

Numerical study on rheology of emulsions and bubbly suspensions with elastoviscoplastic matrix fluids in simple shear

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We investigate the rheology of biphasic fluid systems comprising an elastoviscoplastic (EVP) matrix fluid containing initially spherical, monodisperse, viscous droplets through interface-resolved numerical simulations. The interface is captured using the level-set method, and the EVP phase is modeled with the Saramito constitutive equation. We explore the effects of the volume fraction, capillary, Weissenberg, and Bingham numbers, considering both dilute and semidilute regimes (dispersed phase volume fractions from 0.16% to 20%) at two density and viscosity ratios. The constitutive curve shows a negative curvature, consistent with previous studies on emulsion rheology with coalescing drops. Increasing the capillary and Bingham numbers show qualitatively similar trends: increasing drop deformation with both results in a decrease of the system's relative viscosity. This is further supported by stress budget analysis, which shows that yield stress increases the effective viscosity of the matrix fluid, thereby increasing the effective capillary number and reducing the effective viscosity ratios. The bulk rheology shows a complex relationship with the Weissenberg number (Wi). In dilute systems, droplet deformation increases with Wi , while bubbles exhibit a nonmonotonic trend. Relative viscosity rises with Wi for emulsions up to $Wi = 1.75$ due to localized EVP stresses, then slightly decreases. For bubbly suspensions, the viscosity shows little variation at low Wi but decreases at larger values. At semidilute concentrations, similar trends emerge with EVP stresses dictating the system's bulk rheology. At high Wi , increased EVP stresses, reduced coalescence, larger

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interfacial stress, and near-wall droplet migration collectively increase emulsion relative viscosity.

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I. INTRODUCTION

Yield stress fluids are ubiquitous in industrial and natural environments, ranging from industrial processes such as the processing of foods, polymers, pharmaceuticals, and cosmetic products, to naturally occurring in geophysical flows such as avalanches and mudslides [1]. Mechanically, they behave such that they do not flow when subject to stresses below a critical threshold, the yield stress, τ_y . When yield stress fluids show elastic behavior, prior to and upon yielding, they are commonly referred to as elastoviscoplastic (EVP).

Emulsions containing yield stress fluids arise in numerous industrial applications. Liquid-liquid emulsions can be leveraged for encapsulation to assist in coating and drug delivery, and are also encountered in food, cosmetic, and pharmaceutical processing [2–4]. Bubbly suspensions of yield stress fluids likewise occur in many industrial processes: concrete slurry is aerated to impart favorable density, thermal, acoustic, and fire resistance properties; foamy cements and drilling fluids are used in well construction by the oil industry; chocolate and dairy products are aerated to impart favorable textural properties. Conversely, air bubbles can become undesirably entrained during mixing processes and degrade the material’s mechanical and thermal properties.

Despite the prevalence of yield stress fluids in these multiphase systems, most existing studies have focused on suspensions and emulsions comprising Newtonian carrier fluids, while studies on viscoelastic fluids have largely investigated the dynamics of a single droplet. Therefore, this introduction mainly addresses the existing literature on suspensions with Newtonian carrier fluids, followed by non-Newtonian effects primarily on single-droplet dynamics.

Dispersing particles or a second fluid phase throughout a carrier (or matrix) fluid significantly alters the rheological properties of the material. The pioneering work of Einstein [5,6] showed that, for a suspension of rigid, spherical, noncolloidal particles in a Newtonian matrix fluid at dilute concentrations (i.e., no particle interactions), the effective viscosity, $\tilde{\mu}_{\text{eff}}$, increases linearly with the dispersed phase volume fraction, ϕ , such that

$$\mu_r = \frac{\tilde{\mu}_{\text{eff}}}{\tilde{\mu}(\phi = 0)} = 1 + \frac{5}{2}\phi, \quad (1)$$

where $\tilde{\mu}(\phi = 0)$ denotes the viscosity of the matrix fluid, and μ_r is the relative viscosity (note that dimensional quantities are indicated with the tilde notation $\tilde{\bullet}$ in this work and dimensionless quantities without). Subsequent extensions to Einstein’s formula have added higher-order terms to account for effects such as semidilute and concentrated volume fractions (where particle interactions become significant), particle elasticity, nonspherical particle shapes, hydrodynamic interactions, polydispersity, and electrostatic effects [7–16].

For biphasic emulsions, the focus of the present work, Taylor [17] derived an analogous relation to Einstein’s formula for the effective viscosity of a biphasic emulsion comprising rigid, viscous immiscible drops dispersed in a viscous matrix fluid,

$$\mu_r = 1 + \frac{1 + \frac{5}{2}k_\mu}{1 + k_\mu}\phi, \quad (2)$$

where $k_\mu = \tilde{\mu}_2/\tilde{\mu}_1$ is the drop-to-matrix viscosity ratio, and the subscripts “1” and “2” denote the matrix and dispersed phases, respectively. In the limit of $k_\mu \rightarrow \infty$, the dispersed phase exhibits particle-like behavior and Einstein’s relation [Eq. (1)] is recovered. Conversely, in the limit $k_\mu \rightarrow 0$ (more representative of bubbles), we obtain

$$\mu_r = 1 + \phi. \quad (3)$$

Thus, the inclusion of stiff bubbles increases the viscosity of the fluid due to the additional resistance to flow by interfacial tension and distortion of streamlines surrounding the bubble, which locally increase the strain rate and energy dissipation. The difference with Einstein's relation for solid particles ($k_\mu \rightarrow \infty$), or for a matched viscosity biphasic emulsion ($k_\mu = 1$), is evident: a suspension of rigid bubbles ($k_\mu \rightarrow 0$) increases the relative viscosity to a lesser degree than rigid particles or viscous drops. This is due to the free-slip condition on the bubble surface, reducing the local velocity gradient and dissipation rate around the interface [18], whereas liquid droplets or solid particles produce a stronger perturbation to the surrounding flow.

In contrast, the relative viscosity of a dilute suspension of fully deformable bubbles varies with the volume fraction according to [19]

$$\mu_r = 1 - \frac{5}{3}\phi. \quad (4)$$

It follows from Eq. (4) that inclusions of fully deformable bubbles decrease the relative viscosity. Between the two limiting cases of rigid bubbles [Eq. (3)] and fully deformable bubbles [Eq. (4)], the relative viscosity of the suspension depends on the bubble's deformability. This is characterized by the capillary number, Ca,

$$\text{Ca} = \frac{\tilde{\mu}\tilde{L}_{\text{ref}}\tilde{\gamma}}{\tilde{\sigma}}, \quad (5)$$

which expresses the ratio of the characteristic viscous stress to the interfacial stress, where $\tilde{\gamma}$ denotes a characteristic shear rate of the flow, $\tilde{L}_{\text{ref}}$ a characteristic length scale, and $\tilde{\sigma}$ the interfacial tension. Several theoretical frameworks have been developed to quantify the effect of deformability and volume fraction on suspension rheology. Constitutive equations have been proposed by Rust and Manga [18], Llewellyn [20], Pal [21], and Ohie *et al.* [22], building on the seminal works of Oldroyd [23] and Frankel and Acrivos [24] to predict the rheological properties of bubbly suspensions with intermediate deformability between rigid and fully deformable bubbles.

Owing to the complex dynamics present even in a seemingly simple system comprising a single sheared droplet, and the additional complexity introduced when multiple droplets are present at concentrations where interactions become significant, interface-resolved simulations have emerged as a useful tool for studying the dynamics and rheology of such systems. Access to fully resolved motion and the morphology of the dispersed phase, together with the surrounding flow field, provides valuable insights into the internal structures and dynamics within the suspension that give rise to the observed bulk rheology. Several early studies of a single sheared viscous droplet in planar Couette flow investigated the droplet's transient deformation and breakup with varying capillary number and viscosity ratios [25–28]. The droplet's increasing deformation with shear together with the anisotropy of its deformed shape is found to give rise to shear-thinning viscosity and normal stress differences [25,26].

Beyond dilute concentrations, hydrodynamic interactions between the dispersed phase further influence the dynamics of the biphasic system and its bulk rheology. Zhou and Pozrikidis [29] used the boundary element method to conduct two-dimensional simulations of concentrated emulsions of viscous drops in a viscous matrix fluid. Subsequently, three-dimensional simulations were performed by Loewenberg and Hinch [30] using the same technique. Both studies clearly identified shear-thinning behavior in systems with deformable (high Ca) drops, together with the emergence of normal stress differences arising from anisotropic droplet deformation. Additionally, their results reveal the temporal variations in the shear stress and normal stress differences, highlighting the unsteady nature of the bulk rheology of these systems beyond dilute concentrations. Hydrodynamic interactions between the dispersed phase cause the effective viscosity to deviate from the linear variation with volume fraction at dilute concentrations: it has been shown to increase superlinearly for stiff droplets [31,32], remain approximately linear for deformable droplets [30], or show a nonmonotonic variation for coalescing droplets [33–35].

Droplet coalescence also influences the emulsion's bulk rheology, and its effect cannot be neglected in biphasic emulsions [29,34,35] that are not stabilized by some means (e.g., surfactants).

Rosti *et al.* [34] studied the rheology of emulsions with coalescing drops as a function of volume fraction and capillary number using interface-resolved simulations based on a volume-of-fluid (VOF) method. Their results revealed a nonmonotonic variation of the effective viscosity with ϕ , with an initial increase in μ_r followed by a subsequent decrease due to increased interactions and drop coalescence. De Vita *et al.* [35] studied the effect of coalescence on emulsion rheology using a VOF method and compared cases in which numerical coalescence was allowed to occur with cases where it was suppressed through local repulsive collision forces. Their analysis showed that coalescence decreases μ_{eff} due to a reduction in the interfacial stress contribution to the wall shear stress budget, and they observed the formation of droplet-laden wall layers at high volume fractions when coalescence was suppressed.

Hydrodynamic interactions can further lead to migration and the formation of organized structures of the dispersed phase, creating a heterogeneous spatial distribution. Zhou and Pozrikidis [29] studied the dynamics of concentrated emulsions under shear, finding a critical capillary number which demarcates the persistence of a stable lattice structure which shows a periodic motion, from its breakdown heralded by coalescence onset. Droplet interactions can result in the formation of wall layers, while shear-induced lift forces and diffusion can lead to migration towards the channel center in Couette flows [35,36]. Caserta and Guido [33] observed the formation of drop structures aligned in the flow direction and alternating droplet-rich and depleted regions in the vorticity direction (“vorticity banding”), accompanied by a reduction in effective viscosity. De Vita *et al.* [37] studied the stability of these bands for coalescing and noncoalescing drops with interface-resolved VOF simulations. Shear-induced diffusion in the shear-gradient direction of a concentrated, droplet-rich fringe has also been observed [36], and is attributed to shear-induced close-range droplet interactions [38]. The recent work by Giroto *et al.* [39] on concentrated emulsions provides a promising framework for analyzing droplet-wise Lagrangian statistics to bridge bulk rheology observations with the underlying Lagrangian dynamics and interactions of individual droplets.

Summarizing the studies on biphasic emulsions comprising Newtonian fluids, the emulsion rheology is a function of the dispersed phase concentration, the viscosities of the constituent fluids, the interfacial tension, hydrodynamic interactions between the dispersed phase, inertia [40], and the size and spatial distribution of the dispersed phase. The recent review by Reboucas *et al.* [41] provides a comprehensive overview of the current state of theoretical and numerical approaches towards modeling drop deformation and emulsion rheology.

When one or more of the constituent fluids in an emulsion (or bubbly suspension) exhibits viscoelasticity, the dynamics are additionally influenced by elastic stresses. Several studies have investigated the role of viscoelasticity on the deformation and dynamics of single droplets. By contrast, emulsions involving EVP fluids—materials that simultaneously exhibit elastic, viscous, and plastic behavior—have been far less studied. To contextualize this, we briefly review some related works on single-droplet dynamics in viscoelastic and EVP systems.

Viscoelasticity—of either the matrix or dispersed phase—has been shown to influence droplet deformation. Some early studies [42–44] reported an increase in droplet deformation in a viscoelastic matrix, while others [45] reported the opposite trend. It was later understood that matrix viscoelasticity has a nonmonotonic effect on drop deformation: deformation is suppressed at low elasticity, and increases at higher elasticity [46]. The opposite configuration—a viscoelastic droplet sheared in a Newtonian matrix fluid—has also been investigated numerically in several studies [46–49]. For an Oldroyd-B droplet, the deformation decreases with increasing elasticity, while its orientation angle increases in viscosity-matched systems. Mukherjee and Sarkar [49] further showed that the deformation increases again at large droplet-to-matrix viscosity ratios.

Izbassarov and Tammisola [50] investigated the dynamics of an EVP drop sheared in a Newtonian matrix using the Saramito constitutive equation [51]. At low yield stress, the drop deformation showed a nonmonotonic trend with increasing elasticity, whereas at higher yield stress the deformation increased monotonically with elasticity. A regime map of the Weissenberg and Bingham numbers showed various un-yielded region morphologies within the droplet. Zhang *et al.* [52] subsequently investigated the influence of the non-Newtonian rheology on the deformation and break-up

modes of an EVP droplet sheared in a viscous matrix. Unyielded regions within the drop show a viscous, hyperelastic solid behavior according to the Saramito model, which can contribute to an increase in the suspension's effective viscosity as shown by the analysis of Rosti and Brandt [53].

Migration and the formation of organized structures of the dispersed phase can further be induced by viscoelastic stresses, thereby influencing its spatial distribution. Mukherjee and Sarkar [54] found that matrix viscoelasticity results in a tensile stress pulling droplets towards the wall in a simple shear flow owing to the different streamline curvatures on either side of the droplet. This elastic hoop stress counteracts the wall-lift force and encourages the formation of droplet-laden wall layers. Feneuil *et al.* [55] observed the formation of long bubble chains when shearing a bubbly suspension in an EVP matrix fluid, similar to the particle chaining that occurs in viscoelastic fluids [56]. Furthermore, their numerical simulations showed that increasing the elasticity of the matrix phase inhibited bubble-pair motion in the vorticity direction.

To summarize, a substantial body of literature has investigated the dynamics and rheology of suspensions of droplets or bubbles with Newtonian carrier fluids, revealing numerous phenomena that are dependent on the fluid properties, the deformability and concentration of the dispersed phase, and interactions between the dispersed phase, among others. Viscoelastic effects further influence droplet dynamics and result in migration behaviors and dispersed phase structures that are distinct from those in Newtonian systems; elastoviscoplasticity introduces additional complexity as elastic, viscous, and yielding effects simultaneously occur. However, the dynamics of such systems at concentrations beyond the dilute limit has received comparatively less attention; studies with EVP matrix fluids in such multiphase systems have primarily focused on a single droplet, or on droplet or bubble pairs. Elucidating the internal dynamics and dispersed phase structures that arise in these systems is essential to develop a fundamental understanding of how these influence their bulk rheology.

To this end, interface-resolved simulations offer a powerful approach as they provide fully resolved, three-dimensional flow and stress fields. This is particularly useful for such systems in which non-Newtonian stress fields arise that are not straightforward to experimentally access. This motivates the aim of the present work: we investigate the rheology of biphasic emulsions and bubbly suspensions with EVP matrix phases at both dilute and semidilute concentrations through interface-resolved numerical simulations. Our objective is to investigate how the deformability and viscosity of the dispersed phase, together with the EVP rheology of the matrix phase, influence the system's bulk rheology.

