of the crosses indicates the sense of rotation of the particles and the handedness of their helical trajectories in the laboratory frame: black [red] crosses indicate that particles rotate about the positive [negative] direction of gravity, $d\chi_{\text{yaw}}/dt < [>] 0$.

unstable orientations, the “flutterers” actually behave like “settlers”) therefore appear as fixed points of saddle type, represented by the filled red circles in Fig. 2. For example, for $\chi_{\text{pitch}} = \chi_{\text{roll}} = 0$, the χ_{pitch} axis represents the saddle point’s unstable directions since a small perturbation to χ_{pitch} continues to grow; conversely the χ_{roll} axis represents the stable direction since, following a small perturbation to χ_{roll} , the disk returns to its upright orientation. (The lines $\chi_{\text{pitch}} = \pm\pi/2$ represent coordinate singularities, unavoidable for any choice of Euler angles; along these, a change in χ_{roll} is exactly compensated for by an equal and opposite change to the enslaved χ_{yaw} , so all points on these lines represent the same orientation.)

The phase plane contains eight further fixed points, $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$, which are neutrally stable centers, identified by the crosses. When released with this initial orientation, $\chi_{\text{pitch}}(t)$, $\chi_{\text{roll}}(t)$ remain constant, implying that the disk sediments without changing its inclination, but rotating about the direction of gravity at a constant rate $d\chi_{\text{yaw}}/dt = f_{\text{yaw}}(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$; see Eq. (3). This results in a helical center-of-mass trajectory with constant radius and pitch. The handedness of the trajectories is indicated by the color of the symbols in Fig. 2: red [black] crosses represent fixed points for which $d\chi_{\text{yaw}}/dt$ is positive [negative].

For all other initial orientations, $\chi_{\text{pitch}}(t)$ and $\chi_{\text{roll}}(t)$ change periodically as they traverse along the closed loops in the phase plane. This leads to periodic changes in $d\chi_{\text{yaw}}/dt$ which maintains its sign, resulting in the disk rotating about the direction of gravity in the same direction as for the enclosed fixed point. However, the time it takes for χ_{yaw} to increase by 2π is, in general, not a rational multiple of the time it takes to traverse the closed loop in the $(\chi_{\text{pitch}}, \chi_{\text{roll}})$ -phase plane, resulting in a quasiperiodic motion and complex spiral trajectories in the laboratory-frame, as illustrated in Fig. 3(a) below.

III. RESULTS

A. Numerical/theoretical results

We will now investigate how this behavior changes as we break the disk’s fore-aft symmetry by reducing $R_{\min}$ while keeping $R_{\max} = 1$, thus pinching the disk along its roll axis. For this purpose the figures in row (i) of Fig. 3 show the subregion of the $(\chi_{\text{pitch}}, \chi_{\text{roll}})$ -phase plane that we highlighted by the yellow shading in Fig. 2. As before, black lines indicate the paths of $\chi_{\text{pitch}}(t)$ and $\chi_{\text{roll}}(t)$ through the phase plane, while symbols identify the position of the various fixed points at which $d\chi_{\text{pitch}}/dt = d\chi_{\text{roll}}/dt = 0$. The translucent pink lines show how the fixed points $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$ (the neutrally stable centers for the case of the doubly symmetric U-shaped disk) move through the phase plane as $R_{\min}$ is reduced from $R_{\min} = R_{\max} = 1$ (on the left), to $R_{\min} = 0.4$ (on the right).

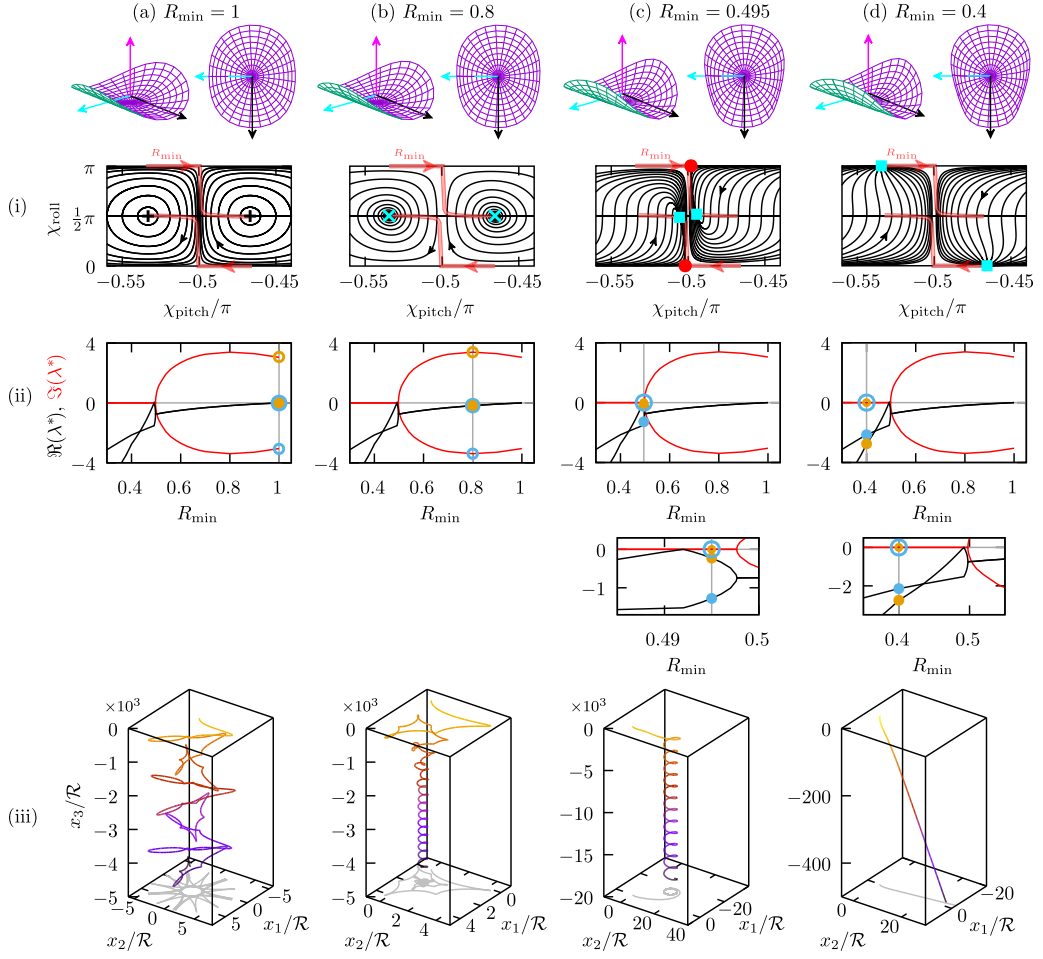


FIG. 3. Behavior of disks (with $R_{\max} = 1$) with an increase in the degree of pinching in each successive column; (a) $R_{\min} = 1$, (b) $R_{\min} = 0.8$, $R_{\min} = 0.495$, (c) $R_{\min} = 0.4$. Row (i) shows the evolution of the disk's orientation in terms of pitch and roll angles in the part of the phase plane highlighted by the yellow region in Fig. 2. Black lines illustrate the direction of the paths in the phase plane, and symbols indicate the locations of the fixed points: centers (black crosses), stable spiral nodes (cyan crosses), stable nodes (filled cyan squares) and saddle points (filled red circles). The translucent pink lines illustrate the motion of the central fixed points through the phase plane as $R_{\min}$ is varied [with the red arrows indicating an increase in $R_{\min}$, i.e., moving from column (d) toward column (a)]. These lines are therefore identical in each of the four phase portraits in row (i), serving only to guide the eye from column to column; the black lines show actual phase-space trajectories for fixed $R_{\min}$. Row (ii) shows the evolution of the eigenvalues λ^* of the Jacobian matrix (9), evaluated at the central fixed points, with the black [red] curves showing the real [imaginary] part as a function of $R_{\min}$. The two complex-conjugate eigenvalues for each value of $R_{\min}$ are identified by the pairs of orange and blue symbols, with the real [imaginary] part represented by filled [hollow] circles. Zoomed plots are shown immediately below for columns (c) and (d) to highlight the structure of the eigenvalues' trajectories close to the bifurcations. Row (iii) shows the trajectories of the disk's center of mass in the laboratory frame, with colors indicating the value of x_3 (relative to the x_3 range in each plot); the projection of the trajectories into the x_1 - x_2 plane is shown with gray lines.