The remainder of this paper is organized as follows: in Sec. II the problem setup and numerical methodology are described. In Sec. III the results are presented and the effects of dispersed-phase deformability and viscosity, as well as the EVP rheology of the matrix phase, on the bulk rheology are discussed. Finally, the main conclusions are summarized in Sec. IV.

II. PROBLEM SETUP AND METHODOLOGY

A. Problem setup

Our problem setup comprises an EVP matrix fluid dispersed with initially spherical, monodisperse viscous droplets (or bubbles). The system is driven by a planar Couette flow generated by two bounding walls moving in opposite directions. Letting $\tilde{R}$ denote the initial droplet (or bubble) radius, we use a computational domain of size $\tilde{L}_x \times \tilde{L}_y \times \tilde{L}_z = 16\tilde{R} \times 10\tilde{R} \times 16\tilde{R}$ in the streamwise (x), wall-normal (or shear-gradient, y), and vorticity (z) directions, respectively. The walls move with velocity $\pm \tilde{u}_w \hat{e}_x$, imposing a nominal shear rate $\dot{\gamma}_0 = 2\tilde{u}_w/\tilde{L}_y$ on the fluid. A sketch of the setup and computational domain is illustrated in Fig. 1.

The dynamics of our problem are governed by the following dimensionless numbers:

$$\text{Re} = \frac{\tilde{\rho}_1 \dot{\gamma}_0 \tilde{R}^2}{\tilde{\mu}_{s,1} + \tilde{\mu}_{p,1}}, \quad \text{Ca} = \frac{(\tilde{\mu}_{s,1} + \tilde{\mu}_{p,1}) \tilde{R} \dot{\gamma}_0}{\tilde{\sigma}},$$

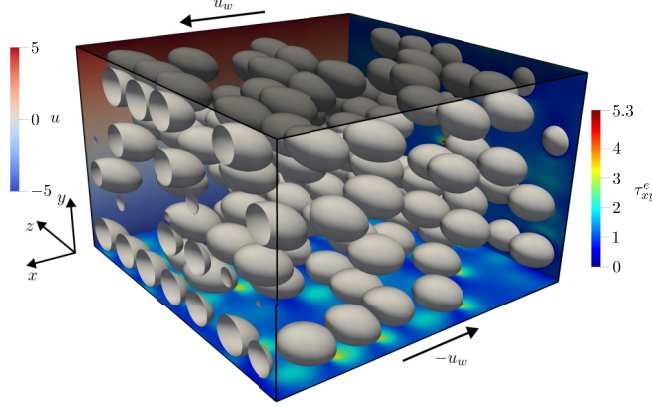


FIG. 1. Schematic of the computational domain and coordinate system. The case visualized shows a volume fraction $\phi = 0.2$.

$$\phi = \frac{4/3\pi\tilde{R}^3N}{\tilde{L}_x\tilde{L}_y\tilde{L}_z}, \quad k_\rho = \frac{\tilde{\rho}_2}{\tilde{\rho}_1}, \quad k_\mu = \frac{\tilde{\mu}_2}{\tilde{\mu}_{s,1} + \tilde{\mu}_{p,1}},$$

$$\text{Wi} = \tilde{\lambda}_1\tilde{\gamma}_0, \quad \beta = \frac{\tilde{\mu}_{s,1}}{\tilde{\mu}_{s,1} + \tilde{\mu}_{p,1}}, \quad \text{Bn} = \frac{\tilde{\tau}_y}{(\tilde{\mu}_{s,1} + \tilde{\mu}_{p,1})\tilde{\gamma}_0},$$

where Re is the Reynolds number, Ca is the capillary number, ϕ is the dispersed phase volume fraction with N the initial number of droplets (or bubbles), k_ρ is the dispersed-to-matrix phase density ratio, k_μ is the dispersed-to-matrix phase total viscosity ratio, Wi is the Weissenberg number with $\tilde{\lambda}_1 = \tilde{\mu}_{p,1}/\tilde{G}_1$ the relaxation time, β is the solvent-to-total viscosity ratio in the EVP matrix, and Bn is the Bingham number. Here the tilde notation $\tilde{\bullet}$ is used to denote dimensional material properties, and the subscripts “1” and “2” pertain to the matrix and dispersed phase, respectively. The reference scales used to nondimensionalize our setup are as follows: lengths are scaled with the reference length scale $\tilde{L}_{\text{ref}} = \tilde{R}$, velocities are scaled with $\tilde{u}_{\text{ref}} = \tilde{R}\tilde{\gamma}_0$, time with $\tilde{t}_{\text{ref}} = 1/\tilde{\gamma}_0$, and all stresses (pressure, viscous stresses, and the EVP extra stresses) are scaled with the characteristic viscous stress, $\tilde{\tau}_{\text{ref}}^v = (\tilde{\mu}_{s,1} + \tilde{\mu}_{p,1})\tilde{\gamma}_0$. The reference velocity scale $\tilde{u}_{\text{ref}} = \tilde{R}\tilde{\gamma}_0$ is chosen to represent a characteristic velocity around the droplet. Following prior studies [25,30,35,40], scaling with the initial droplet radius characterizes the droplet-scale physics and parametrizes their influence on the bulk rheology, i.e., the droplet Reynolds number characterizes droplet-scale inertial effects [40], and the capillary number quantifies droplet deformability [18,25,29,30].

The present study is performed at a fixed Reynolds number $\text{Re} = 0.1$, chosen so that inertial effects are negligible [35,40], and at a fixed $\beta = 0.09$. We conduct parametric studies over $\phi = \{0.0016, 0.05, 0.1, 0.15, 0.2\}$, $\text{Ca} = \{0.1, 0.2, 0.3, 0.4\}$, $\text{Wi} = \{0.1, 0.5, 1, 2\}$ and $\text{Bn} = \{0.1, 0.25, 0.5, 0.75, 1\}$. These studies are performed for two density and viscosity ratios, $k_\rho = k_\mu = 1$ and $k_\rho = k_\mu = 1/40$; the former corresponds to a matched-viscosity system representative of liquid-liquid emulsions, and the latter, with $k_\mu \ll 1$ is chosen to reproduce some salient features of bubbly suspensions. The maximum value of Ca and Bn are chosen such that a single sheared droplet does not break up. Buoyancy is neglected so that we may focus on how the matrix-phase rheology and the dispersed-phase deformability and viscosity influence the system’s bulk rheology; this assumption is appropriate for the low Galilei number flows considered, as discussed further in Sec. II C. Simulations are advanced until a statistically steady state is reached. For dilute conditions, we monitor the temporal evolution of drop deformation and wall shear stress, while for semidilute concentrations we track the evolution of the interfacial area and wall shear stress. Typically this required $t \approx 20$ for the dilute case and $t \approx 100$ for semidilute concentrations (see Appendix D for some selected cases).

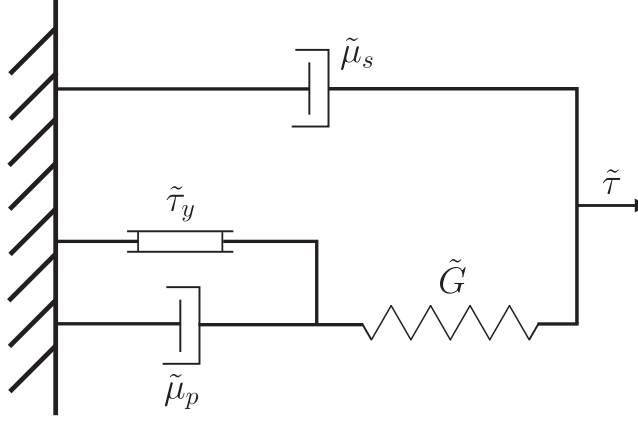


FIG. 2. Mechanical model of the one-dimensional Saramito constitutive equation.

B. Governing equations and numerical methodology

We study the dynamics of emulsions and bubbly suspensions in planar Couette flow through interface-resolved numerical simulations of the governing equations. The incompressible Navier-Stokes equations govern the fluid dynamics of our problem. In the dimensionless form, they comprise the continuity equation,

$$\nabla \cdot \mathbf{u} = 0, \quad (6)$$

and the momentum conservation equation in the one-fluid formulation for two immiscible fluids,

$$\rho \frac{D\mathbf{u}}{Dt} = \frac{1}{\text{Re}} \left[-\nabla p + \nabla \cdot (2\mu \mathbf{D}) + \nabla \cdot \boldsymbol{\tau}^e + \frac{1}{\text{Ca}} \kappa \delta(\psi) \mathbf{n} \right], \quad (7)$$

where $\mathbf{u}$ denotes the velocity vector, ρ the dimensionless density, μ a dimensionless Newtonian viscosity in both fluid phases, p the pressure, $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ the strain rate tensor, and $D\mathbf{u}/Dt \equiv \partial \mathbf{u} / \partial t + \mathbf{u} \cdot \nabla \mathbf{u}$ denotes the material derivative. The third term on the right-hand side of Eq. (7) accounts for the extra EVP stresses in the matrix phase; the fourth term accounts for stresses due to surface tension, localized at the fluid interface through a Dirac delta function $\delta(\psi)$ of the level-set field ψ , acting in the direction of the local curvature normal, $\mathbf{n}$, with κ the local dimensionless curvature.

To model the matrix phase as an EVP fluid, we use the Saramito constitutive equation [51],

$$\text{Wi} \overset{\nabla}{\boldsymbol{\tau}^e} + \max \left(0, \frac{|\boldsymbol{\tau}_d^e| - \text{Bn}}{|\boldsymbol{\tau}_d^e|} \right) \boldsymbol{\tau}^e = 2(1 - \beta) \mathbf{D}, \quad (8)$$

where $|\boldsymbol{\tau}_d^e| = \sqrt{\frac{1}{2} \boldsymbol{\tau}_d^e : \boldsymbol{\tau}_d^e}$, with $\boldsymbol{\tau}_d^e = \boldsymbol{\tau}^e - \frac{1}{3} \text{tr} \boldsymbol{\tau}^e \mathbf{I}$ the deviatoric part of the EVP extra stress. $\overset{\nabla}{\boldsymbol{\tau}^e}$ denotes the upper-convected derivative

$$\overset{\nabla}{\boldsymbol{\tau}^e} \equiv \frac{\partial \boldsymbol{\tau}^e}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\tau}^e - (\nabla \mathbf{u}) \boldsymbol{\tau}^e - \boldsymbol{\tau}^e (\nabla \mathbf{u}^T). \quad (9)$$

The function $F = \max(0, (|\boldsymbol{\tau}_d^e| - \text{Bn})/|\boldsymbol{\tau}_d^e|)$ models the yielding behavior of the EVP material: when $F = 0$, the material deforms as a viscoelastic Kelvin-Voigt solid, and when $F \gg 0$, the material flows as a viscoelastic fluid. A mechanical model of the one-dimensional Saramito constitutive equation is illustrated in Fig. 2.

Last, a numerical method is required to resolve the fluid-fluid interface. Here the level-set method [57] is used to capture the interface between the fluid phases, governed by an advection equation of

the level-set function, ψ ,

$$\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla \psi = 0, \quad (10)$$

where ψ is the signed distance from the interface. In this work, we adopt the convention that $\psi > 0$ in the matrix fluid, $\psi < 0$ in the dispersed fluid, and $\psi = 0$ at the interface. We reinitialize ψ every 20 time steps in order to preserve its signed-distance property by reshaping ψ into a signed-distance function through solution of a time-dependent Hamilton-Jacobi equation,

$$\frac{\partial \psi}{\partial \mathcal{T}} + S(\psi_0)(|\nabla \psi| - 1) = 0 \quad (11)$$

In Eq. (11), $S(\psi_0)$ denotes the mollified sign function of the initial level-set field ψ_0 . Equation (11) is solved iteratively in pseudo-time, $\mathcal{T}$, until the residual $|\psi - \psi_0|$ falls below a threshold Δx^3 , where Δx is the grid spacing; this typically requires a few time steps. We also employ the mass correction scheme proposed by Ge *et al.* [58], which is particularly important here due to the long integration times required for the semidilute cases ($t \approx 100$) to reach a statistically stationary state.

We use the one-fluid formulation to solve the governing equations, in which a single Navier-Stokes equation is solved over the entire computational domain. The variation in the material properties across the fluid phases is captured using spatially varying material property fields,

$$\Theta(\mathbf{x}, t) = \Theta_1 H(\psi) + \Theta_2 [1 - H(\psi)], \quad (12)$$

where Θ denotes some material property (ρ , μ , G , μ_p , τ_y , ...), and the subscripts “1” and “2” refer to the matrix and dispersed phase, respectively. The notation $\Theta(\mathbf{x}, t)$ is used to emphasize that Θ is a spatially varying, time-dependent material property field, which is constructed based on a regularized Heaviside function of the level-set field, $H(\psi)$. In our problem setup, we have the following dimensionless material property fields:

$$\begin{aligned} \rho(\mathbf{x}, t) &= H(\psi) + k_\rho [1 - H(\psi)], \\ \mu(\mathbf{x}, t) &= \beta H(\psi) + k_\mu [1 - H(\psi)], \\ \mu_p(\mathbf{x}, t) &= (1 - \beta) H(\psi), \\ G(\mathbf{x}, t) &= H(\psi), \\ \tau_y(\mathbf{x}, t) &= H(\psi), \end{aligned} \quad (13)$$

where each field is normalized with the material property in the matrix fluid (the total viscosity of the matrix fluid is used to normalize the viscosity fields). The variations in the material properties across the fluid interface are smeared through the regularized Heaviside function $H(\psi)$, and the pressure jump between the two fluid phases due to surface tension is computed through the continuum surface force method [59].

We use periodic boundary conditions in the flow (x) and vorticity (z) directions, and the domain is bounded in the wall-normal direction (y) by two no-slip walls moving in opposing directions. At the walls, no-slip boundary conditions are imposed for the velocities, Neumann boundary conditions are imposed for the pressure, and a linear extrapolation boundary condition is used for the extra stress tensor and level-set fields.

The computational domain of size $L_x \times L_y \times L_z = 16R \times 10R \times 16R$ is discretized using $N_x \times N_y \times N_z = 256 \times 160 \times 256$ grid points in the x , y , and z directions. This computational domain has been employed in several previous studies of suspension rheology [34,35,60,61], chosen to be sufficiently large in the periodic directions to ensure boundary effects are negligible. This resolution provides 16 grid points per initial droplet radius, which has been shown in previous studies [35,61] to satisfactorily resolve the topological changes in a statistical sense, as well as the stress fields and droplet dynamics of droplets sheared in viscoelastic and EVP matrix fluids [48,50,62]. Note that as the droplets coalesce and enlarge, their dynamics are more accurately resolved. Additional

TABLE I. Material parameters of the Saramito constitutive equation fit to the rheological data for a 0.07% w/w Carbopol gel used in the experimental study by Holenberg *et al.* [71]. The $\tilde{\cdot}$ tilde notation is used to denote dimensional material properties, and the “1” subscript is used to emphasize that the material properties are fit for the single-phase EVP matrix fluid in the present setup.

Parameter	$\tilde{G}_1$	$\tilde{\mu}_{p,1}$	β	$\tilde{\tau}_{y,1}$
Value	18.6	1.32	0.09	0.98
Units	Pa	Pa s	[–]	Pa

simulations using a larger domain and higher resolution are performed and reported in Appendix D, confirming that the results are independent of the choice of domain size and grid resolution.

Our code has been used in numerous previous studies of multiphase flows of Newtonian and EVP fluids [50,55,58,63], and one can find the numerical details of the implementation in [58,63] along with a large number of validation cases. We briefly summarize here the numerical methods used to solve the governing equations. In Eqs. (6)–(11), spatial derivatives are discretized using a second-order central difference scheme, with three exceptions: the advection term in Eq. (8) is discretized with the fifth-order weighted essentially nonoscillatory (WENO) scheme [64], as is the advection term in Eq. (11), and the advection term in Eq. (10) is discretized with a fifth-order high-order upstream central (HOU) scheme [65]. For temporal integration, Eqs. (7) and (8) are advanced in time using the second-order Adams-Bashforth scheme, Eq. (10) using a third-order, total-variation-diminishing (TVD) Runge-Kutta scheme [66], and Eq. (11) is solved in pseudo-time with the same TVD Runge-Kutta scheme. The projection method [67] is used to impose the divergence-free constraint on $\mathbf{u}$; a constant coefficient pressure-Poisson equation is derived using the pressure-splitting method of Dodd and Ferrante [68], which is solved using a direct, fast Poisson solver [69,70].

C. Choice of fluid parameters

In this section we motivate our choice of base case dimensionless numbers, i.e., $\text{Re} = 0.1$, $\text{Ca} = 0.2$, $\text{Wi} = 0.5$, $\beta = 0.09$, $\text{Bn} = 0.1$. The Saramito model parameters, i.e., $\tilde{G}_1$, $\tilde{\mu}_{p,1}$, β , and $\tilde{\tau}_{y,1}$ for the single-phase matrix fluid (indicated by the “1” subscript), are fitted to rheological data for a 0.07% w/w aqueous Carbopol 940 solution used in the work of Holenberg *et al.* [71]. The fitting is performed by subjecting the single-phase matrix fluid to oscillatory shear and computing the corresponding Saramito predictions of the storage and loss moduli, G' and G'' . The sum of squared error between the model and experimental data is used as the objective function to be minimized; further details are provided in Appendix B. The optimized Saramito model parameters are listed in Table I, and Fig. 3 compares the experimental measurements with the model predictions based on these fitted parameters.

With this choice of fluid parameters, we can estimate representative values of Re , Ca , Wi , and Bn for the present setup. Taking $\tilde{\rho}_1 = 1000 \text{ kg m}^{-3}$, $\tilde{R} \approx 1 \text{ mm}$, $\tilde{\sigma} = 0.072 \text{ N m}^{-1}$, and applying a shear rate of $\dot{\gamma}_0 \approx 7 \text{ s}^{-1}$ we obtain

$$\begin{aligned}
 \text{Re} &= \frac{1000 \times 10^{-6} \times 7}{1.45} \approx 4.83 \times 10^{-3}, \\
 \text{Ca} &= \frac{1.45 \times 10^{-3} \times 7}{0.072} \approx 0.14, \\
 \text{Wi} &= 0.071 \times 7 \approx 0.5, \\
 \text{Bn} &= \frac{0.98}{1.45 \times 7} \approx 0.1,
 \end{aligned}$$

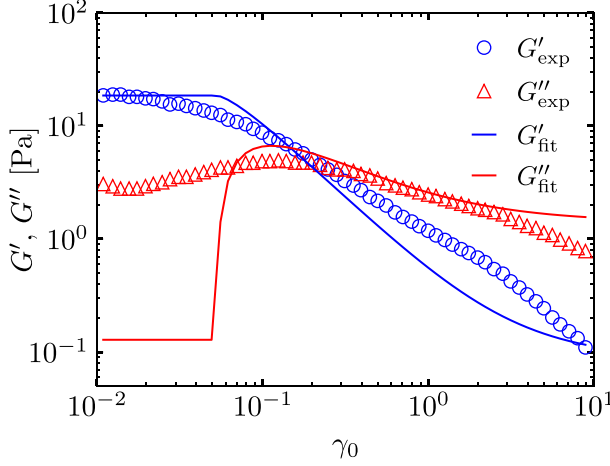


FIG. 3. Comparison of the oscillatory shear strain amplitude sweep showing G' and G'' of an aqueous 0.07% w/w Carbopol gel. Experimental measurements [71] are shown by the marker points, and the Saramito model predictions are depicted by the solid lines. The Saramito model parameters fit to the experimental data are listed in Table I.

which motivates the Re , Ca , Wi , and Bn we use for our base case. We note that Re is significantly smaller in reality than the $Re = 0.1$ used here, but we use a value that is numerically tractable and sufficiently small such that inertial effects are negligible [35,40]. Likewise, the viscosity ratio $k_\mu = 1/40$ we use is larger than that for air bubbles in Carbopol gels; this choice again reflects a compromise between numerical tractability and retaining a sufficiently small viscosity ratio to capture the qualitative rheology of bubbly suspensions. In Sec. III we show that this choice of density and viscosity ratio is sufficient to capture some salient features of gas bubbles in EVP liquids. We further estimate a characteristic Galilei number, $Ga = \bar{\rho}_1^2 \bar{g} \bar{R}^3 / (\bar{\mu}_{s,1} + \bar{\mu}_{p,1})^2 \approx 5 \times 10^{-3}$ for our system, which quantifies the ratio of gravitational to viscous stresses. As $Ga \ll 1$ for our setup, the choice to neglect buoyancy is justified.

We make two remarks here regarding the rheological fitting: first, the most notable discrepancy between the experimental data and model prediction is G'' at low γ_0 in the linear viscoelastic regime. For the Saramito model, the viscous response at low γ_0 (in which the material remains unyielded) comes from the solvent viscosity, which predicts a much smaller G'' than is typically measured in gels [72]—a well-known limitation of the Saramito model [51,73,74]. However, with the simple shear protocol used in this study, the fluid experiences very large strains of order $\gamma = t\dot{\gamma}_0 \approx 100$. These strains are much larger than those in the small γ_0 regime where the Saramito model shows this limitation, and are more representative of the large deformations relevant to flow applications.

Second, whereas the experimental γ_0 sweep shows continued strain thinning of G' and G'' at large γ_0 , the fitted curves show a plateau at large γ_0 . Incorporating the Herschel-Bulkley-like shear-thinning behavior in the Saramito model [75] could improve this agreement further as EVP fluids are typically shear thinning. In the present study, however, we focus on the effects of matrix phase viscoelasticity and yield stress on the bulk rheology, and will neglect the additional complexities of introducing Herschel-Bulkley-like shear thinning.

D. Verification

To verify our solver with a problem relevant to our setup, we compare with the results of Cardinaels *et al.* [76] for a Newtonian droplet in a viscoelastic, Oldroyd-B matrix fluid sheared in planar Couette flow. Droplet deformation and orientation are quantified using the Taylor deformation

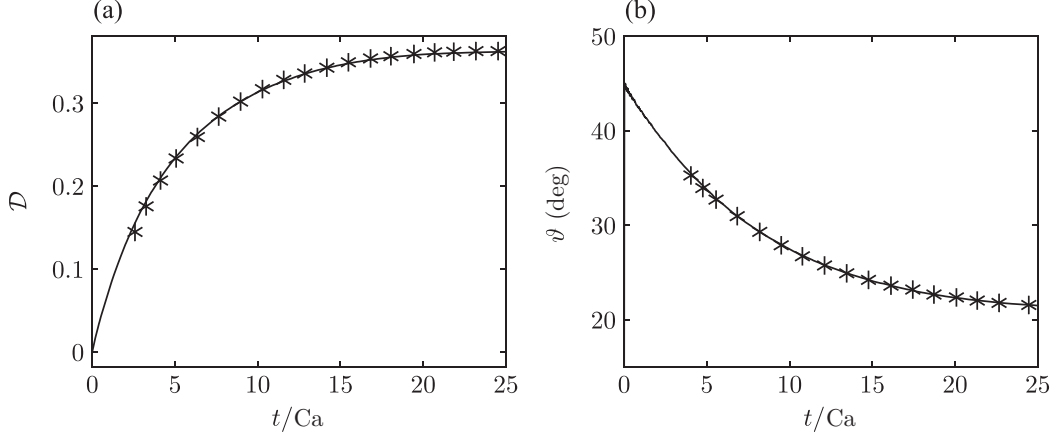


FIG. 4. Temporal evolution of (a) the Taylor deformation parameter, $\mathcal{D}$, and (b) the orientation angle, ϑ , for a Newtonian droplet in a viscoelastic matrix fluid sheared in planar Couette flow. The solid line corresponds to our solution, and the asterisk markers (*) show the results of Cardinaels *et al.* [76]. Here $\text{Re} = 0.25$, $\text{Ca} = 0.2$, $k_\rho = 1$, $k_\mu = 1.5$, $\text{Wi} = 0.625$, $\beta = 0.68$, $2R/L_y = 0.76$.

parameter, $\mathcal{D}$, and the orientation angle, ϑ . The deformation parameter is defined as

$$\mathcal{D} = \frac{L - B}{L + B}, \quad (14)$$

where L is the maximum distance between the drop centroid and the interface, and B is the minimum distance. The orientation angle, ϑ , is the angle between the flow-direction axis and the line joining the drop centroid to the furthest point on the interface. In Fig. 4 we show the temporal evolution of $\mathcal{D}$ and ϑ and find good agreement with the results of Cardinaels *et al.* [76], demonstrating that the solver accurately computes droplet deformation in viscoelastic simple-shear flows. For additional verification cases, the reader is referred to Izbassarov *et al.* [63].

III. RESULTS

A. Single-droplet dynamics and dilute regime rheology

To explore the rheology of dilute systems, we first examine the dynamics of a single Newtonian droplet ($k_\rho = k_\mu = 1$) and bubble ($k_\rho = k_\mu = 1/40$) sheared in an EVP matrix fluid.

Figure 5 gives an overview of the droplet shape and stress distributions at steady state as Ca , Wi , and Bn are varied. The color map shows the shear component of the extra stress, τ_{xy}^e , on the central x - y plane, normalized by the steady state τ_{xy}^e of the single-phase matrix fluid. In the top row, Ca is varied across $\{0.1, 0.2, 0.3\}$; the middle row shows variations of Wi across $\{0.5, 1, 1.5\}$; and the bottom row shows variations in Bn across $\{0.1, 0.5, 1\}$.

Figure 5 shows that the stress fields appear qualitatively similar, displaying alternating high- and low-stress regions around the drop surface with the maximum stress at the droplet tips. This behavior is consistent with prior studies on Newtonian droplets sheared in viscoelastic matrix fluids [46,48]. Expectedly, drop deformation and flow alignment increase with Ca [Figs. 5(a)–5(c)], and the stress magnitude decreases owing to the weaker flow perturbation (except at the tips, which persistently show large local stresses). The variation of drop deformation with Bn shows a qualitatively similar trend [Figs. 5(g)–5(i)] as that observed with Ca .

By contrast, Figs. 5(d)–5(f) show a qualitatively different variation of the drop shape as Wi increases. Whereas variations with Ca and Bn lead to increasingly ellipsoidal shapes, the deformation induced by increasing Wi is weaker, and the drop assumes a qualitatively different shape

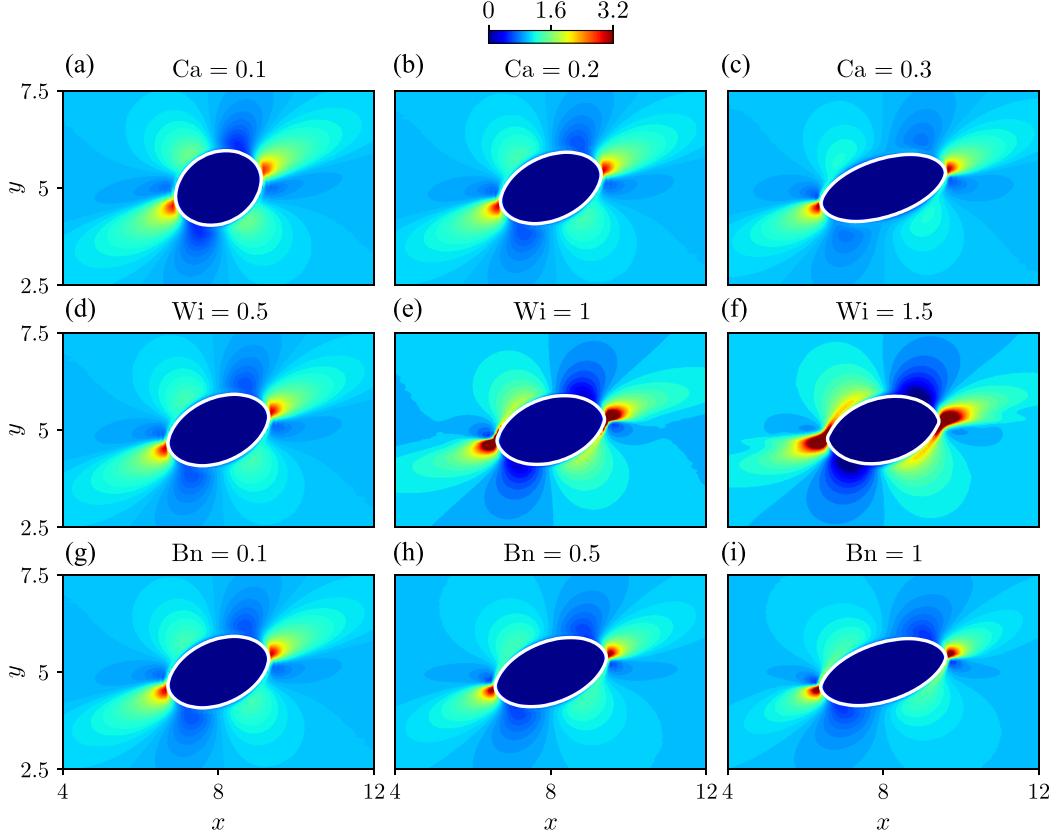


FIG. 5. The $\tau_{xy}^e / \tau_{xy}^e(\phi = 0)$ field for droplets ($k_\rho = k_\mu = 1$) on the central x - y plane. The τ_{xy}^e field is normalized with the steady extra shear stress of the single-phase matrix fluid, $\tau_{xy}^e(\phi = 0)$. The drop surface ($\psi = 0$ level set) is shown by the white isocontour. The drop shape is shown at steady state. The nondimensional parameters vary as follows (top row): (a) $Ca = 0.1$, (b) $Ca = 0.2$, (c) $Ca = 0.3$, remaining dimensionless numbers: $Wi = 0.5$, $Bn = 0.1$; (middle row): (d) $Wi = 0.5$, (e) $Wi = 1$, (f) $Wi = 1.5$, remaining dimensionless numbers: $Ca = 0.2$, $Bn = 0.1$; (bottom row): (g) $Bn = 0.1$, (h) $Bn = 0.5$, and (i) $Bn = 1$, remaining dimensionless numbers: $Ca = 0.2$, $Wi = 0.5$.

with cusplike tips. Furthermore, the magnitude of shear stress increases significantly with Wi . This arises from the increasing elastic effects with larger Wi : as Wi increases, regions of extensional and compressional flow around the drop surface (namely, near the droplet tips and equator) [46] generate large normal stresses which subsequently produce an intense shear-stress distribution shown in Fig. 5(f). We expect this amplified shear-stress distribution to increase the emulsion's effective viscosity; this is analyzed later in this section.