The lines are provided to guide the eye and establish a link between the four phase portraits shown in row (i) of Fig. 3. Analytic expressions for the location of the fixed points are derived in Appendix C.

For $R_{\min} = 1$ we see the behavior already observed in Fig. 2: The central fixed point in the phase plane is a neutrally stable center, and the eigenvalues λ^* of the Jacobian matrix

$$\mathbf{J} = \begin{pmatrix} \partial f_{\text{pitch}} / \partial \chi_{\text{pitch}} & \partial f_{\text{pitch}} / \partial \chi_{\text{roll}} \\ \partial f_{\text{roll}} / \partial \chi_{\text{pitch}} & \partial f_{\text{roll}} / \partial \chi_{\text{roll}} \end{pmatrix}, \quad (9)$$

evaluated at the fixed point, are purely imaginary as shown in row (ii) of Fig. 3.

As we reduce $R_{\min}$, the fixed points $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$ move toward the line $\chi_{\text{pitch}} = -\pi/2$ and the associated eigenvalues become complex conjugates with a negative real part, changing the character of the fixed points from centers to stable spiral nodes. This implies that, unlike the case of the doubly symmetric U-shaped disk, the disk's inclination now evolves toward these fixed points, $(\chi_{\text{pitch}}, \chi_{\text{roll}}) \rightarrow (\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$, irrespective of its initial orientation. As the inclination approaches the fixed points, the disk's rate of rotation about the direction of gravity also approaches a constant, $d\chi_{\text{yaw}}/dt \rightarrow f_{\text{yaw}}(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$. Thus, following some initial transients, the disk's center of mass ultimately sediments along a helical trajectory, as shown in Fig. 3(b.iii). Note that the character of the initial transients is reminiscent of the behavior of the doubly symmetric U-shaped disks: as the paths of $(\chi_{\text{pitch}}(t), \chi_{\text{roll}}(t))$ spiral toward their fixed point $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$ in the phase plane, the disk's reorientation alternates between pitching and rolling-dominated phases, with accompanying enslaved changes in the yaw angle. Hence, during the transient phase, the trajectory of the disk's center of mass still follows a complex spiral path as it approaches its ultimate helical trajectory. We note that the position of the saddle points in the phase plane (indicated by filled red circles in Fig. 2) also changes with $R_{\min}$, but for $R_{\min} = 0.8$ they are still outside the subregion of the phase plane shown in Fig. 3(b.i). If the disk is released with an inclination that corresponds to one of these saddle points, then it will in fact behave like a “drifter,” i.e., it will sediment downwards with a constant horizontal drift, but without any reorientation. These trajectories are, of course, unstable, and so this behavior will not be observable in experiments, justifying the decision of Ref. [26] to base their classification into “flutterers,” “drifters” and “settlers” on the particle shape rather than a particle's possible trajectories.

The plots in Fig. 3(c) show that, as we reduce $R_{\min}$ further, the two fixed points $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$ move further toward the line $\chi_{\text{pitch}} = -\pi/2$ and the complex conjugate eigenvalues coalesce on the real axis, following which they initially move in opposite directions: as $R_{\min}$ is reduced, the eigenvalue indicated by the orange [blue] symbol in row (ii) of Fig. 3 moves toward [away from] zero along the real axis.

When the eigenvalues become purely real, the stable spiral nodes turn into stable nodes, identified by the filled cyan squares in Fig. 3(c.i). This means that the approach of $(\chi_{\text{pitch}}(t), \chi_{\text{roll}}(t))$ toward $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$ becomes monotonic rather than oscillatory, resulting in the much simpler trajectory of the disk's center of mass during the initial transients: the disk no longer alternates between pitching and rolling-dominated reorientations and therefore approaches its final helical trajectory without any complicated transient spiral motion; see the projection of the center-of-mass trajectory onto the (x_1, x_2) plane in Fig. 3(c.iii). We note that the radius of the helical trajectory increases with a reduction in $R_{\min}$. Furthermore, two of the nine saddle points shown in Fig. 2 have now moved so far along the χ_{roll} axis that they are now located close to $\chi_{\text{roll}} = -\pi/2$ (filled red circles).

As we reduce $R_{\min}$ further, the two fixed points $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$ that have approached the line $\chi_{\text{pitch}} = -\pi/2$, move rapidly downwards [upwards] toward $\chi_{\text{roll}} = 0$ [π], where they approach the two saddle points (shown in red) that have just moved into our subregion of the phase plane. Recall now that the symmetries of f_{pitch} and f_{roll} , Eqs. (7) and (8), imply that each stable node at $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$ has a counterpart at $(\chi_{\text{pitch}}^*, -\chi_{\text{roll}}^*)$. The two close-ups in Fig. 4 show that, as $R_{\min}$

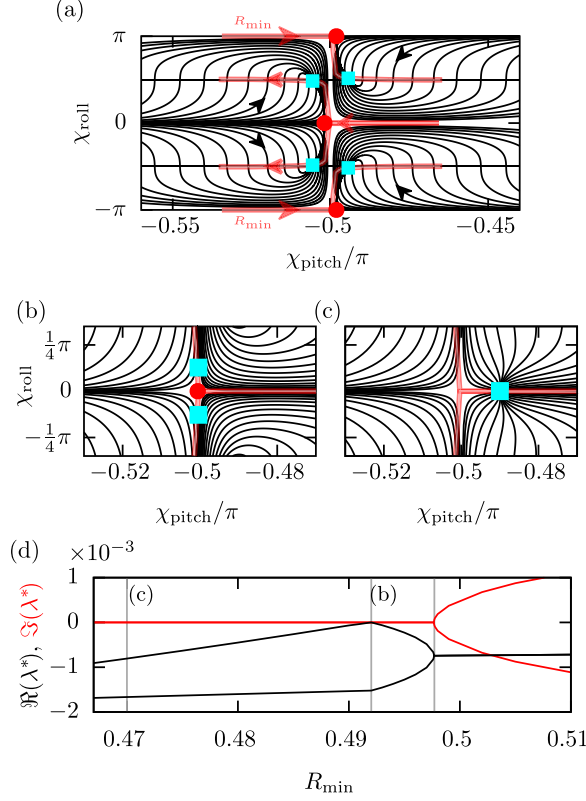


FIG. 4. (a)–(c) Detail of the phase plane, showing the coalescence of the two stable nodes (filled cyan squares at $(\chi_{\text{pitch}}^*, \pm\chi_{\text{roll}}^*)$) with the saddle point (filled red circle at $\chi_{\text{roll}} = 0$). (a) $R_{\text{min}} = 0.495$; this is a repetition of Fig. 3(c.ii) but shown over the full range $\chi_{\text{roll}} \in [-\pi, +\pi]$ to highlight the coalesce of the two stable nodes at $\pm\chi_{\text{roll}}$. The subsequent plots are for (b) $R_{\text{min}} = 0.492$ and (c) $R_{\text{min}} = 0.47$, and are thus located between columns (c) and (d) in Fig. 3. Following the coalescence, only a single stable node remains at $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^* = 0)$. (d) The real (black) and imaginary (red) parts of the eigenvalues at the fixed points as a function of R_{min} close to the bifurcation region, with the gray vertical dashed lines at $R_{\text{min}} = 0.492$ [corresponding to the phase portrait in column (c)], $R_{\text{min}} = 0.47$ [corresponding to the phase portrait in column (b)], and $R_{\text{min}} = 0.497$ (the location of the node to spiral-node transition).

decreases, these two stable nodes coalesce with the saddle point, resulting in the emergence of a single stable node that then moves along the χ_{roll} axis as R_{min} is reduced yet further.

Equation (5) shows that $\chi_{\text{roll}} = 0$ implies that $d\chi_{\text{yaw}}/dt = 0$. Once the disk’s inclination has approached its fixed point $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^* = 0)$, the disk therefore ceases to rotate, and all three Tait-Bryan angles remain constant. Equation (4) then implies that, following the decay of the initial transients, the disk’s center of mass sediments along a straight trajectory, with constant velocity, $d\mathbf{r}_M/dt = \text{const.}$, and without any further reorientation, as illustrated in Fig. 3(d.iii). The disk has thus turned from a “flutterer” into a “drifter.” The transition from a helical to a straight center-of-mass trajectory is achieved by the radius of the helix becoming infinite as $\chi_{\text{roll}}^* \rightarrow 0$.