In Fig. 6 we plot the deformation and orientation relative to the flow direction for both droplets and bubbles, quantified respectively through the Taylor deformation parameter, $\mathcal{D}$, and the orientation angle, ϑ . The variations of $\mathcal{D}$ and ϑ as a function of Ca [Figs. 6(a) and 6(d)], Wi [Figs. 6(b) and 6(e)], and Bn [Figs. 6(c) and 6(f)] are shown. Variation with Ca and Bn show a similar trend for droplets and bubbles, with both showing a monotonic increase in deformation and flow alignment; as drop deformation increases with Ca (or Bn), the stress distribution around the drop surface results in a larger torque which increases flow alignment. At low Ca , $\mathcal{D}$ is approximately equal for droplets and bubbles, with droplets showing slightly lower ϑ . At $Ca > 0.25$, the variation of $\mathcal{D}$ shows a distinct change between droplets and bubbles. For droplets, $\mathcal{D}$ increases superlinearly

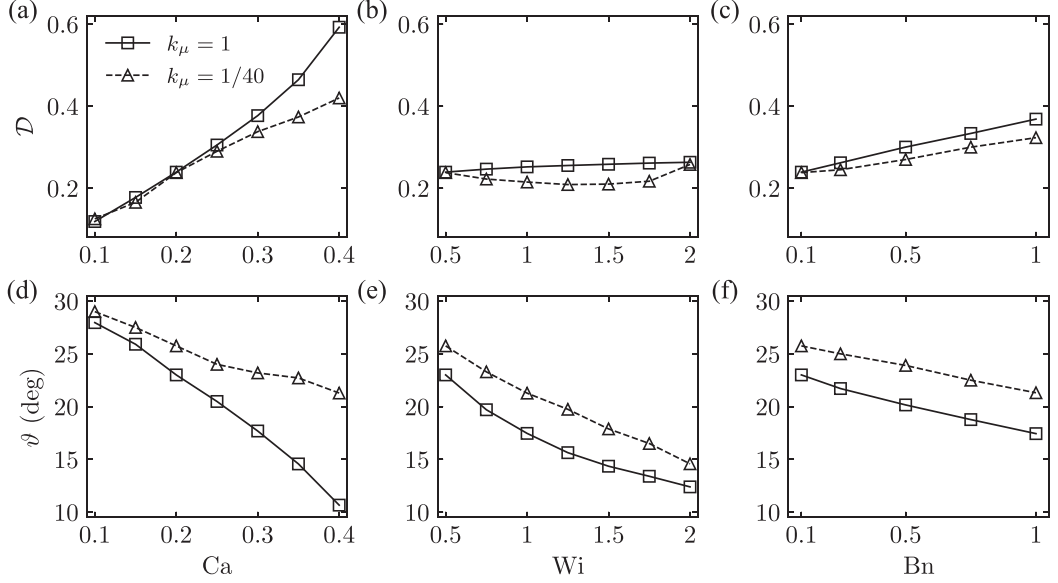


FIG. 6. The Taylor deformation parameter, $\mathcal{D}$ (solid line), and orientation angle, ϑ (dashed line), of a single droplet or bubble as a function of (a), (d) Ca , (b), (e) Wi , (c), (f) Bn . The square markers correspond to $k_\rho = k_\mu = 1$ (droplets), and the triangular markers to $k_\rho = k_\mu = 1/40$ (bubbles). The remaining dimensionless numbers are (a), (d) $Wi = 0.5$, $Bn = 0.1$, (b), (e) $Ca = 0.2$, $Bn = 0.1$, (c), (f) $Ca = 0.2$, $Wi = 0.5$.

as the critical Ca for breakup is approached [28]. In contrast, bubble deformation appears more stabilized for $Ca \geq 0.25$, reflected in the reduced rate of increase of $\mathcal{D}$ with Ca . This is also evident in the variation of ϑ in Fig. 6(d): whereas droplets show a superlinear decrease at large Ca , for bubbles, a change in slope occurs at $Ca \approx 0.25$, after which ϑ decreases more slowly.

Figure 7 shows the isocontour of the interface on the central x - y plane to compare the qualitative deformed shapes for a droplet [Fig. 7(a)] and a bubble [Fig. 7(b)] at $Ca = 0.3$. The color map shows the τ_{xy}^e stress distribution. Whereas the droplet shows a deformed, ellipsoidal shape with rounded tips, the bubble shows cusped, pointed tips [this can also be somewhat seen in Fig. 5(f) at larger

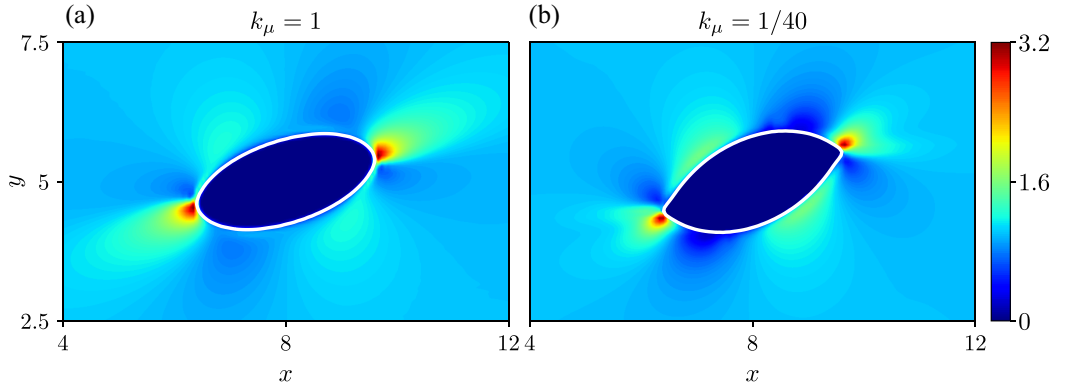


FIG. 7. The τ_{xy}^e field for a single (a) droplet ($k_\rho = k_\mu = 1$) and (b) bubble ($k_\rho = k_\mu = 1/40$) on the central x - y plane. The interface ($\psi = 0$ level set) is shown by the white isocontour. The dimensionless numbers are $Ca = 0.3$, $Wi = 0.5$, $Bn = 0.1$.

Wi for droplets]. This is reminiscent of the cusped tail formed by rising bubbles in viscoelastic [77] and EVP [78] fluids. The formation of cusplike tips arises from the strong extensional flows at the bubble tips [46], which generates tensile normal stresses that locally deform the tips and create the cuspid shape.

The variations of $\mathcal{D}$ and ϑ with Bn show qualitatively similar trends as those observed with varying Ca. This can be attributed to two effects. First, considering viscoplastic effects: the yield stress imposes a stress threshold that must be exceeded for yielding to occur; once yielded, part of the stress budget driving the flow is expended in overcoming the yield stress, resulting in a larger shear stress being required to sustain the nominal shear rate. This increases the fluid's effective viscosity (see Appendix C for the variation of the shear viscosity as a function of Wi and Bn for the single-phase matrix fluid in simple shear). Consequently, this increases the effective capillary number, $Ca_{\text{eff}} = \mu_{\text{eff}} R \dot{\gamma}_0 / \sigma$.

The second effect is that elastic effects have been shown to increase with finite and increasing Bn [79–81]. The Saramito constitutive equation [Eq. (8)] shows a modulation of the relaxation function by a nonlinear coefficient of $(|\tau_d^e| - Bn)/|\tau_d^e|$ in the yielded state compared to the Oldroyd-B model, leading to an enhancement of elastic effects with increasing Bn. For example, under simple shear, the Saramito model predicts an increase in normal stresses with Bn [51]. Differences between the trends with Ca [Figs. 6(a) and 6(d)] and those with Bn [Figs. 6(c) and 6(f)] suggest this behavior: the increase in $\mathcal{D}$ with Bn shows an early deviation between droplets and bubbles, whereas they are approximately constant up to $Ca = 0.25$ with increasing Ca. Further, the difference in ϑ between droplets and bubbles remains approximately constant with Bn, whereas it increases with Ca. These qualitative differences indicate that the different drop dynamics cannot be attributed solely to an increase in the effective capillary number, with increasing elastic effects playing a possible role in this variation.

Finally, we note that no significant unyielded structures are observed in our simulations: in most cases, the volume of unyielded material constitutes less than 1% of the computational domain. This is a consequence of the homogeneous shear imposed on the fluid in planar Couette flow, which uniformly deforms the material until yielding occurs. Small perturbations of the stress field by the dispersed phase are not sufficient to maintain unyielded material around the drop vicinity. Therefore, the primary effect of the yield stress in the present setup is to increase the effective viscosity of the matrix fluid.

Figure 6(b) shows that droplet deformation increases monotonically with Wi, whereas bubble deformation exhibits a nonmonotonic dependence, characterized by an initial decrease followed by a subsequent increase. Such nonmonotonic variation of $\mathcal{D}$ with Wi is well known and consistent with earlier studies of Newtonian drops sheared in Oldroyd-B matrix fluids [46,48]. This behavior arises from a subtle interplay between two competing mechanisms: the development of normal viscoelastic stresses associated with the flow kinematics, which promotes deformation, and viscoelastic modifications to the flow field that reduce the viscous stresses acting on the interface, thereby suppressing deformation [46]. At sufficiently large Wi, the former dominates, leading to an increase in $\mathcal{D}$. Figure 6(e) shows a monotonic decrease of ϑ with Wi for both droplets and bubbles. This trend can be attributed to the distribution of viscoelastic stresses around the interface, which generates a torque on the drop and tends to align it with the flow direction [46,48].

It is instructive to compare the present trends in droplet deformation with Wi with previous studies of a Newtonian drop sheared in an Oldroyd-B matrix [48] to highlight their differences. First, in the present study a monotonic increase in $\mathcal{D}$ is observed for droplets over the range of Wi considered. Prior studies have shown that $\mathcal{D}$ varies nonmonotonically with Wi at the capillary number considered here, with an increase occurring at $Wi \approx 0.8$ [48]. Aggarwal and Sarkar [48] further reported that the minimum of $\mathcal{D}$ as a function of Wi shifts to lower Wi as Ca increases. The monotonic increase in droplet $\mathcal{D}$ observed here therefore suggests enhanced deformation due to the presence of a finite yield stress, arising from both an increase in the effective capillary number and enhancement of elastic effects at finite Bingham number.

Second, bubble deformation shows a nonmonotonic variation with Wi over the same range, with the rise in $\mathcal{D}$ occurring at larger Wi than previously reported for sheared droplets (i.e., matched viscosity systems) [48]. This reflects a complex interplay between viscoelastic stresses and the flow kinematics in the vicinity of the bubble surface. The initial decrease in bubble deformation indicates that sufficiently large normal stresses do not develop to increase deformation; instead, the dominant effect of viscoelasticity at low Wi is to modify the local flow kinematics and attenuate the viscous stresses responsible for deformation [46]. At $Wi \approx 1.25$, $\mathcal{D}$ increases, indicating the development of strong viscoelastic stresses around the bubble that promote deformation. As Wi increases further, cusped bubble tips form [see Fig. 7(b)], further modifying the local stress field. The large increase in $\mathcal{D}$ in the range $1.5 \leq Wi \leq 2$ suggests development of large normal stresses at the bubble surface that rapidly increase its deformation.

Next, we analyze the system's bulk shear rheology, characterized through the effective shear viscosity μ_{eff} :

$$\mu_{\text{eff}} = \frac{\tau_w}{\dot{\gamma}_0} = \frac{\langle \mu \frac{\partial u}{\partial y} |_0 \rangle + \langle \tau_{xy}^e |_0 \rangle}{\dot{\gamma}_0}, \quad (15)$$

where τ_w is the wall shear stress, the notation $\langle \cdot \rangle$ denotes averaging in the x and z directions, and $|_0$ denotes that the quantity is evaluated at the lower wall. The contribution of the EVP stress in the matrix phase to the wall shear stress budget is accounted for by the addition of $\tau_{xy}^e|_0$. The relative viscosity, μ_r , is defined as

$$\mu_r = \frac{\mu_{\text{eff}}}{\mu_{\text{eff}}(\phi = 0)}, \quad (16)$$

i.e., the ratio between the effective shear viscosity of the biphasic system, and that of the single-phase matrix fluid.

Following the relations of Einstein [6] and Taylor [17], we assume that μ_r is a linear function in ϕ in the dilute regime in the absence of interactions between the dispersed phase:

$$\mu_r = 1 + a_\mu \phi. \quad (17)$$

Einstein's relation for rigid particles corresponds to $a_\mu = 2.5$; Taylor's relation for biphasic viscous systems corresponds to $a_\mu = 1.75$ for a matched-viscosity system, and $a_\mu = 1$ for a bubbly suspension ($k_\mu \rightarrow 0$). $a_\mu > 0$ signifies an increase in μ_r with ϕ (the effective viscosity of the biphasic system is greater than that of the single-phase matrix fluid), whereas $a_\mu < 0$ indicates a decrease in μ_r with ϕ (the effective viscosity is lower than that of the matrix fluid).

Figure 8 shows a_μ as a function of Ca [Fig. 8(a)], Wi [Fig. 8(b)], and Bn [Fig. 8(c)] for droplets and bubbles; the values corresponding to Einstein and Taylor's relations are shown by the horizontal lines. We observe that a_μ decreases monotonically with both Ca and Bn for both viscosity ratios. This is consistent with the previous discussion on the coupling between droplet (or bubble) deformation and the perturbation of the local flow field; as deformation and flow alignment of the dispersed phase increase, the flow and stress fields are perturbed to a lesser degree, thereby reducing the increase in effective viscosity.

At small Ca , a_μ approaches the value predicted by Taylor's relation for rigid viscous drops sheared in a Newtonian medium, for both droplets and bubbles. This already indicates deviation from Taylor's relation for the present system with EVP matrix fluids, as $a_\mu = 1$ is predicted by Taylor's relation for a bubble-laden system ($k_\mu \rightarrow 0$) (specifically, $a_\mu \approx 1.04$ by Taylor's relation for the present $k_\mu = 1/40$ considered). Furthermore, a_μ becomes negative for bubbles at large Ca , i.e., the effective shear viscosity of the bubble suspension is less than that of the matrix fluid. This reflects the roles of deformability and the interfacial condition in determining the bulk rheology. As bubble deformability increases with Ca , large free-slip surfaces at the interface attenuate the local velocity gradients and stress fields, thereby significantly reducing dissipation and decreasing the system's effective viscosity [18,19].

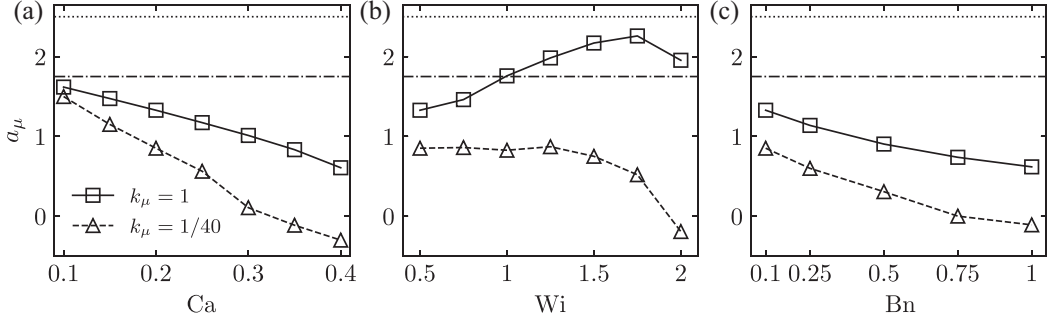


FIG. 8. a_μ coefficient of the linear μ_r relation ($\mu_r = 1 + a_\mu \phi$) as a function of (a) Ca , (b) Wi , (c) Bn . The horizontal dash-dot line shows Taylor's relation [Eq. (2)] for droplets ($k_\mu = 1$), and the dotted line shows Einstein's relation [Eq. (1)] for particles. The square markers correspond to the present results for droplets ($k_p = k_\mu = 1$), and the triangular markers to bubbles ($k_p = k_\mu = 1/40$). The remaining dimensionless numbers are (a) $Wi = 0.5$, $Bn = 0.1$, (b) $Ca = 0.2$, $Bn = 0.1$, (c) $Ca = 0.2$, $Wi = 0.5$.

Figure 8(b) shows a_μ as a function of Wi for droplets and bubbles, revealing markedly differing trends from those observed with Ca and Bn . For droplet-laden emulsions, a nonmonotonic behavior is observed, in which a_μ increases with Wi up to $Wi \approx 1.75$ and decreases thereafter at $Wi = 2$. Notably, despite the increased drop deformation and flow alignment observed in Figs. 6(b) and 6(e), the emulsion's relative viscosity increases. At $Wi > 1$, a_μ increases above that for Taylor's relation for a viscosity-matched emulsion. This behavior is attributed to the intense EVP shear stress distribution that develops around the droplet at large Wi [see Fig. 5(f)], induced by the local flow perturbation by the drop and the EVP rheology of the matrix fluid. EVP stresses are generated at the droplet surface owing to the local flow kinematics: strong extensional flows near the droplet tips and equatorial region give rise to large normal stresses. These normal stresses balance nonlinearly with the shear stress, resulting in an intense shear stress distribution around the drop surface, as shown in Figs. 5(d)–5(f). The larger relaxation time associated with increasing Wi allows these stresses to persist in the flow and consequently increase the emulsion's effective viscosity. This contrasts with Newtonian systems, in which variations in the effective viscosity primarily arise from modification of the viscous shear-stress distribution by the dispersed phase, which is strongly coupled to its deformability.

For bubbles, a_μ initially does not show significant variation with Wi and remains approximately constant up to $Wi \approx 1.25$, after which it rapidly decreases. This is consistent with the increased deformation and flow alignment for $Wi > 1.25$ seen in Figs. 6(b) and 6(e). Further, examining the τ_{xy}^e field around the bubble in Fig. 7(b) reveals differences in the stress distribution compared with the droplet case [Fig. 7(a)]. First, the large stresses which form at the bubble are more localized at the tips than that for the droplet which show a larger spatial extent. Consequently, these stresses contribute less to the wall shear-stress budget than in the droplet case. Second, regions of low stress are observed just upstream of the bubble tips compared with the droplet case. These low-stress regions balance the high stress regions, thereby attenuating the increase in effective viscosity. At $Wi = 2$, a_μ decreases significantly and becomes negative, indicating that the bubbly suspension's effective viscosity decreases below that of the matrix fluid.

B. Semidilute regime rheology

Beyond the dilute regime, complex stress fields arise due to drop-drop interactions, heterogeneities in droplet spatial and size distribution, and topological changes, all of which collectively influence the system's bulk rheology. To investigate the bulk rheology of our system at semidilute concentrations, we perform simulations at dispersed phase volume fractions

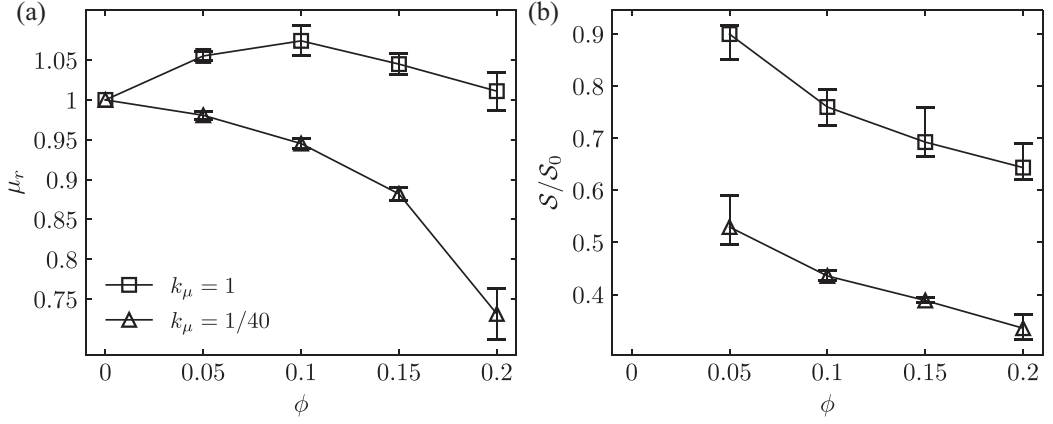


FIG. 9. Variation of (a) μ_r and (b) S/S_0 as a function of ϕ . The square markers correspond to droplets ($k_p = k_\mu = 1$) and the triangles to bubbles ($k_p = k_\mu = 1/40$). The error bars indicate the range of temporal variation within the averaging window. Here $Ca = 0.2$, $Wi = 0.5$, $Bn = 0.1$.

$\phi = \{0.05, 0.1, 0.15, 0.2\}$. We monitor the temporal evolution of the wall shear stress, τ_w , and the interfacial area, $\mathcal{S}$, and advance the simulations until a statistically steady state is reached, typically at $t \approx 100$. Thereafter, data are sampled and temporally averaged over an interval of $t \approx 10$ – 20 to obtain statistically converged results.

We first investigate how the relative viscosity varies with the volume fraction of the dispersed phase, extending the analysis to the semidilute regime in which interactions between the dispersed phase become significant. Figure 9(a) shows μ_r as a function of ϕ for both the droplet-laden (square markers) and bubble-laden (triangular markers) cases. For bubble-laden suspensions, μ_r decreases monotonically with ϕ . For the droplet-laden cases, μ_r shows nonmonotonic variation with an initial increase up to a maximum followed by a subsequent decrease. This behavior is consistent with previous studies with coalescing droplets for Newtonian systems [34,35,61].

To quantify the deformation of the dispersed phase and infer the extent of topological changes due to coalescence and breakup, we use the interfacial area, $\mathcal{S}$. We calculate $\mathcal{S}$ as [82]

$$\mathcal{S} = \int_{\Omega} |\nabla \psi| \delta(\psi) dV, \quad (18)$$

where Ω denotes the computational domain. A larger $\mathcal{S}$ may indicate either a greater number of droplets (due to increased breakup or suppressed coalescence) or more deformation of the dispersed phase; conversely, a smaller $\mathcal{S}$ may indicate either a reduced number of droplets (due to increased coalescence or inhibited breakup) or less dispersed phase deformation.

Figure 9(b) shows the interfacial area decreases monotonically with volume fraction for both droplet- and bubble-laden cases, indicating greater coalescence at larger ϕ owing to increased interactions and collision frequency. Moreover, the interfacial area for the bubbly suspensions is approximately half that for the droplet-laden cases, indicating a significantly higher coalescence rate for bubbles. We confirm this by presenting instantaneous visualizations of the interface in Fig. 10 (shown by isosurfaces of the $\psi = 0$ level set). Comparing the droplet and bubble size distributions in Figs. 10(a) and 10(b) shows several notable features. Most significantly, a larger number of droplets than bubbles is observed, indicating a larger coalescence rate in the bubbly suspension. Second, the droplets show a diverse size distribution with a spectrum of small and large droplets as well as large, deformed, ligamentous structures. Bubbles, on the other hand, coalesce into a fewer number of large bubbles. The formation of these long, ligamentous droplets, as well as these large bubbles, suggests a stabilizing effect on breakup by the matrix phase viscoelasticity. Last, as seen for a single

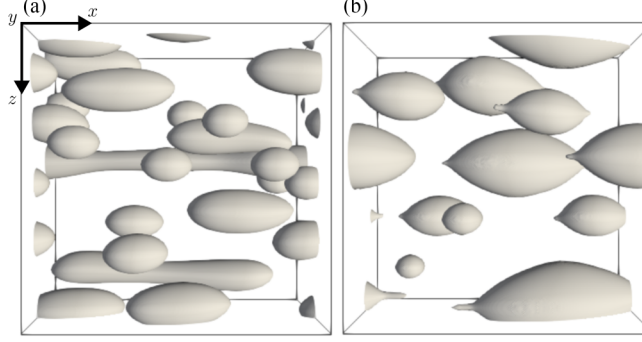


FIG. 10. Instantaneous visualizations of the interface ($\psi = 0$) for (a) droplets ($k_p = k_\mu = 1$) at $t = 90$ and (b) bubbles ($k_p = k_\mu = 1/40$) at $t = 90$. Here $\phi = 0.1$, $Ca = 0.2$, $Wi = 0.5$, $Bn = 0.1$.

bubble [Fig. 7(b)], the bubbly suspension shows the formation of pointed bubble tips arising from deformation by local normal stresses.

To analyze the contributions of different mechanisms to the wall shear-stress, we seek to formulate its stress budget. Following Yang *et al.* [83] and Rosti *et al.* [34], it can be shown that the wall shear stress, $\tau_w = \langle \mu \partial u / \partial y \rangle|_0 + \langle \tau_{xy}^e \rangle|_0$, can be decomposed into a sum of bulk-averaged contributions of the terms appearing in the momentum balance [Eq. (7)]. Their analysis can be extended to include the contribution of the EVP extra stress, τ^e . The wall-shear stress budget is decomposed as follows (details of the derivation are given in Appendix A),

$$\langle 2\mu D_{xy} \rangle|_0 + \langle \tau_{xy}^e \rangle|_0 = -\text{Re}\langle \rho uv \rangle + \langle 2\mu D_{xy} \rangle + \langle \tau_{xy}^e \rangle + \frac{1}{Ca} \langle \mathcal{G} \rangle, \quad (19)$$

where the notation $\langle \cdot \rangle$ denotes averaging over the computational domain, $\mathcal{G}(y) = \int_0^y \langle f_x \rangle d\xi$, with f_x the x component of the surface tension stress. On the right-hand side of Eq. (19), the first term represents the contribution of inertial forces to the stress budget (negligible here due to the low Reynolds number of the present setup), the second term arises from viscous stresses, the third term from the EVP extra stress, and the fourth term represents the contribution of the interfacial stress.

Figures 11(a) and 11(b) show the variation of the wall shear-stress budget components, τ^i , normalized with the wall shear stress of the single-phase matrix fluid ($\tau_w(\phi = 0)$), as a function of ϕ for the droplet-laden [Fig. 11(a)] and bubble-laden [Fig. 11(b)] cases: these contributions sum to the total wall shear stress. The colored bars show the contributions of the constituent terms τ^i in Eq. (19) to the total wall shear-stress budget. The color legends of the bar charts denote the EVP extra stress τ^e (blue), the matrix phase solvent viscous stress $\tau^{v,m}$ (orange), the dispersed phase viscous stress $\tau^{v,d}$ (green), and the interfacial stress τ^σ (red).

The EVP stress of the matrix fluid shows the largest contribution to the wall shear-stress budget for all cases; the contribution of τ^e decreases with increasing ϕ . For the droplet-laden case, the increased viscous stress within the dispersed phase ($\tau^{v,d}$) partially offsets the decrease in the τ^e contribution owing to the matched viscosity between the droplets and matrix fluid. For the bubbly suspension, there is negligible contribution from the dispersed phase viscous stress to the stress budget, highlighting their behavior as voidlike inclusions. The interfacial stress contribution, τ^σ , increases initially with ϕ for both droplet- and bubble-laden cases, but subsequently decreases as coalescence frequency rises. We note that the contribution of the advection term is of order 10^{-4} , confirming that the inertial stress is negligible in the present low Re setup; we therefore will not consider its contribution in this discussion.

We examine the influence of the droplet spatial distribution on the stress distribution, and hence wall shear stress. In Fig. 11(f) we present profiles of the average droplet concentration, $\langle H(\psi) \rangle$, across the channel height, overlaid with the average EVP stress profile, $\langle \tau_{xy}^e \rangle$, for two different

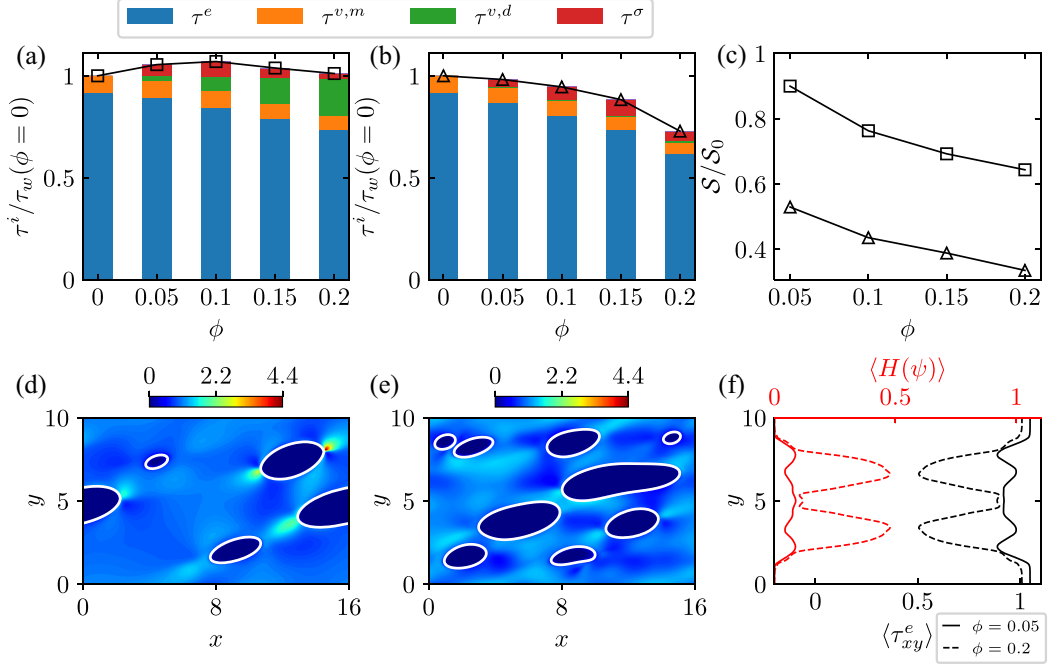


FIG. 11. Stress budget analysis for (a) droplets ($k_\mu = 1$), (b) bubbles ($k_\mu = 1/40$) as a function of ϕ ; (c) interfacial area as a function of ϕ for droplets (square markers) and bubbles (triangle markers); color map of the τ_{xy}^e field on the x - y plane for droplets ($k_\rho = k_\mu = 1$) at (d) $\phi = 0.05$, (e) $\phi = 0.2$; (f) profiles of $\langle \tau_{xy}^e \rangle$ (black lines) and $\langle H(\psi) \rangle$ (red lines) for droplets at $\phi = 0.05$ (solid line) and $\phi = 0.2$ (dashed line). The bar charts in (a) and (b) show the contribution of the extra stress (blue, τ^e), matrix phase viscous stress (orange, $\tau^{v,m}$), dispersed phase viscous stress (green, $\tau^{v,d}$), and interfacial stress (red, τ^σ) to the stress budget. Here $Ca = 0.2$, $Wi = 0.5$, $Bn = 0.1$.

volume fractions. It is evident that the EVP stress profile is strongly modulated in regions of high droplet concentration. The EVP stress decreases in regions of high droplet concentration due to the presence of the viscous phase, while the stress profile is distorted around concentrated droplet regions and droplet interfaces in a manner that depends on the local droplet concentration, the viscosity ratio, and the interfacial tension. Here they collectively act to decrease the wall shear stress with increasing ϕ .