The coalescence of two stable nodes with a saddle point, illustrated in Figs. 4(a)–4(c), indicates the occurrence of a (pitchfork) bifurcation, implying that the Jacobian matrix (9) is singular and thus has a zero eigenvalue. This corresponds to the eigenvalue indicated by the orange symbol in the row (ii) of Fig. 3 reaching the origin of the complex plane, following which it changes direction and moves away from zero, like the other negative eigenvalue. This behavior is apparent in Fig. 4(c), which shows the real and imaginary parts of the eigenvalues as a function of R_{min} in the vicinity of

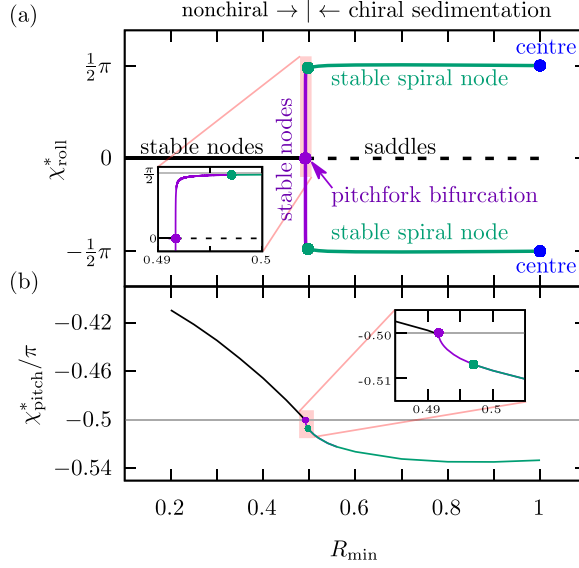


FIG. 5. Plots of the angles of the fixed points at which the disk (with $R_{\max} = 1$) sediments with a constant inclination with respect to gravity as a function of $R_{\min}$; (a) χ_{roll}^* and (b) χ_{pitch}^* (shown for the stable fixed point only). The character of the fixed point is indicated by the line color and the label next to each section of the curve in (a). A nonzero value of χ_{roll}^* implies that the sedimenting disk rotates about the direction of gravity, resulting in a helical trajectory of its center of mass. The insets in each plot show closeups near the bifurcation points.

Figs. 4(a)–4(c), clearly highlighting the two bifurcations that occur within a very narrow range of $R_{\min}$.

Figure 5(a) illustrates the pitchfork character of the bifurcation using a plot of χ_{roll}^* as a function of $R_{\min}$. Reading this plot from left to right, we start with a strongly pinched disk for which the only fixed point is the stable node associated with $\chi_{\text{roll}}^* = 0$. As we increase $R_{\min}$ we reach the pitchfork bifurcation (at $R_{\min} = 0.492$; the filled magenta circle) beyond which the disk has two stable inclinations, both with the same “nose down” negative pitch angle, and equal and opposite values of χ_{roll}^* . The magnitude of χ_{roll}^* initially increases very rapidly with an increase in $R_{\min}$ until it approaches the limit $\chi_{\text{roll}}^* \rightarrow \pm\pi/2$ which is attained for the doubly symmetric U-shaped disk characterized by $R_{\min} = R_{\max} = 1$. The point at which the character of the fixed point changes from a stable node to a stable spiral node is identified by the filled green circle. The transition occurs close to the point where the rate of change of χ_{roll}^* with $R_{\min}$ changes rapidly, though there is no obvious causal relation between these two events; the closeup inset in Fig. 5(a) shows that the $\chi_{\text{roll}}^*(R_{\min})$ -curve is very steep but has a finite slope and that the transition from a stable node to a stable spiral node (marked by the green symbol) occurs after the sharp bend in the curve. Figure 5(b) shows the behavior of χ_{pitch}^* with $R_{\min}$. The pitchfork bifurcation associated with the transition from nonchiral and chiral behavior (marked with the purple symbol) occurs when $\chi_{\text{pitch}}^* = -\pi/2$; the location of the node to spiral-node transition (green symbol) is not associated with any particular feature in this curve. The figure also demonstrates the weak dependence of the equilibrium pitch angle with the degree of pinching: χ_{pitch}^* changes by only $\sim 7^\circ$ across the range shown.

So far we have restricted our attention to regions of the phase plane in which the pinching of the disk turns the central fixed points from centers into stable nodes. In Fig. 6(a) we return to the full phase plane and plot the paths $(\chi_{\text{pitch}}(t), \chi_{\text{roll}}(t))$ for the case when $R_{\min} = 0.32$, $R_{\max} = 0.6$, for which the central fixed points are spiral nodes. The plot illustrates how the symmetries/anti-symmetries encoded in Eqs. (7) and (8) render the four fixed points located in the region where

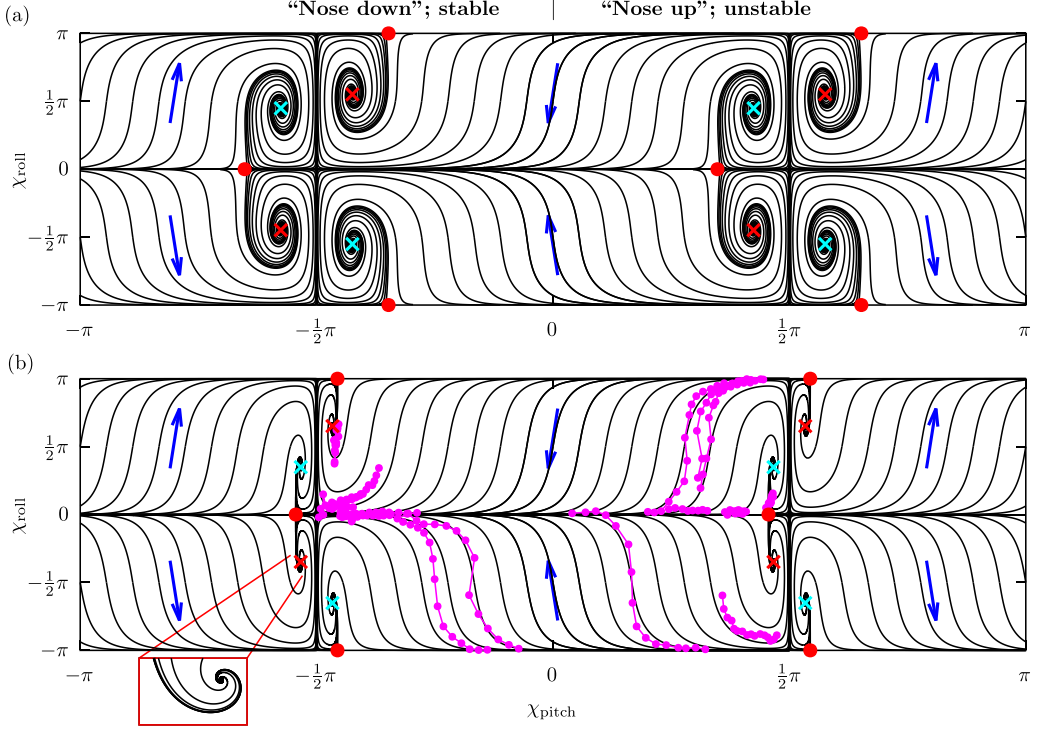


FIG. 6. Phase portrait for a disk with $R_{\min} = 0.32$, $R_{\max} = 0.6$: Solid black lines show the paths in the phase plane, while the symbols identify fixed points where $d\chi_{\text{pitch}}/dt = d\chi_{\text{roll}}/dt = 0$. X-shaped symbols represent stable ($\chi_{\text{pitch}} < 0$) and unstable ($\chi_{\text{pitch}} > 0$) spiral nodes. The former correspond to disks sedimenting in a “nose down” orientation while rotating about the direction of gravity in a positive (cyan symbols) or negative (red symbols) sense. The corresponding points in the right half of the phase plane correspond to unstable fixed points in which the disk sediments in a “nose up” orientation. Filled red symbols represent saddle points. Trajectories and fixed points are computed via the framework in Sec. II using a resistance matrix computed via (a) FEM (as per Refs. [25,30]); (b) fitting to experimental data (as per Ref. [24]), with magenta symbols, connected by magenta lines, showing the raw experimental results; the red inset shows a closeup of one of the stable fixed points.