The $\langle H(\psi) \rangle$ profiles show that droplets migrate closer to the channel center with increasing ϕ , forming two concentrated layers approximately halfway between the wall and channel centerline. As droplets coalesce, the resulting larger and more deformable droplets experience a wall-lift force in Couette flow which drives migration towards the channel center [36,54]. Instantaneous visualizations of the τ_{xy}^e field on 2D x - y planes are shown in Figs. 11(d) and 11(e) for $\phi = 0.05$ and $\phi = 0.2$, respectively. Increased coalescence with ϕ leads to the formation of larger droplets and concentrated regions near the channel center, accompanied by reduced τ_{xy}^e stresses in the vicinity of the droplets. As in Sec. III A, we note that significant unyielded structures are not observed in this setup due to the homogeneous shear imposed on the fluid by planar Couette flow. This is illustrated by the τ_{xy}^e stress distribution shown in Figs. 11(d) and 11(e): here the matrix phase stress largely exceeds the Bingham number ($Bn = 0.1$).

We next examine the effect of droplet deformability on the bulk rheology through varying capillary number. Figure 12(a) shows that μ_r decreases monotonically with Ca for both droplets and bubbles. The bulk rheology shows a shear-thinning behavior that is typical of emulsions with a deformable dispersed phase [29,30]. Instantaneous visualizations of the τ_{xy}^e field for droplets

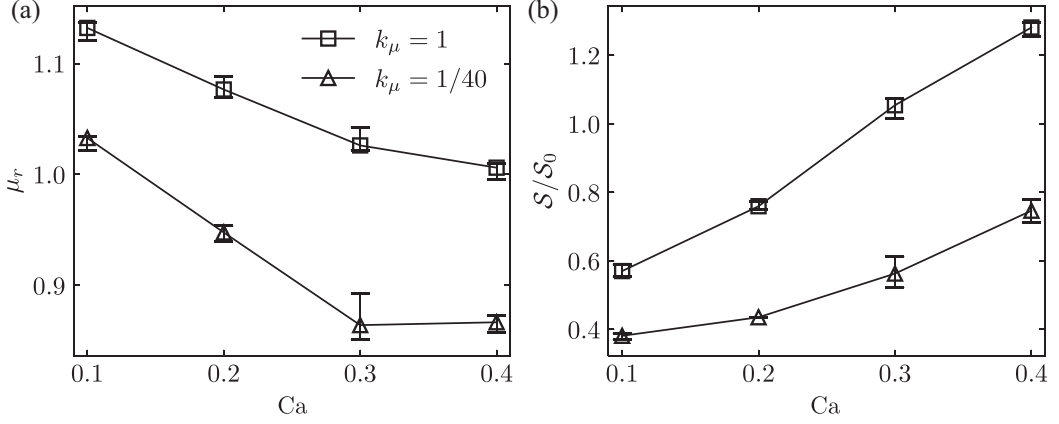


FIG. 12. Variation of (a) μ_r and (b) S/S_0 as a function of Ca for semidilute concentrations. The square markers correspond to droplets ($k_p = k_\mu = 1$) and the triangles correspond to bubbles ($k_p = k_\mu = 1/40$). The error bars indicate the range of temporal variation within the averaging window. Here $\phi = 0.1$, $Wi = 0.5$, $Bn = 0.1$.

confirm that large local stresses form around the stiff drops at small Ca [Fig. 14(a)], but decrease considerably as Ca increases [Figs. 14(b) and 14(c)]. The viscosity of the droplet-laden case remains greater than that of the single-phase matrix fluid ($\mu_r > 1$), but approaches it as Ca increases. For bubbles, $\mu_r > 1$ at $Ca = 0.1$, but decreases below the single-phase value at $Ca = 0.2$ due to increased deformation. The free-slip-like interfacial condition for the bubbly suspension results in lower μ_r than in the droplet-laden case. For large Ca ($Ca \geq 0.3$), μ_r appears to plateau for the bubbly suspension, suggesting that we approach a limiting case for fully deformable bubbles. This plateau is predicted fairly well by Eq. (4) with $\mu_r(\phi = 0.1) = 1 - 5/3 \cdot 0.1 \approx 0.833$.

Figure 12(b) shows the normalized interfacial area as a function of Ca . It can be seen that S/S_0 increases monotonically for both droplet- and bubble-laden cases. For droplets, S/S_0 increases above unity at $Ca \geq 0.3$; lower coalescence rates and greater deformability at large Ca lead to the formation of long, ligamentous structures of the dispersed phase. To support this, we present instantaneous visualizations of the interface ($\psi = 0$ iso-surfaces) in Fig. 13 to examine the instantaneous drop size distribution for droplets [Figs. 13(a) and 13(b)] and bubbles [Figs. 13(c) and 13(d)] at two capillary numbers ($Ca = \{0.2, 0.3\}$). Comparing Figs. 13(a) and 13(b), we observe that droplets exhibit both increased deformation and a higher drop count with Ca , confirming that the increase in S/S_0 arises from a combination of increased deformation and reduced coalescence. A similar

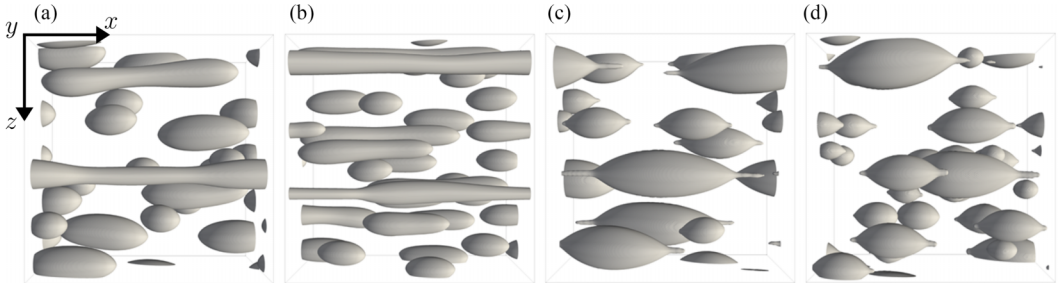


FIG. 13. Instantaneous visualizations of the interface ($\psi = 0$) for (a) $Ca = 0.2$, $k_p = k_\mu = 1$ (droplet) at $t = 90$, (b) $Ca = 0.3$, $k_p = k_\mu = 1$ (droplet) at $t = 90$; (c) $Ca = 0.2$, $k_p = k_\mu = 1/40$ (bubble) at $t = 105$, (d) $Ca = 0.3$, $k_p = k_\mu = 1/40$ (bubble) at $t = 105$. Here $\phi = 0.1$, $Wi = 0.5$, $Bn = 0.1$.

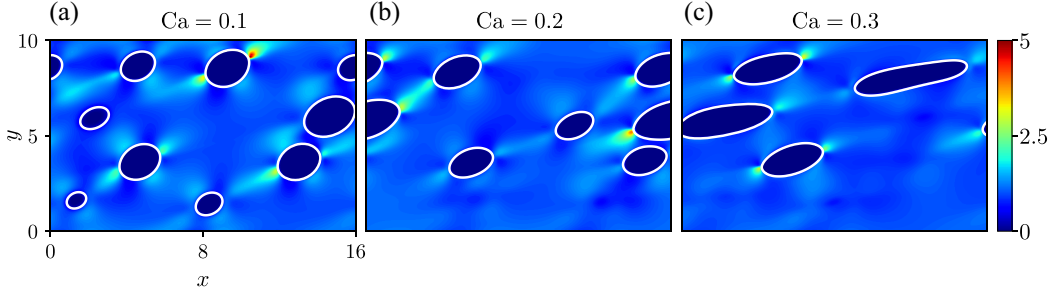


FIG. 14. Visualization of the τ_{xy}^e stress field on the x - y plane at $z/R = 15$ for (a) $Ca = 0.1$, (b) $Ca = 0.2$, and (c) $Ca = 0.3$. The white isocontours represent the interface ($\psi = 0$). Here $\phi = 0.1$, $k_\rho = k_\mu = 1$ (droplets), $Wi = 0.5$, $Bn = 0.1$.

trend is observed for bubbles when comparing Figs. 13(c) and 13(d). Further, comparing Figs. 13(a) and 13(b) with Figs. 13(c) and 13(d) shows a higher drop count for the droplet-laden emulsions compared to the bubbly suspensions, indicating a higher coalescence rate for bubbly suspensions, consistent with previous studies [84]. Last, as in Fig. 10, cusped bubble tips are observed for the bubbly suspension.

C. The effect of matrix phase viscoelasticity

We next explore the effect of matrix phase viscoelasticity on the bulk rheology of emulsions and bubbly suspensions by varying the Weissenberg number while all other parameters are held constant.

We first examine the variation in interfacial area as a function of Wi in Fig. 15(c) to infer information about the dispersed phase morphology. The most significant contrast here is that droplets and bubbles show opposing trends, with S/S_0 increasing monotonically for droplets, whereas it decreases for bubbles. The increase in S/S_0 for droplets may indicate both an increase in drop deformation [as seen with a single drop in Fig. 6(b)], as well as a reduction in the coalescence rate. The instantaneous droplet distribution presented in Fig. 16 at $Wi = 0.5$ [Fig. 16(a)] and $Wi = 1$ [Fig. 16(b)] shows a slight increase in droplet count at higher Wi , suggesting somewhat reduced coalescence. Cusped tips form for some droplets at sufficiently large Wi owing to increased EVP stresses. Conversely, the surface area for bubbles decreases monotonically with Wi —below 0.4 at $Wi \geq 1$ —due to high coalescence and suppressed breakup.

In Figs. 15(a) and 15(b) the wall shear stress budget decomposition is presented for droplets [Fig. 15(a)] and bubbles [Fig. 15(b)], the normalized stress budget components summing to the relative viscosity. Overall, μ_r shows qualitatively similar trends as in the dilute regime analyzed in Sec. III A: droplet-laden emulsions show a nonmonotonic variation with an initial increase followed by a decrease at $Wi = 2$, whereas bubbly suspensions show little variation at low Wi followed by a decrease at larger Wi . Examining the stress budget provides further insight into the observed variations: the stress of the matrix phase (the sum of the EVP extra stress and solvent stress) makes up the largest contributions and shows significant variations with Wi in the stress budget. This highlights the distinct mechanism by which matrix phase viscoelasticity influences the bulk rheology: whereas variations in μ_r with Ca and Bn arise from the coupling between the system's effective viscosity and the deformability of the dispersed phase, variation of μ_r with Wi here is dictated by the flow kinematics in the vicinity of the dispersed phase and the resultant EVP stress distribution. Instantaneous visualizations of the τ_{xy}^e stress distribution on the x - y plane are presented in Figs. 15(d) and 15(e), showing in particular the formation of an intense shear layer and stress bridge between the two drops [Fig. 15(e)]. These hydrodynamic interactions between droplets generate large stresses within the domain which further increase the effective viscosity. Variations in the interfacial stress contribution are also observed owing to the varying dispersed

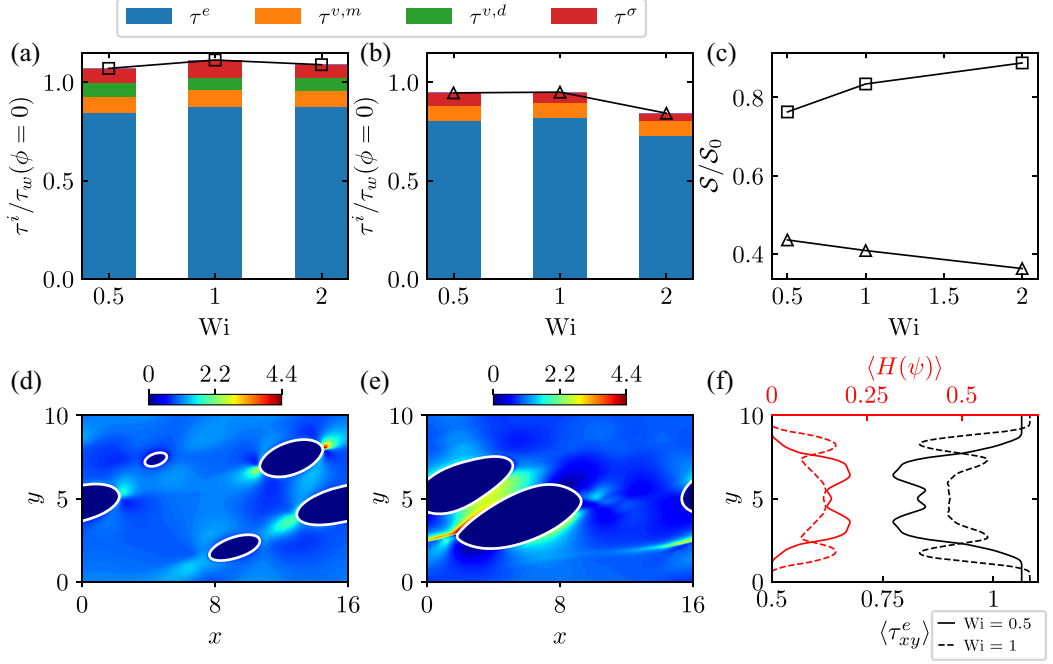


FIG. 15. Stress budget analysis for (a) droplets ($k_\mu = 1$), (b) bubbles ($k_\mu = 1/40$) as a function of Wi ; (c) interfacial area as a function of Wi for droplets (square markers) and bubbles (triangle markers); color map of the τ_{xy}^e field on the x - y plane at (d) $Wi = 0.5$, (e) $Wi = 1$ for droplets ($k_\rho = k_\mu = 1$); (f) profiles of $\langle \tau_{xy}^e \rangle$ (black line) and $\langle H(\psi) \rangle$ (red line) for droplets at $Wi = 0.5$ (solid line) and $Wi = 1$ (dashed line). The bar charts in (a) and (b) show the contribution of the extra stress (blue, τ^e), matrix phase viscous stress (orange, $\tau^{v,m}$), dispersed phase viscous stress (green, $\tau^{v,d}$), and interfacial stress (red, τ^σ) to the stress budget. Here $\phi = 0.1$, $Ca = 0.2$, $Bn = 0.1$.

phase morphology with Wi : in particular, Fig. 15(a) shows increased contribution of the interfacial stress with Wi for droplet-laden emulsions, consistent with the higher droplet count in Fig. 16 and larger S/S_0 in Fig. 15, which further increase the wall stress over this range of Wi .

The variation in relative viscosity with Wi for the bubbly suspension [Fig. 15(b)] shows a qualitatively similar trend as for dilute concentrations [Fig. 8(b)]. Examination of the stress budget

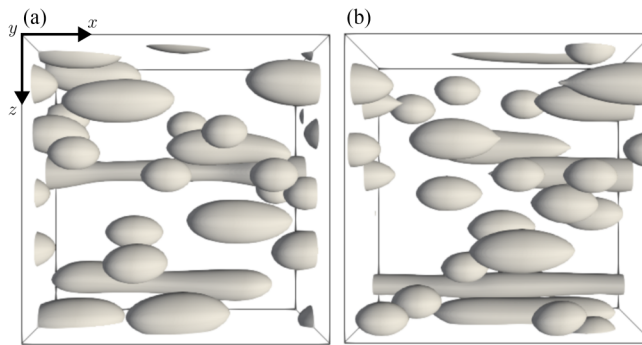


FIG. 16. Instantaneous visualizations of the interface ($\psi = 0$) for droplets at (a) $Wi = 0.5$ at $t = 90$ and (b) $Wi = 1$ at $t = 90$. Here $\phi = 0.1$, $Ca = 0.2$, $k_\rho = k_\mu = 1$, $Bn = 0.1$.

again indicates that this is due to the wall shear stress being dominated by the EVP stress component, which shows the most significant variations and contribution. There is, however, one important distinction: while dilute bubbly suspensions showed a greater relative viscosity than the matrix fluid at low Wi [$\mu_r > 1$; see Fig. 8(b)], here $\mu_r < 1$ at semidilute concentrations, i.e., the system's effective viscosity is always lesser than that of the matrix fluid. Inspection of the interfacial area in Fig. 15(c) shows low S/S_0 , indicating significant coalescence of the dispersed phase. The formation of large, deformable bubbles, which create large free-slip surfaces, significantly attenuates the EVP shear stress distribution around the bubble surface, resulting in a reduced EVP stress contribution to the wall stress budget and, consequently, a lower effective viscosity. This highlights the influence of bubble interactions and coalescence on the bulk rheology at semidilute concentrations.

Finally, we examine the variations in the wall-normal stress profile, τ_{xy}^e . As before, the droplet distribution strongly influences the stress profile; however, we also observe notable changes in the average droplet distribution with increasing Wi . The $\langle H(\psi) \rangle$ profile shows a flattened peak at the channel center together with smaller, secondary peaks closer to the walls at larger Wi . The central peak is explained by lift forces driving large, deformable droplets towards the channel center in Couette flow, whereas the near-wall peaks arise from an increase in tensile elastic stresses pulling the droplets towards the wall. Specifically, the imbalance in streamline curvature between the unconfined and confined sides of a droplet generates a hoop stress that pulls it towards the wall [54]. Overall, this leads to a more uniform distribution of the mean droplet concentration across the wall-normal direction, in contrast to Fig. 11(f), which showed increasing drop accumulation at the center with ϕ . Increased near-wall droplet concentration further contributes to the rise in wall shear stress with Wi through local shear enhancement.

We observe that, unlike Fig. 11(f), which shows a strong coupling between the drop concentration and stress profiles, Fig. 15(f) at $Wi = 1$ shows that the stress peaks spread over a broader region. This indicates that the stress distribution becomes increasingly governed by the nonlinearity of the EVP stress distribution and their interaction with the local flow kinematics and perturbation, rather than by local drop concentration alone. Further, these enhanced elastic effects and nonlinearities at larger Wi lead to the formation of distinct stress structures, such as local stress concentrations [e.g., at the droplet tips in Fig. 5(f)] and intense stress bridges between droplets [Fig. 15(e)].

We note that the present study explores Weissenberg numbers up to four times the base-case value ($Wi = 0.5$) obtained from our fitting procedure. While EVP fluids typically exhibit low elasticity, non-negligible elastic effects often arise in such fluids that can significantly influence the flow dynamics [71,74,78]. Hence, this parametric study aims to elucidate the role of elasticity on the bulk rheology and dispersed phase dynamics of such multiphase systems. Further, the recent work by Abdelgawad *et al.* [85] demonstrates a promising approach for synthesizing yield stress fluids with independently tunable extensional and shear properties. Accordingly, the present parametric study provides potential insights into the rheology of biphasic fluid systems comprising such highly elastic yield stress fluids.

D. The effect of matrix phase yield stress

Finally, we explore the effect of increasing matrix phase yield stress. The stress budget and relative viscosity variation with Bn are shown in Figs. 17(a) and 17(b). As previously discussed, the effective viscosity of the matrix phase ($\tilde{\mu}_{1,\text{eff}}$) increases with Bn . This gives rise to several notable effects that are apparent in the stress budget analysis. First, a decrease in the interfacial stress contribution (τ^σ) is observed owing to an increase in the effective capillary number. This is further shown by the monotonic increase in S/S_0 with Bn in Fig. 17(c). Moreover, we note that this large increase in S/S_0 with Bn also suggests reduced coalescence, particularly for the droplet-laden cases with $S/S_0 > 1$ at large Bn .

Second, the sum of the EVP stress (τ^e) and the solvent viscous stress ($\tau^{v,m}$) remains approximately constant; this indicates that the EVP stress distribution around the drop is not significantly

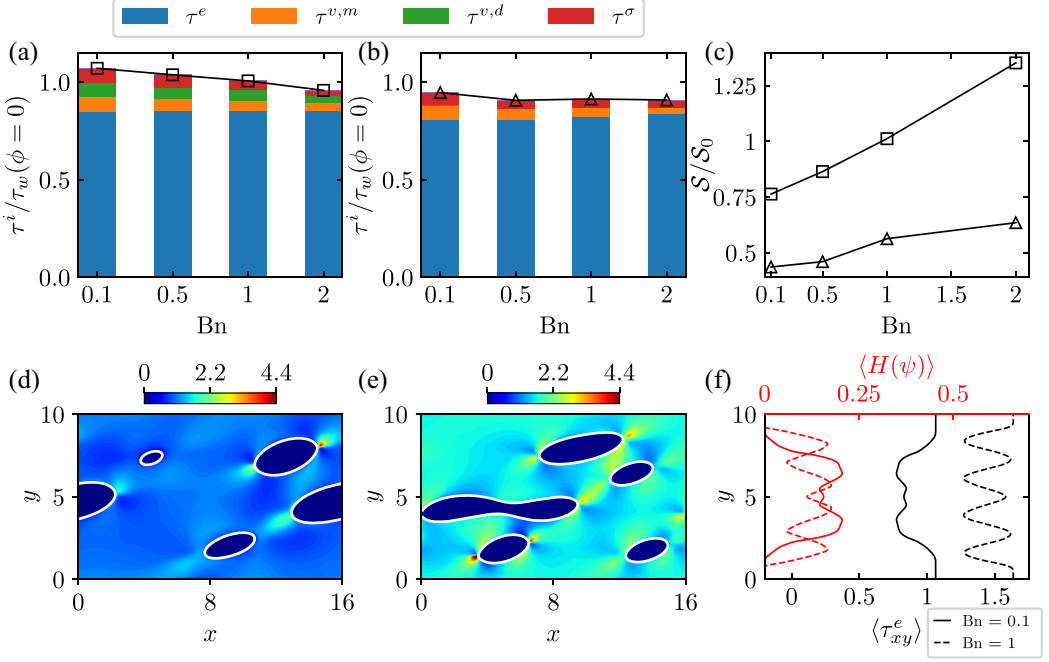


FIG. 17. Stress budget analysis for (a) droplets ($k_\mu = 1$), (b) bubbles ($k_\mu = 1/40$) as a function of Bn ; (c) interfacial area as a function of Bn for droplets (square markers) and bubbles (triangle markers); color map of the τ_{xy}^e field on the x - y plane at (d) $Bn = 0.1$, (e) $Bn = 1$; (f) profiles of $\langle \tau_{xy}^e \rangle$ (black line) and $\langle H(\psi) \rangle$ (red line) for droplets at $Bn = 0.1$ (solid line) and $Bn = 1$ (dashed line). The bar charts in (a) and (b) show the contribution of the extra stress (blue, τ^e), matrix phase viscous stress (orange, $\tau^{v,m}$), dispersed phase viscous stress (green, $\tau^{v,d}$), and interfacial stress (red, τ^σ) to the stress budget. Here $\phi = 0.1$, $Ca = 0.2$, $Wi = 0.5$.

affected (in a normalized sense) with increasing Bn , similar to that observed for the single-droplet case [see Figs. 5(g)–5(i)]. Examining the EVP stress and solvent contributions, an increase in τ^e is observed, with a corresponding decrease in $\tau^{v,m}$. This is again a consequence of the increase in effective viscosity of the matrix fluid ($\tilde{\mu}_{1,\text{eff}}$) which decreases the effective solvent-to-total viscosity ratio, $\beta_{\text{eff}} = \tilde{\mu}_1 / \tilde{\mu}_{1,\text{eff}}$ and accordingly rebalances the normalized sum $\tau^e + \tau^{v,m}$.

Third, a decrease in the contribution of the dispersed phase viscous stress ($\tau^{v,d}$) to the stress budget is observed with increasing Bn . As the effective viscosity of the matrix fluid increases with Bn , the effective viscosity ratio, $k_{\mu,\text{eff}} = \tilde{\mu}_2 / \tilde{\mu}_{1,\text{eff}}$, decreases and, consequently, the dispersed phase viscous stress contributes less to the normalized stress budget. Furthermore, it is observed that $\mu_r < 1$ at the largest Bn i.e., the system's effective viscosity decreases below that of the matrix fluid's. While μ_r also showed monotonic decrease with the capillary number [see Fig. 12(a)], it did not decrease below unity for the droplet-laden cases in our parametric study of Ca . Comparing Fig. 17(c) with Fig. 12(b), we infer that the dispersed phase shows approximately similar deformation as S/S_0 is comparable at the largest Ca and Bn investigated. Therefore, this indicates the reduction in the effective k_μ (which does not occur with increasing Ca) results in μ_r decreasing below unity at high Bn . Conversely, the bubbly suspension shows an initial decrease in μ_r with Bn , after which μ_r remains approximately constant for $Bn \geq 0.5$. Examination of the stress budget shows an initial decrease in the interfacial stress component (τ^σ), with minor variations thereafter. As in the droplet-laden cases, variations in the effective β with Bn re-balance the τ^e and $\tau^{v,m}$ components, while their sum remains approximately constant. Significant bubble coalescence is inferred from the low

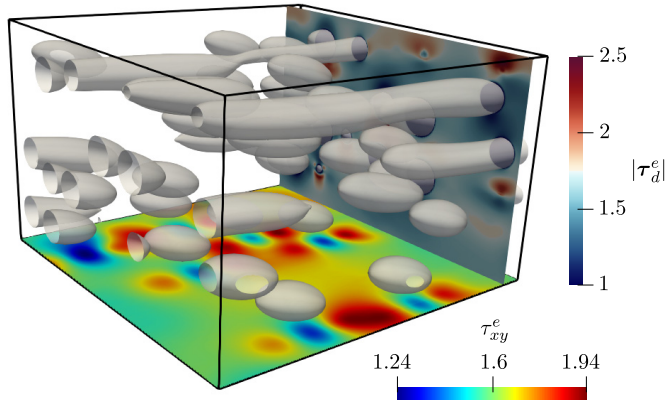


FIG. 18. Visualizations of the droplet surface at $t = 100$. The droplets form a cellular arrangement throughout the domain with a low orientation angle. Some drops coalesce and form long, ligamentous structures. Here $\phi = 0.1$, $Ca = 0.2$, $k_p = k_\mu = 1$, $Wi = 0.5$, $Bn = 1$.

interfacial area in Fig. 17(c), resulting in the formation of large, deformable bubbles. This results in $\mu_r < 1$ for all Bn considered in this parametric study [Fig. 17(b)].

Instantaneous visualizations of the τ_{xy}^e field on the x - y plane are shown in Figs. 17(d) and 17(e) at $Bn = 0.1$ and $Bn = 1$, respectively. At $Bn = 1$, a general increase in magnitude of the τ_{xy}^e distribution is observed throughout the domain due to the increase in effective viscosity of the matrix fluid. Correspondingly, larger deformation is observed in the droplets visualized owing to the increase in effective Ca .

Figure 17(f) shows profiles of $\langle \tau_{xy}^e \rangle$ and $\langle H(\psi) \rangle$, revealing most notably the formation of a cellular pattern of the dispersed phase across the channel height at $Bn = 1$. An instantaneous visualization of the droplets at $Bn = 1$ is shown in Fig. 18, from which it is evident that they appear distributed in a cellular arrangement along the shear-gradient direction, without migration towards any preferential position. As the effective capillary number increases with Bn , large droplet deformation and increased flow orientation is observed. The dispersed phase comprises a combination of small droplets and large, coalesced, ligamentous structures. The latter suggests that the yield stress has a stabilizing effect on droplet breakup. It is noted that similar elongated, ligamentous droplets at high Bingham numbers have also been reported in the opposite configuration of a single EVP drop sheared in a Newtonian matrix fluid [52].

We can hypothesize several reasons behind the formation of this cellular structure. First, droplet coalescence appears to be greatly reduced at high Bn with the large number of droplets observed in Fig. 18. This reduces the number of large drops that may migrate to the centerline due to deformability-induced migration. This reduced coalescence may be a result of the increase in effective capillary number with Bn , which decreases coalescence frequency by reducing the film drainage rate between colliding droplets [86]. Second, the increase in effective viscosity of the matrix fluid with Bn further reduces the film drainage rate [61], as well as decreases the cross-flow migration velocity of droplets [55]. Third, there may be a modulation of the film drainage rate between colliding droplets due to elastic effects, which is enhanced with finite, increasing Bn .

Finally, we present a color map of the $|\tau_d^e|$ field on the y - z plane in Fig. 18, where the color bar's lower limit corresponds to the Bingham number. We observe that $|\tau_d^e| \gg Bn$ everywhere in the matrix phase on the visualized plane, showing that (1) the matrix phase fluid is essentially fully yielded here due to the homogeneous, nominal shear, and (2) the inhibited cross-flow migration is not caused by the presence of unyielded structures surrounding and immobilizing the droplets.

IV. SUMMARY AND DISCUSSION

We have investigated the rheology of biphasic fluid systems comprising a Newtonian dispersed phase in an elastoviscoplastic matrix through interface resolved numerical simulations. Two dispersed-to-matrix phase density and viscosity ratios were considered: a matched system ($k_\rho = k_\mu = 1$, called droplets), and a system in which the dispersed phase density and viscosity are significantly less than that of the matrix fluid ($k_\rho = k_\mu = 1/40$, called bubbles). In our parametric study, we varied the droplet concentration between dilute and semidilute concentrations ($0.0016 \leq \phi \leq 0.2$), the non-Newtonian parameters of the matrix fluid (Wi and Bn), and the capillary number, Ca.

For dilute concentrations, an increase in deformation and flow alignment of a single droplet were observed with increasing Ca and Bn. In the case of increasing Bn, the effective viscosity of the matrix fluid increases, resulting in a corresponding increase in the effective capillary number. This increase in drop deformation and flow alignment is accompanied by a reduction in the system's relative viscosity, owing to weaker perturbation of the surrounding stress field and diminished contribution of the interfacial stress to the wall shear stress budget. For bubble-laden cases, a decrease of the effective viscosity below that of the matrix fluid is observed beyond a critical capillary number as the deformed bubbles act as voidlike inclusions that provide large, free-slip interfaces.