$\chi_{\text{pitch}} < 0$ (corresponding to the disk sedimenting with a “nose down” inclination) stable. Conversely, the fixed points in the region where the disk sediments with a “nose up” inclination, $\chi_{\text{pitch}} > 0$, are unstable. The global phase portrait shows that the four stable fixed points are globally attractive, so disks released with any initial inclination ultimately evolve toward one of them. However, apart from reflections (and the associated change of handedness), the four fixed point actually represent the same type of sedimentation: the pairs $(\chi_{\text{pitch}}^*, \pm\chi_{\text{roll}}^*)$, correspond to disks that sediment with the same pitch angle but have rotated by equal and opposite amounts about their roll axis. Equation (6) shows that these disks rotate in opposite directions about the direction of gravity, implying that they sediment along helical center-of-mass trajectories that have the same shape (in terms of radius and pitch) but opposite handedness. Furthermore, the orientations associated with the two fixed points located “diagonally across” the points $(\pm\pi/2, 0)$ can be shown to have identical inclinations, with their orientation only differing by a rotation about the yaw axis by π .

Thus, following the decay of initial transients, sedimenting pinched disks approach center-of-mass trajectories of the same shape—either helical or straight, depending on the amount of pinching. Their initial orientation only affects the handedness of the helical trajectories.

B. Experiments

We showed in Ref. [25] that for doubly symmetric U-shaped disks (without any pinching), the prediction of closed trajectories in the $(\chi_{\text{pitch}}, \chi_{\text{roll}})$ -phase plane (as shown in Fig. 2) is consistent with the experimental observations of Ref. [24].

Figures 6(a) and 6(b) show the corresponding comparison for a pinched polyamide nylon disk of thickness $b = 237 \mu\text{m}$, area $A = \pi \mathcal{R}^2 = 4.52 \text{ cm}^2$, curvatures $R_{\text{min}} = 0.32$ and $R_{\text{max}} = 0.6$, sedimenting in a cuboidal tank of internal dimensions $90 \times 40 \times 40 \text{ cm}^3$. We note that the geometry of the disk used in the experiments differed slightly from that shown in Fig. 1(a) because of the way the disk was fabricated in the experiments; see Appendix D for details.

In the experiments we monitored the disk's orientation while it sedimented over a vertical distance of approximately $30\mathcal{R}$, staying at least $10\mathcal{R}$ from the free surface and the side and bottom boundaries of the tank. The Reynolds number reached a maximum value of $\text{Re} \sim 10^{-2}$, implying that inertial effects are likely to be negligible. Since the disk's reorientation takes place over very large vertical distances [note the different vertical and horizontal scales in Fig. 3(iii)], each of the 15 experiments only provided a relatively short segment of the paths in the $(\chi_{\text{pitch}}, \chi_{\text{roll}})$ -phase plane; these are shown in magenta in Fig. 6(b). The black lines in that figure show the extrapolated behavior, obtained by fitting the entries of the resistance matrix to the experimental data and then integrating Eqs. (2) from a range of initial conditions; see Refs. [24,31] for details. The raw data and fitted matrices are available at Ref. [32].

Experiments and computations show that the symmetry breaking induced by the pinching of the disk changes the stability of the interior fixed points: instead of the orientation following neutrally stable periodic orbits as observed for doubly symmetric U-shaped disks, the “pinched” disk approaches a constant inclination in which it sediments along a helical trajectory in a nose down orientation, irrespective of its initial inclination. The extrapolated paths through the phase plane (black) agree well with the short segments of raw experimental data (magenta) and allow the identification of the position of the fixed points [saddle points, and stable and unstable centers, all labeled as in Fig. 6(a)]. In the experiments, the fixed points are closer to the lines $\chi_{\text{pitch}} = \pm\pi/2$, but the topology of the paths in the phase plane, and thus the disk's overall behavior, is consistent with that predicted by the computations.

IV. DISCUSSION

We have investigated the inertialess sedimentation of a “pinched,” conically deformed circular disk—an achiral particle with a single plane of symmetry—utilizing both experimental observations and a theoretical analysis based on a resistance-matrix formalism. This formalism and our phase-space analysis was crucial to the comprehensive understanding of the dynamics elucidated in this work; such insight would have been infeasible to ascertain through costly time-dependent 3D numerical simulations.

It is known from our previous study [24,25] that for the special case where the disk is deformed onto a cylindrical surface (resulting in a particle with two planes of symmetry), the disk generally sediments along complex spiral trajectories whose shape and handedness depend on the disk's initial orientation, the characteristic behavior exhibited by “flutterers.” For a particular initial inclination, the trajectories of such disks are perfect helices. Our current study showed that symmetry breaking by pinching makes the helical trajectories globally attractive, so that they are approached from all initial inclinations, with the disk sedimenting in a “nose down” inclination while rotating about the direction of gravity. (There are also isolated, unstable inclinations, associated with the saddle points in the $(\chi_{\text{pitch}}, \chi_{\text{roll}})$ -phase plane, where the disks actually behave like “drifters” and thus sediment with a constant horizontal drift but without reorienting.) As the amount of pinching is increased, the radius of the helical trajectories increases and ultimately becomes infinite, resulting in straight trajectories along which the disk sediments with constant velocity and without any reorientation, indicating that the disk has become a “drifter.” It is also worth re-emphasizing that the phase space

for this system is 2D, as discussed in Sec. II A and Appendix A, and so Figs. 3–5 illustrate the complete sequence of possible dynamical transitions; there cannot be any “hidden” transitions in a third dimension of phase space.

Our study shows that the sedimentation of achiral particles along chiral trajectories is a remarkably robust phenomenon that persists over a wide range of disk shapes and can readily be observed experimentally. The fact that for sufficiently strong pinching the character of the trajectories undergoes a qualitative change means that the conical disks considered here span the range of behaviors from modes with complex spirals trajectories to “drifting” modes where the trajectory of the center of mass is a straight line inclined with respect to gravity. The transition between the two regimes has important consequences for the long-term behavior of dilute suspensions of such particles (with negligible particle-particle interactions): suspensions made of particles that move along spiral trajectories sediment with zero net horizontal dispersion, whereas suspensions made of more strongly “pinched” particles will spread out horizontally while sedimenting without any rotation. This has the potential to be exploited in applications that aim to control the collective hydrodynamics of particles in such suspensions; see, e.g., Ref. [19].

An interesting open question is the identification of the physical mechanism that gives rise to the phenomena described here. Specifically, what causes the change in behavior from chiral to nonchiral trajectories as the disk is pinched more strongly? Why does the pinched disk adopt a “nose down” orientation? Etc. In the case of a symmetric U-shaped disk we were able to associate the unexpected occurrence of spiral trajectories with the fact that the upright-U orientation is stable to rolling and unstable to pitching (with the reverse applying in the upside-down-U orientation). This implied that, in general, a sedimenting U-shaped disk undergoes a periodic sequence of alternating rolling and pitching motions, with concomitant changes to the yaw angle. Since the disk’s stability/instability to rolling/pitching is directly encoded in the signs of certain terms in translation-rotation coupling matrix ($C_{12}, C_{21} < 0$; see Appendix A for details of the notation), we have a direct link between the physical mechanism and the mathematical analysis. For the case of a pinched disk, the physical interpretation of the mathematical analysis is much more complex: the shape of the trajectory is determined by the eigenvalues of the 2×2 Jacobian matrix (9) whose entries are complicated functions of many entries in the resistance matrix. It is therefore difficult to identify a simple mechanism that explains the observed behavior, and there may, of course, not be one.

We note that our analysis is directly applicable to any family of particles that initially have two planes of symmetry and are then subjected to a symmetry-breaking deformation to a shape with a single plane of symmetry. The analysis requires only the nonzero entries in the resistance matrix. Unfortunately, it is not possible to reverse the argument and associate the different types of behavior directly with specific particle shapes: the matrix entries can (and have to) be computed for each specific particle shape; however, it is not possible to solve the inverse problem of recovering a unique particle shape from a given set of matrix entries.

We also reiterate that the behavior reported here applies only in a regime where fluid inertia can be neglected; thus one cannot appeal to the arguably more intuitive behavior observed at high Reynolds numbers such as the aerodynamic gliding of (square) U-shaped disks analyzed by Ref. [33], or the fluttering and tumbling of flat disks studied by, e.g., Ref. [9].

In conjunction with our previous work [24,25], we now have a comprehensive understanding of the low Reynolds-number sedimentation of rigid disks whose shapes (U-shaped or conical) represent isometric deformations of planar disks. We are currently extending the analysis to the case with fluid-structure interaction, i.e., the case where the shape of the disk is determined by the fluid traction. For sufficiently thin disks their near-inextensibility will constrain their deformation to the approximately isometric deformations considered here. We therefore believe that the results of this study will provide guidance to the interpretation of the results obtained in that much more complicated scenario.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [32].

APPENDIX A: DERIVATION OF THE GOVERNING EQUATIONS

In the absence of inertia, the quasisteady motion of a rigid particle in an unbounded fluid is governed by a linear relation between the external force and torque (about the center of mass) acting on the particle, and its resulting instantaneous rate of rotation and the velocity of its center of mass. In the following, we nondimensionalize all lengths on the disk's linear dimension $\mathcal{R}$; the velocities on the typical sedimentation velocity $\mathcal{U} = \Delta\rho g \mathcal{R}^2 / \mu$, obtained by balancing viscous and buoyancy forces; time on $\mathcal{R}^2 / (\pi \mathcal{U} h)$; the rate of rotation on $\mathcal{R} / \mathcal{U}$; and the external force and torque on $\Delta\rho g \mathcal{R}^3$ and $\Delta\rho g \mathcal{R}^4$, respectively. Using tildes to distinguish nondimensional variables from their dimensional counterparts, the relation between the external force $\tilde{\mathbf{F}}$ and torque $\tilde{\mathbf{T}}_M$ and the velocity of the particle's center of mass, $\tilde{\mathbf{U}}_M = d\tilde{\mathbf{r}}_M/d\tilde{t}$, and its rate of rotation $\tilde{\boldsymbol{\omega}}$ is then given by

$$\begin{bmatrix} \tilde{\mathbf{F}} \\ \tilde{\mathbf{T}}_M \end{bmatrix} = \tilde{\mathbf{R}} \begin{bmatrix} \tilde{\mathbf{U}}_M \\ \tilde{\boldsymbol{\omega}} \end{bmatrix}. \quad (\text{A1})$$

If the vectors are decomposed into the body-fitted coordinate system $\mathcal{F}_b$, aligned with the particle's principal axes (see Fig. 1), then the entries in the resistance matrix $\tilde{\mathbf{R}}$ are constant and depend only on the particle's shape. For particles that have a reflectional symmetry about the (x_2, x_3) plane in the body-fitted coordinate system the matrix has the form

$$\tilde{\mathbf{R}} = \begin{pmatrix} K_{11} & 0 & 0 & 0 & C_{12} & C_{13} \\ 0 & K_{22} & K_{23} & C_{21} & 0 & 0 \\ 0 & K_{23} & K_{33} & C_{31} & 0 & 0 \\ 0 & C_{21} & C_{31} & \Omega_{11} & 0 & 0 \\ C_{12} & 0 & 0 & 0 & \Omega_{22} & \Omega_{23} \\ C_{13} & 0 & 0 & 0 & \Omega_{23} & \Omega_{33} \end{pmatrix}; \quad (\text{A2})$$

see Ref. [11]. Here, K_{ij} and Ω_{ij} are the elements of the translation and rotation tensors, while the coefficients C_{ij} represent the coupling tensor which characterises the tendency of the particle to rotate when translating and vice versa.

To transform Eq. (A1) into quantities in the lab-frame of reference, we recall that the disk's orientation is described in terms of an intrinsic sequence of rotations by the Tait-Bryan angles χ_{yaw} , χ_{pitch} , and χ_{roll} . Following the notation from Fig. 1(b) of the main text, this corresponds to

$$\mathcal{F}_{\text{lab}} = \mathcal{F}_0 \xrightarrow{\chi_{\text{yaw}} \text{ about } x_3^{\mathcal{F}_{\text{lab}}}} \mathcal{F}_1 \xrightarrow{\chi_{\text{pitch}} \text{ about } x_2^{\mathcal{F}_1}} \mathcal{F}_2 \xrightarrow{\chi_{\text{roll}} \text{ about } x_1^{\mathcal{F}_2}} \mathcal{F}_3 = \mathcal{F}_b. \quad (\text{A3})$$

Hence, for an arbitrary vector $\mathbf{a}$ with components $a_i^{\mathcal{F}_j}$ in frame $\mathcal{F}_j$, its components in the subsequent frame are given by

$$[a_1^{\mathcal{F}_{j+1}}, a_2^{\mathcal{F}_{j+1}}, a_3^{\mathcal{F}_{j+1}}]^T = \mathcal{R}_{\mathcal{F}_j}^{\mathcal{F}_{j+1}} [a_1^{\mathcal{F}_j}, a_2^{\mathcal{F}_j}, a_3^{\mathcal{F}_j}]^T, \quad (\text{A4})$$

where the matrices $\mathcal{R}_{\mathcal{F}_j}^{\mathcal{F}_{j+1}}$ are standard orthogonal rotation matrices that depend only on the angle describing the corresponding rotation. For example,

$$\mathcal{R}_{\mathcal{F}_0}^{\mathcal{F}_1} = \begin{pmatrix} \cos(\chi_{\text{yaw}}) & \sin(\chi_{\text{yaw}}) & 0 \\ -\sin(\chi_{\text{yaw}}) & \cos(\chi_{\text{yaw}}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{A5})$$

and

$$\mathcal{R}_{\mathcal{F}_1}^{\mathcal{F}_0} = (\mathcal{R}_{\mathcal{F}_0}^{\mathcal{F}_1})^{-1} = (\mathcal{R}_{\mathcal{F}_0}^{\mathcal{F}_1})^T. \quad (\text{A6})$$

Thus, translating the velocity of the disk's center of mass from $\mathcal{F}_{\text{lab}}$ to $\mathcal{F}_b$ is achieved by

$$\begin{aligned} [\tilde{U}_{M,1}^{\mathcal{F}_b}, \tilde{U}_{M,2}^{\mathcal{F}_b}, \tilde{U}_{M,3}^{\mathcal{F}_b}]^T &= \mathcal{R}_{\mathcal{F}_{\text{lab}}}^{\mathcal{F}_b} [\tilde{U}_{M,1}^{\mathcal{F}_{\text{lab}}}, \tilde{U}_{M,2}^{\mathcal{F}_{\text{lab}}}, \tilde{U}_{M,3}^{\mathcal{F}_{\text{lab}}}]^T \\ &= \mathcal{R}_{\mathcal{F}_{\text{lab}}}^{\mathcal{F}_b} \left[\frac{d\tilde{r}_{M,1}^{\mathcal{F}_{\text{lab}}}}{d\tilde{t}}, \frac{d\tilde{r}_{M,2}^{\mathcal{F}_{\text{lab}}}}{d\tilde{t}}, \frac{d\tilde{r}_{M,3}^{\mathcal{F}_{\text{lab}}}}{d\tilde{t}} \right]^T, \end{aligned} \quad (\text{A7})$$

where

$$\mathcal{R}_{\mathcal{F}_{\text{lab}}}^{\mathcal{F}_b} = \mathcal{R}_{\mathcal{F}_2}^{\mathcal{F}_b} \mathcal{R}_{\mathcal{F}_1}^{\mathcal{F}_2} \mathcal{R}_{\mathcal{F}_{\text{lab}}}^{\mathcal{F}_1}. \quad (\text{A8})$$

Similarly, the rate of rotation is translated into the rate of change of the Tait-Bryan angles via

$$\begin{aligned} [\tilde{\omega}_1^{\mathcal{F}_b}, \tilde{\omega}_2^{\mathcal{F}_b}, \tilde{\omega}_3^{\mathcal{F}_b}]^T &= \mathcal{R}_{\mathcal{F}_2}^{\mathcal{F}_b} \mathcal{R}_{\mathcal{F}_1}^{\mathcal{F}_2} \left[0, 0, \frac{d\chi_{\text{yaw}}}{d\tilde{t}} \right]^T + \mathcal{R}_{\mathcal{F}_2}^{\mathcal{F}_b} \left[\frac{d\chi_{\text{pitch}}}{d\tilde{t}}, 0, 0 \right]^T + \left[0, \frac{d\chi_{\text{roll}}}{d\tilde{t}}, 0 \right]^T \\ &= \mathcal{S} \left[\frac{d\chi_{\text{pitch}}}{d\tilde{t}}, \frac{d\chi_{\text{roll}}}{d\tilde{t}}, \frac{d\chi_{\text{yaw}}}{d\tilde{t}} \right]^T, \end{aligned} \quad (\text{A9})$$

where

$$\mathcal{S} = \begin{pmatrix} \cos(\chi_{\text{roll}}) & 0 & -\cos(\chi_{\text{pitch}}) \sin(\chi_{\text{roll}}) \\ 0 & 1 & \sin(\chi_{\text{pitch}}) \\ \sin(\chi_{\text{roll}}) & 0 & \cos(\chi_{\text{pitch}}) \cos(\chi_{\text{roll}}) \end{pmatrix}. \quad (\text{A10})$$

Thus, Eq. (A1) is transformed into

$$\left(\mathcal{R}_{\mathcal{F}_2}^{\mathcal{F}_b} \mathcal{R}_{\mathcal{F}_1}^{\mathcal{F}_2} \mathcal{R}_{\mathcal{F}_{\text{lab}}}^{\mathcal{F}_1} \left[\frac{d\tilde{r}_{M,1}^{\mathcal{F}_{\text{lab}}}}{d\tilde{t}}, \frac{d\tilde{r}_{M,2}^{\mathcal{F}_{\text{lab}}}}{d\tilde{t}}, \frac{d\tilde{r}_{M,3}^{\mathcal{F}_{\text{lab}}}}{d\tilde{t}} \right]^T \right) = \tilde{\mathbf{R}}^{-1} \left(\mathcal{R}_{\mathcal{F}_2}^{\mathcal{F}_b} \mathcal{R}_{\mathcal{F}_1}^{\mathcal{F}_2} \mathcal{R}_{\mathcal{F}_{\text{lab}}}^{\mathcal{F}_1} [0, 0, -1]^T \right), \quad (\text{A11})$$

where the vectors on the right-hand side result from the only external force in the laboratory frame: gravity acting vertically in the $-\mathbf{e}_z$ direction, with no external torque.

Some straightforward if lengthy algebra then shows that the evolution of χ_{roll} and χ_{pitch} is governed by the system of two autonomous ODEs

$$\frac{d\chi_{\text{pitch}}}{d\tilde{t}} = \frac{1}{\mathbb{D}} (\cos(\chi_{\text{pitch}})(\mathbb{A} \cos^2(\chi_{\text{roll}}) + \mathbb{B}) + \mathbb{C} \sin(\chi_{\text{pitch}}) \cos(\chi_{\text{roll}})), \quad (\text{A12})$$

$$\frac{d\chi_{\text{roll}}}{d\tilde{t}} = \frac{1}{\mathbb{D}} \frac{\sin(\chi_{\text{roll}})}{\cos(\chi_{\text{pitch}})} (\mathbb{E} \cos^2(\chi_{\text{pitch}}) + \mathbb{A} \sin(\chi_{\text{pitch}}) \cos(\chi_{\text{pitch}}) \cos(\chi_{\text{roll}}) + \mathbb{G}). \quad (\text{A13})$$

With $\chi_{\text{pitch}}(t)$ and $\chi_{\text{roll}}(t)$ determined from these equations, the evolution of the yaw angle then follows from the solution of the ODE

$$\frac{d\chi_{\text{yaw}}}{d\tilde{t}} = -\frac{1}{\mathbb{D}} \frac{\sin(\chi_{\text{roll}})}{\cos(\chi_{\text{pitch}})} (\mathbb{H} \cos(\chi_{\text{roll}}) \cos(\chi_{\text{pitch}}) + \mathbb{G} \sin(\chi_{\text{pitch}})). \quad (\text{A14})$$

Finally, the trajectory of the disk's center of mass is determined by

$$\begin{aligned}
 \frac{d\tilde{r}_{M,1}^{\mathcal{F}_{\text{lab}}}}{dt} &= ((-2\mathbb{K}\mathbb{M} \sin(\chi_{\text{yaw}}) \cos(\chi_{\text{roll}}) \cos(\chi_{\text{pitch}})^2 + (-\mathbb{K}\mathbb{J} \sin(\chi_{\text{yaw}}) \sin(\chi_{\text{pitch}}) \cos(\chi_{\text{roll}})^2 \\
 &\quad + \mathbb{P} \cos(\chi_{\text{yaw}}) \sin(\chi_{\text{roll}}) \cos(\chi_{\text{roll}}) - (\mathbb{L}\mathbb{O} \sin(\chi_{\text{roll}})^2 - \mathbb{K}\mathbb{N}) \sin(\chi_{\text{pitch}}) \sin(\chi_{\text{yaw}})) \\
 &\quad \times \cos(\chi_{\text{pitch}}) + \mathbb{K}\mathbb{M}(\cos(\chi_{\text{roll}}) \sin(\chi_{\text{yaw}}) + \sin(\chi_{\text{roll}}) \sin(\chi_{\text{pitch}}) \cos(\chi_{\text{yaw}}))) / (\mathbb{K}\mathbb{L}), \\
 \frac{d\tilde{r}_{M,2}^{\mathcal{F}_{\text{lab}}}}{dt} &= ((2\mathbb{K}\mathbb{M} \cos(\chi_{\text{yaw}}) \cos(\chi_{\text{roll}}) \cos(\chi_{\text{pitch}})^2 + (\mathbb{K}\mathbb{J} \cos(\chi_{\text{yaw}}) \sin(\chi_{\text{pitch}}) \cos(\chi_{\text{roll}})^2 \\
 &\quad + \mathbb{P} \sin(\chi_{\text{yaw}}) \sin(\chi_{\text{roll}}) \cos(\chi_{\text{roll}}) + (\mathbb{L}\mathbb{O} \sin(\chi_{\text{roll}})^2 - \mathbb{K}\mathbb{N}) \sin(\chi_{\text{pitch}}) \cos(\chi_{\text{yaw}})) \\
 &\quad \times \cos(\chi_{\text{pitch}}) - \mathbb{K}\mathbb{M}(\cos(\chi_{\text{roll}}) \cos(\chi_{\text{yaw}}) - \sin(\chi_{\text{roll}}) \sin(\chi_{\text{pitch}}) \sin(\chi_{\text{yaw}}))) / (\mathbb{K}\mathbb{L}), \\
 \frac{d\tilde{r}_{M,3}^{\mathcal{F}_{\text{lab}}}}{dt} &= (-((\mathbb{J}\mathbb{K} \cos(\chi_{\text{roll}})^2 + \mathbb{L}\mathbb{O} \sin(\chi_{\text{roll}})^2) \cos(\chi_{\text{pitch}})^2 \\
 &\quad - 2\mathbb{K}\mathbb{M} \sin(\chi_{\text{pitch}}) \cos(\chi_{\text{roll}}) \cos(\chi_{\text{pitch}}) + \mathbb{K}\mathbb{N} \sin(\chi_{\text{pitch}})^2)) / (\mathbb{K}\mathbb{L}).
 \end{aligned}$$

The coefficients in these ODEs depend on the entries in resistance matrix:

$$\begin{aligned}
 \mathbb{A} &= ((C_{21}K_{23} - C_{31}K_{22})C_{13}^2 + (-C_{21}^2K_{33} + 2C_{21}C_{31}K_{23} - C_{31}^2K_{22} + \Omega_{11}(K_{22}K_{33} - K_{23}^2))C_{13} \\
 &\quad - \Omega_{33}K_{11}(C_{21}K_{23} - C_{31}K_{22}))\Omega_{22} + K_{11}(C_{21}K_{23} - C_{31}K_{22})\Omega_{23}^2 \\
 &\quad - 2\left((C_{21}K_{23} - C_{31}K_{22})C_{13} - \frac{1}{2}C_{21}^2K_{33} + C_{21}C_{31}K_{23} - \frac{1}{2}C_{31}^2K_{22} + \frac{1}{2}\Omega_{11}(K_{22}K_{33} - K_{23}^2)\right) \\
 &\quad \times C_{12}\Omega_{23} + \Omega_{33}C_{12}^2(C_{21}K_{23} - C_{31}K_{22}), \\
 \mathbb{B} &= -(C_{12}\Omega_{23} - C_{13}\Omega_{22})((-K_{22}K_{33} + K_{23}^2)\Omega_{11} + C_{21}^2K_{33} - 2C_{21}C_{31}K_{23} + C_{31}^2K_{22}), \\
 \mathbb{C} &= -(C_{21}K_{33} - C_{31}K_{23})(C_{13}^2 - K_{11}\Omega_{33})\Omega_{22} + C_{12}^2\Omega_{33} - 2C_{12}C_{13}\Omega_{23} + K_{11}\Omega_{23}^2, \\
 \mathbb{D} &= ((C_{13}^2 - K_{11}\Omega_{33})\Omega_{22} + C_{12}^2\Omega_{33} - 2C_{12}C_{13}\Omega_{23} + K_{11}\Omega_{23}^2)((-K_{22}K_{33} + K_{23}^2)\Omega_{11} \\
 &\quad + C_{21}^2K_{33} - 2C_{21}C_{31}K_{23} + C_{31}^2K_{22}), \\
 \mathbb{E} &= (-C_{12}\Omega_{33} + C_{13}\Omega_{23})(-C_{21}^2K_{33} + 2C_{21}C_{31}K_{23} - C_{31}^2K_{22} + \Omega_{11}(K_{22}K_{33} - K_{23}^2)) \\
 &\quad - (C_{21}K_{33} - C_{31}K_{23})((-C_{13}^2 + K_{11}\Omega_{33})\Omega_{22} - K_{11}\Omega_{23}^2 + 2C_{12}C_{13}\Omega_{23} - C_{12}^2\Omega_{33}), \\
 \mathbb{G} &= (C_{21}K_{33} - C_{31}K_{23})((-C_{13}^2 + K_{11}\Omega_{33})\Omega_{22} - K_{11}\Omega_{23}^2 + 2C_{12}C_{13}\Omega_{23} - C_{12}^2\Omega_{33}), \\
 \mathbb{J} &= -C_{21}^2 + K_{22}\Omega_{11}, \\
 \mathbb{K} &= (-C_{13}^2 + K_{11}\Omega_{33})\Omega_{22} - K_{11}\Omega_{23}^2 + 2C_{12}C_{13}\Omega_{23} - C_{12}^2\Omega_{33}, \\
 \mathbb{L} &= -C_{21}^2K_{33} + 2C_{21}C_{31}K_{23} - C_{31}^2K_{22} + \Omega_{11}(K_{22}K_{33} - K_{23}^2), \\
 \mathbb{M} &= -C_{21}C_{31} + K_{23}\Omega_{11}, \\
 \mathbb{N} &= -C_{31}^2 + K_{33}\Omega_{11}, \\
 \mathbb{O} &= \Omega_{22}\Omega_{33} - \Omega_{23}^2, \\
 \mathbb{P} &= (((-K_{11}K_{22} + K_{22}K_{33} - K_{23}^2)\Omega_{33} + C_{13}^2K_{22})\Omega_{22} + (K_{11}K_{22} - K_{22}K_{33} + K_{23}^2)\Omega_{23}^2 \\
 &\quad - 2C_{12}C_{13}K_{22}\Omega_{23} + C_{12}^2K_{22}\Omega_{33})\Omega_{11} + ((-C_{31}^2K_{22} + 2C_{21}C_{31}K_{23} + C_{21}^2(K_{11} - K_{33}))\Omega_{33} \\
 &\quad - C_{13}^2C_{21}^2)\Omega_{22} + ((K_{33} - K_{11})C_{21}^2 - 2C_{21}C_{31}K_{23} + C_{31}^2K_{22})\Omega_{23}^2 + 2C_{12}C_{13}C_{21}^2\Omega_{23} \\
 &\quad - C_{12}^2C_{21}^2\Omega_{33}.
 \end{aligned}$$

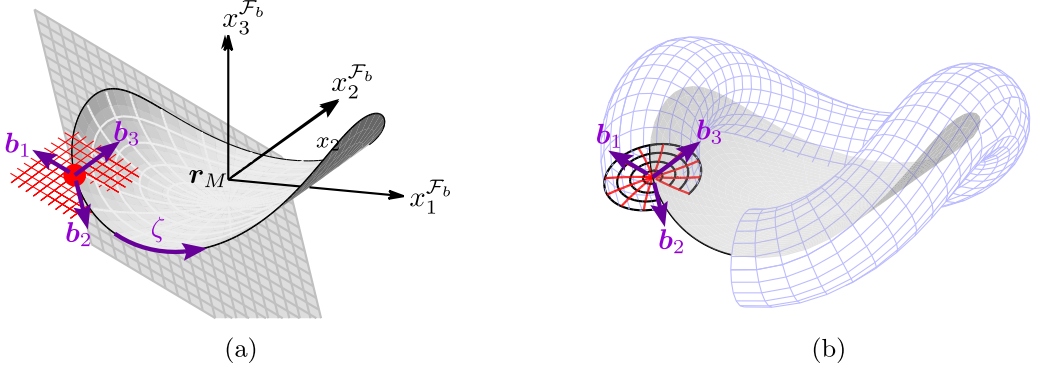


FIG. 7. (a) Illustration of the tangent plane to the disk whose edge is parameterized by the coordinate ζ . The three orthogonal unit vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are chosen such that $\mathbf{b}_2$ is tangential to the edge of the disk; $\mathbf{b}_1$ is tangential to the disk but normal to its edge; $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$ is normal to the disk and its edge. (b) Toroidal region within which the finite-element solution is augmented by suitable singular functions; red lines are contours of constant θ , and black circular lines and contours of constant ρ .

APPENDIX B: AUGMENTED FINITE-ELEMENT METHOD

The full details of the implementation and validation of the numerical method are given in Refs. [25,30]; here we give the motivation and a technical overview.

A flat object moving in a viscous fluid generates singularities in the pressure and shear stress along the sharp edge of the object [28,29]. This presents a challenge to a standard numerical approach, since these singularities contribute significantly to the hydrodynamic drag and torque acting on the body, but they cannot be fully resolved even if adaptive mesh refinement is applied.

To remedy this, we augment the standard finite-element basis functions with appropriate singular functions. For a smoothly deformed circular disk, such as the one shown in Fig. 7, the flow field in the vicinity of its curved edge, whose position we describe by the vector $\mathbf{r}_{\partial D}(\zeta)$, resembles the flow past the straight edge of a semi-infinite flat plate oriented tangential to the disk, as shown in Fig. 7(a). We therefore introduce a body-fitted toroidal coordinate system (ζ, ρ, θ) that is aligned with the edge of the disk and defined relative to the three orthogonal unit vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. Here $\mathbf{b}_2 = (\mathbf{dr}_{\partial D}/d\zeta)/|\mathbf{dr}_{\partial D}/d\zeta|$ is tangential to the edge of the disk; $\mathbf{b}_1$ is tangential to the disk but normal to its edge; $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$ is normal to the disk and its edge. The coordinates are chosen so that $\rho \geq 0$ and $0 \leq \theta \leq 2\pi$ describe the position within a radial slice whose normal is $\mathbf{b}_2$, as shown in Fig. 7(b). In a finite toroidal region surrounding the edge of the disk we then represent the velocities and pressure as

$$\{\mathbf{u}, p\} = \{\mathbf{u}^{[\text{FE}]}, p^{[\text{FE}]}\} + \sum_{i=1}^3 C_i(\zeta) \{\mathbf{u}_i^{[\text{sing}]}(\rho, \theta), p_i^{[\text{sing}]}(\rho, \theta)\}, \quad (\text{B1})$$

where the $\mathbf{u}_i^{[\text{sing}]}, p_i^{[\text{sing}]}$ are the velocity and pressure fields arising from a translation of a (planar straight) edge with unit velocity in the local $\mathbf{b}_i$ direction; they are scaled by the a-priori unknown amplitudes $C_i(\zeta)$. Outside this region, the velocities and pressures were represented by the finite-element basis functions alone, with continuity across the interface between the augmented and nonaugmented regions imposed by Lagrange multipliers. We expanded the amplitudes $C_i(\zeta)$ using one-dimensional Hermite finite elements and determined their discrete amplitudes by formulating the entire problem as a PDE-constrained optimization problem which ensured that within the augmented region, the finite-element part of the solution $\{\mathbf{u}^{[\text{FE}]}, p^{[\text{FE}]}\}$ was as smooth as possible. This was done by minimizing the L^2 -norm of the inter-element jump in the traction associated

with the finite-element part of the solution across the faces of adjacent tetrahedral elements in the augmented region.

APPENDIX C: FIXED POINTS

The fixed points in the $(\chi_{\text{pitch}}, \chi_{\text{roll}})$ -phase plane are determined by the inclinations $(\chi_{\text{pitch}}^*, \chi_{\text{roll}}^*)$ for which Eqs. (A12) and (A13) vanish simultaneously. Given the product form of Eq. (A13), the fixed points come in two families. For the first family, Eq. (A13) vanishes because $\sin(\chi_{\text{roll}}^*) = 0$. The corresponding solution of Eq. (A12) depends on whether χ_{roll}^* is an even or odd multiple of π , yielding the two sets of fixed points

$$(\chi_{\text{pitch}}^{*,I+}, \chi_{\text{roll}}^{*,I+}) = \left(\arctan\left(\frac{\mathbb{A} + \mathbb{B}}{\mathbb{C}}\right) + \pi n, (2m + 1)\pi \right) \quad (\text{C1})$$

and

$$(\chi_{\text{pitch}}^{*,I-}, \chi_{\text{roll}}^{*,I-}) = \left(-\arctan\left(\frac{\mathbb{A} + \mathbb{B}}{\mathbb{C}}\right) + \pi n, 2m\pi \right) \quad (\text{C2})$$

for any integers n, m . For $R_{\min} = R_{\max}$, these fixed points are the saddle points represented by the red circles in Fig. 2.

For the other family of solutions, Eq. (A13) vanishes because its second factor is zero. This yields a further two sets of fixed points

$$(\chi_{\text{pitch}}^{*,II\pm}, \chi_{\text{roll}}^{*,II\pm}) = (\pm \arctan(ab) + n\pi, \arccos(\pm b) + 2m\pi), \quad (\text{C3})$$

and their symmetric counterparts at $(\chi_{\text{pitch}}^{*,II\pm}, -\chi_{\text{roll}}^{*,II\pm})$, again for any integers n, m . Here the coefficients a and b are given by

$$a = \frac{(\mathbb{G} + \mathbb{E})\mathbb{C} - \mathbb{B}\mathbb{A} - \sqrt{[(\mathbb{G} + \mathbb{E})\mathbb{C} - \mathbb{B}\mathbb{F}]^2 + 4\mathbb{A}\mathbb{B}\mathbb{G}(\mathbb{G} + \mathbb{E})}}{2\mathbb{B}\mathbb{G}} \quad (\text{C4})$$

and

$$b = \sqrt{-\frac{\mathbb{B}}{\mathbb{A} + a\mathbb{C}}}. \quad (\text{C5})$$

For $R_{\min} = R_{\max}$, these fixed points are the centers represented by the crosses in Fig. 2.

The pitchfork bifurcation occurs when the two fixed points described by Eq. (C3) approach their symmetric counterparts, i.e., when $\chi_{\text{roll}}^{*,II+} \rightarrow 0$ and $\chi_{\text{roll}}^{*,II-} \rightarrow \pi$. This occurs when $b \rightarrow 1$ which implies that $a \rightarrow -(\mathbb{A} + \mathbb{B})/\mathbb{C}$. Hence, $(\chi_{\text{pitch}}^{*,II+}, \pm \chi_{\text{roll}}^{*,II+}) \rightarrow (\chi_{\text{pitch}}^{*,I-}, \chi_{\text{roll}}^{*,I-})$ and $(\chi_{\text{pitch}}^{*,II-}, \pm \chi_{\text{roll}}^{*,II-}) \rightarrow (\chi_{\text{pitch}}^{*,I+}, \chi_{\text{roll}}^{*,I+})$, corresponding to the coalescence between the two stable nodes and the saddle point described in the main part of the paper.

APPENDIX D: EXPERIMENTAL SETUP

The full details of the experimental procedure are provided in Refs. [24,31,34]; here we therefore only provide a brief overview.

We manufactured the disk by precision-cutting a thin polyamide nylon sheet (Goodfellow; thickness $236.7 \pm 0.3 \mu\text{m}$, density $\rho_d = 1130 \text{ kg m}^{-3}$, refractive index $n_d = 1.53$) into a circular sample of radius $R = 12.00 \pm 0.05 \text{ mm}$ using a lathe. The sheet was then thermoformed into a U-shaped profile with a central radius of curvature $0.5R$ by pressing it into a milled aluminium mould, heated to 200°C . Since we had to be able to withdraw the punch from the die, the mould had a semi-circular depth profile which resulted in two small areas of the disk remaining outside the mould and thus retaining their original flat shape. The “pinched” geometry was then produced by manually pinching the U-shaped sheet to generate an approximately linear variation in curvature,

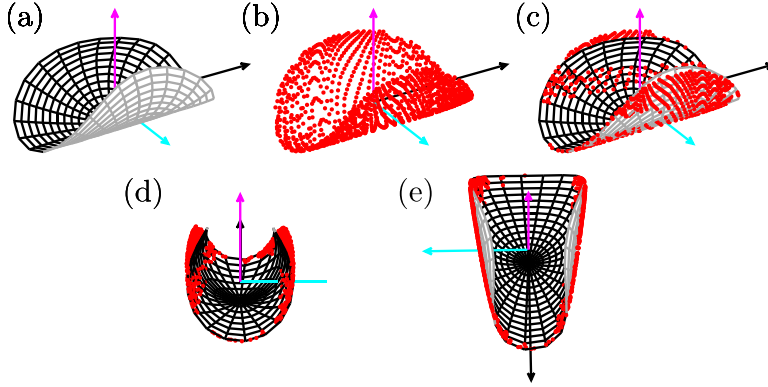


FIG. 8. Comparison of the ideal conical disk (black grid) and the geometry of the disk used in the experiment (red point cloud), for the comparison presented in Fig. 6. Arrows show the body-fitted axes: pitch (cyan), roll (black) and yaw (magenta). (a) Idealized conical geometry, with $R_{\min} = 0.32$, $R_{\max} = 0.6$; (b) point cloud of the experimental shape computed from image analysis (see Refs. [24,31] for details); (c)–(e) ideal conical shape overlaid with the experimental point cloud, showing three orientations for clarity.

with the two ends having radii of $R_{\min} = 0.32R$ and $R_{\max} = 0.60R$; the two small flat parts retained their flat shape during this process. Figure 8 shows a direct comparison between the shape of the particle used in the experiments and the ideal conical shape used in the theoretical/computational analysis.

We investigated the sedimentation of the disk by immersing it in 1000 cSt silicone oil (Allcosil, J. Allcock & Sons) with a dynamic viscosity of $\mu = 1.02 \pm 0.01$ Pa s, density $\rho_f = 972.7 \pm 0.5$ kg m $^{-3}$ and refractive index $n_f = 1.403$ at 22 ± 1 °C, within a Perspex tank of internal dimensions $90 \times 40 \times 40$ cm 3 , filled to a height of 75 cm. Before each experiment, the silicone oil was homogenized using a two-blade propeller to eliminate small density variations caused by residual temperature gradients and air entrapment. The disk was placed approximately 10 cm below the free surface using a meter-long spatula to minimise boundary effects, and its initial orientation was adjusted manually with a 20 cm rod. Image acquisition from the top-view camera commenced 20 s after rod removal, ensuring that all induced fluid motion had subsided. Since the oil’s refractive index was closely matched to that of the disk, the particle was rendered nearly invisible under ambient illumination. To visualise its sedimentation, the disk was illuminated by a high-definition projector (Optoma HD143X) positioned laterally to the tank, which projected oblique light sheets. Rayleigh scattering at the intersections between the light plane and the disk surface revealed the outline of the disk within the illuminated cross-section. The scattered light pattern was recorded from above using a JAI GO-5000M-USB camera fitted with a Kowa LM16HC lens (RMA Electronics) in a 768×768 pixel 2 window, with the camera placed 25.9 cm above the liquid surface. The projector cast 20 stationary planes of light each 290 μ m thick and spaced 29.5 mm apart at the tank’s center, so that at any given time the disk intersected only one light plane. As the disk settled through the fluid, the scattered light captured at 15 frames per second provided a sequence of optical cross-sections used to reconstruct the disk’s evolving three-dimensional orientation as well as the position of its center of mass. The reconstruction was achieved using a neural autoencoder with orientational accuracy of $\pm 1^\circ$ [31,34].

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