At semidilute concentrations, the relative viscosity showed a nonmonotonic dependence on ϕ for the droplet-laden emulsions: an initial increase in μ_r is observed, followed by a subsequent decrease as the coalescence frequency increases and large, deformable droplets form, consistent with observations reported in previous studies [33,35]. The bubble-laden cases showed a monotonic decrease in μ_r with ϕ . Parametric study of the capillary number showed a monotonic decrease in μ_r with Ca for both droplet- and bubble-laden systems, conferring a shear-thinning behavior. The interfacial area showed monotonic increase with Ca for both droplet- and bubble-laden cases due to increased deformability and reduced coalescence frequency, which led to the formation of large, deformed structures of the dispersed phase.

Increasing the yield stress (Bn) of the matrix fluid showed a decrease in μ_r at both dilute and semidilute concentrations, and for both droplet- and bubble-laden cases. This is explained by the increase in the effective viscosity of the matrix phase with Bn, which increases the effective capillary number. The stress budget analysis highlighted that increase in Bn not only affects the effective capillary number, but all relevant quantities dependent on the matrix phase viscosity, notably the relevant viscosity ratios. At the largest Bn, we observed that the effective viscosity of the droplet-laden emulsion decreases below that of the matrix fluid due to a combination of increased drop deformation and reduction in the effective viscosity ratio ($k_{\mu,\text{eff}}$). Significant unyielded structures were not observed in our setup, with the matrix fluid essentially fully yielded due to the homogeneous shear imposed by the planar Couette flow. Overall, the effect of the yield stress was to increase the effective viscosity of the matrix fluid in the present setup, resulting in an increase in the effective capillary number and viscosity ratios. Moreover, for the droplet-laden case, we observed the formation of regularly spaced, cellular structures of the dispersed phase at high Bn, comprising a spectrum of both small droplets and deformed, ligamentous structures of the dispersed phase. The reason for the formation of this cellular arrangement may lie in the inhibition of droplet cross-flow migration by the yield stress, as well as droplet deformability, viscous effects, and elastic effects working to modulate cross-flow migration and film drainage.

Increasing the elasticity (Wi) of the matrix fluid showed qualitatively similar trends for semidilute and dilute regimes, for both droplet- and bubble-laden cases. Specifically, droplet-laden cases showed a nonmonotonic variation, with an initial increase in μ_r followed by a decrease at $Wi = 2$, whereas bubbly suspensions showed little variation at low Wi followed by a decrease at $Wi = 2$. Examination of the stress budget showed that the matrix EVP stress shows the largest variations with Wi; consequently, variations in the bulk rheology here are most significantly influenced by the EVP stress distribution arising from the local flow kinematics in the drop vicinity and the relaxation

time, i.e., the local Wi . The interfacial stress component also showed an increase with Wi due to reduced coalescence and a larger droplet count. Furthermore, increasing Wi showed increased near-wall droplet concentration owing to tensile elastic stresses pulling droplets towards the wall. Overall, we summarize that the increase in μ_r with Wi for droplet-laden emulsions is due to the three effects here: (1) an increase in the EVP stress distribution around the dispersed phase with Wi , (2) a larger interfacial stress contribution due to reduced coalescence, and (3) droplet migration towards the walls which further elevate the near-wall stresses. The bubbly suspensions showed little variation in μ_r at low Wi , followed by a decrease at large Wi , qualitatively similar to the trends in the dilute case. Significant bubble coalescence resulted in $\mu_r < 1$ for all Wi considered in this parametric study, in contrast to the dilute case, underscoring the importance of accounting for dispersed phase interactions and topological changes at semidilute concentrations.

This study provides insights into the rheological behavior of biphasic fluid systems composed of a viscous dispersed phase in an EVP matrix, with particular emphasis on wall shear stress and dispersed phase structures and dynamics at semidilute concentrations. By analyzing the wall shear stress budget, the spatial distribution of the dispersed phase, and qualitative features of the droplet size distribution, we elucidate the interplay between the non-Newtonian rheology of the matrix fluid, the viscosity and deformability of the dispersed phase, and their collective influence on the bulk rheology and dispersed phase dynamics of these multiphase systems.

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The authors report no conflict of interest.

DATA AVAILABILITY

The data that support the findings of this article are openly available [87].

APPENDIX A: DERIVATION OF THE STRESS BUDGET DECOMPOSITION

We follow the approaches by Yang *et al.* [83] and Rosti *et al.* [34] on formulating a stress budget for biphasic systems in planar Couette flow. Averaging in the streamwise and vorticity directions and in time, the streamwise component of the momentum equation reads

$$0 = -\rho \text{Re} \frac{\partial \langle uv \rangle}{\partial y} + \frac{\partial \langle 2\mu D_{xy} \rangle}{\partial y} + \frac{1}{\text{Ca}} \langle f_x \rangle + \frac{\partial \langle \tau_{xy}^e \rangle}{\partial y} \quad (\text{A1})$$

$$= \frac{\partial}{\partial y} \left(-\rho \text{Re} \langle uv \rangle + \langle 2\mu D_{xy} \rangle + \frac{1}{\text{Ca}} \mathcal{G}(y) + \langle \tau_{xy}^e \rangle \right), \quad (\text{A2})$$

where $\mathcal{G}(y) = \int_0^y \langle f_x \rangle d\xi$, with f_x the x component of the interfacial stress, and D_{xy} the xy component of the strain rate tensor, $\mathbf{D}$. This can be integrated along y to some arbitrary position ζ , giving

$$\begin{aligned} & \cancel{-\rho \text{Re} \langle uv \rangle|_0} + \langle 2\mu D_{xy} \rangle|_0 + \frac{1}{\text{Ca}} \langle \mathcal{G} \rangle|_0 + \langle \tau_{xy}^e \rangle|_0 \\ &= -\rho \text{Re} \langle uv \rangle|_\zeta + \langle 2\mu D_{xy} \rangle|_\zeta + \frac{1}{\text{Ca}} \langle \mathcal{G} \rangle|_\zeta + \langle \tau_{xy}^e \rangle|_\zeta. \end{aligned} \quad (\text{A3})$$

The first and third terms on the left-hand side vanish, respectively due to the no-penetration condition at the wall and the absence of droplets at the wall in our simulations (which we verify *a posteriori*).

This gives

$$\langle 2\mu D_{xy} \rangle|_0 + \langle \tau_{xy}^e \rangle|_0 = -\rho \text{Re}\langle uv \rangle|_\zeta + \langle 2\mu D_{xy} \rangle|_\zeta + \frac{1}{\text{Ca}} \langle \mathcal{G} \rangle|_\zeta + \langle \tau_{xy}^e \rangle|_\zeta. \quad (\text{A4})$$

As the choice of ζ is arbitrary, Eq. (A4) can be averaged in the y direction, finally giving

$$\begin{aligned} \langle 2\mu D_{xy} \rangle|_0 + \langle \tau_{xy}^e \rangle|_0 &= \frac{1}{L_y} \int_0^{L_y} \left(-\rho \text{Re}\langle uv \rangle|_\zeta + \langle 2\mu D_{xy} \rangle|_\zeta + \frac{1}{\text{Ca}} \langle \mathcal{G} \rangle|_\zeta + \langle \tau_{xy}^e \rangle|_\zeta \right) dy \\ &= -\text{Re}\langle \rho uv \rangle + \langle 2\mu D_{xy} \rangle + \frac{1}{\text{Ca}} \langle \mathcal{G} \rangle + \langle \tau_{xy}^e \rangle \end{aligned} \quad (\text{A5})$$

where $\langle \cdot \rangle$ denotes averaging over the entire computational domain.

APPENDIX B: PARAMETER FITTING METHODOLOGY

In Sec. II C we fitted the material properties of the Saramito constitutive equation to the oscillatory shear data of Holenberg *et al.* [71] for a 0.07% w/w Carbopol gel. Following a similar approach as Fraggadakis *et al.* [74], here we report the procedure used to fit the material properties.

When subject to oscillatory shear, the velocity field $\tilde{\mathbf{u}} = \tilde{\gamma}_0 \tilde{y} \hat{e}_x = \gamma_0 \tilde{\omega} \cos(\tilde{\omega} \tilde{t}) \tilde{y} \hat{e}_x$ is imposed on the material. For the 2D equations, this gives the following strain rate tensor, $\tilde{\mathbf{D}}$,

$$\tilde{\mathbf{D}} = \frac{1}{2} \begin{bmatrix} 0 & \tilde{\gamma}_0 \\ \tilde{\gamma}_0 & 0 \end{bmatrix}, \quad (\text{B1})$$

where $\tilde{\gamma}_0 = \gamma_0 \tilde{\omega} \cos(\tilde{\omega} \tilde{t})$.

With the strain rate in Eq. (B1) imposed on the Saramito constitutive equation

$$\frac{1}{\tilde{G}} \tilde{\boldsymbol{\tau}}^e + \frac{F}{\tilde{\mu}_p} \tilde{\boldsymbol{\tau}}^e = 2\tilde{\mathbf{D}}, \quad (\text{B2})$$

where

$$F = \max \left(0, \frac{|\tilde{\boldsymbol{\tau}}_d^e| - \tilde{\tau}_y}{|\tilde{\boldsymbol{\tau}}_d^e|} \right),$$

this yields the following coupled ODEs for the extra stresses:

$$\frac{1}{\tilde{G}} \left[\frac{\partial \tilde{\tau}_{xx}^e}{\partial \tilde{t}} - 2\tilde{\tau}_{xy}^e \tilde{\gamma}_0 \right] + \frac{F}{\tilde{\mu}_p} \tilde{\tau}_{xx}^e = 0, \quad (\text{B3})$$

$$\frac{1}{\tilde{G}} \left[\frac{\partial \tilde{\tau}_{xy}^e}{\partial \tilde{t}} - \tilde{\tau}_{yy}^e \tilde{\gamma}_0 \right] + \frac{F}{\tilde{\mu}_p} \tilde{\tau}_{xy}^e = \tilde{\gamma}_0, \quad (\text{B4})$$

$$\frac{1}{\tilde{G}} \left[\frac{\partial \tilde{\tau}_{yy}^e}{\partial \tilde{t}} \right] + \frac{F}{\tilde{\mu}_p} \tilde{\tau}_{yy}^e = 0 \quad (\text{B5})$$

with the initial condition $\tilde{\tau}_{xx}^e(\tilde{t} = 0) = \tilde{\tau}_{xy}^e(0) = \tilde{\tau}_{yy}^e(0) = 0$ (note that this implies that $\tilde{\tau}_{yy}^e(\tilde{t}) = 0$ for all $\tilde{t}$).

The oscillatory stress response of the material is characterized through the storage and loss moduli, G' and G'' , calculated as the coefficients of the first harmonics of the fast Fourier transform (FFT) of the stress signal. The sum of the extra stress and viscous stress contribution from solvent viscosity are used to compute G' and G'' . In practice, this was computed by solving Eqs. (B3)–(B5) until they reach a state of steady oscillations (i.e., the amplitude of successive peaks vary by less than 2%), after which ten full stress cycles were collected with which G' and G'' were computed.

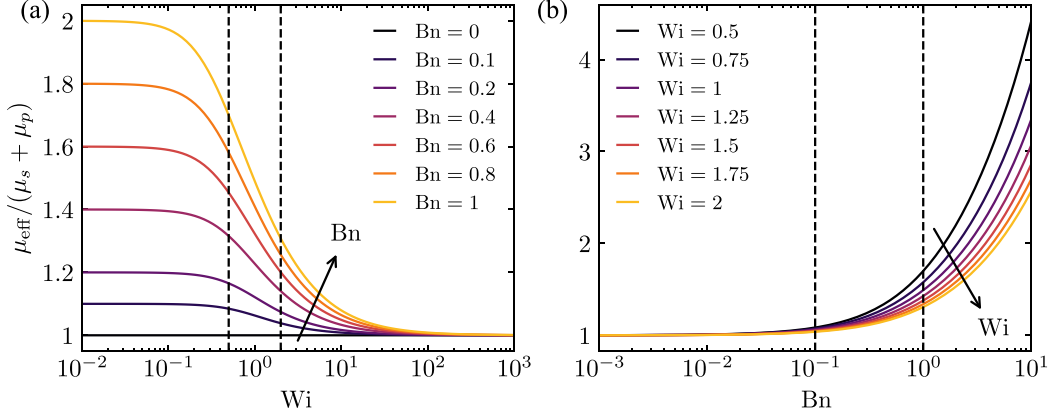


FIG. 19. Effective shear viscosity (μ_{eff}) of single-phase EVP fluid as a function of Wi and Bn at $\beta = 0.09$. The dashed vertical lines indicate the range of Wi and Bn explored in the present study.

Having obtained G' and G'' of the model prediction, they can be compared to that of the experimental measurement and their difference quantified by defining an objective function. In our fitting procedure, we used the following objective function

$$C(\tilde{G}, \tilde{\mu}_p, \tilde{\mu}_s, \tilde{\tau}_y) = \sum_{i=1}^N \left(\frac{G'_{\text{sim}}(\gamma_{0,i}) - G'_{\text{exp}}(\gamma_{0,i})}{G'_{\text{exp}}(\gamma_{0,i})} \right)^2 + \left(\frac{G''_{\text{sim}}(\gamma_{0,i}) - G''_{\text{exp}}(\gamma_{0,i})}{G''_{\text{exp}}(\gamma_{0,i})} \right)^2, \quad (\text{B6})$$

where the subscripts “sim” and “exp” denote the model prediction and experimental measurement, respectively. In Eq. (B6), the summation is over all N strain amplitude data points in the strain amplitude sweep data of Holenberg *et al.* [71]. We minimized the objective function in Eq. (B6) to obtain the best-fit material properties $\tilde{G}$, $\tilde{\mu}_p$, $\tilde{\mu}_s$, $\tilde{\tau}_y$ reported in Table I (note here the subscript “1” is dropped for the material properties for brevity as a single-phase EVP fluid is considered). The above fitting procedure was implemented in a MATLAB script. MATLAB’s `fmincon` function for solving constrained, nonlinear optimization problems was used to minimize the objective function.

APPENDIX C: SINGLE-PHASE SHEAR VISCOSITY

The shear viscosity, μ_{eff} , of a single-phase EVP fluid, normalized with the total viscosity $\mu_p + \mu_s$, is plotted as a function of Wi and Bn in Fig. 19 at $\beta = 0.09$ (the same as that used in the present study). The dashed vertical lines in Figs. 19(a) and 19(b) mark the range of Wi and Bn explored in the present study. μ_{eff} shows a decrease with Wi in Fig. 19(a) (contrast this to the limiting case of $Bn = 0$ for an Oldroyd-B fluid, for which the steady shear viscosity $\mu_{\text{eff}} = (\mu_p + \mu_s)$ is independent of Wi). Two plateaus are observed at low and high Wi , which correspond to the limiting regimes of a viscoplastic fluid and elastic solid, respectively. The low Wi plateau increases with Bn , indicating the increase in effective viscosity of the EVP material due to the yield stress, which is observed for low to intermediate Wi . At large Wi , μ_{eff} becomes independent of Bn as the material response is predominantly elastic with plastic or relaxation effects negligible. The increase in μ_{eff} with Bn owing to the increase in effective viscosity is also shown in Fig. 19(b), as well as the decrease in μ_{eff} with Wi . At low Bn , μ_{eff} becomes independent of Wi as the yield stress becomes negligible i.e., the material behaves as an Oldroyd-B fluid, whose steady shear viscosity is independent of Wi .

APPENDIX D: CONVERGENCE TESTS

To illustrate the time required for the cases in our parametric study to reach a statistically stationary state, Fig. 20 shows the temporal evolution of $\mathcal{S}/\mathcal{S}_0$ for some representative cases for

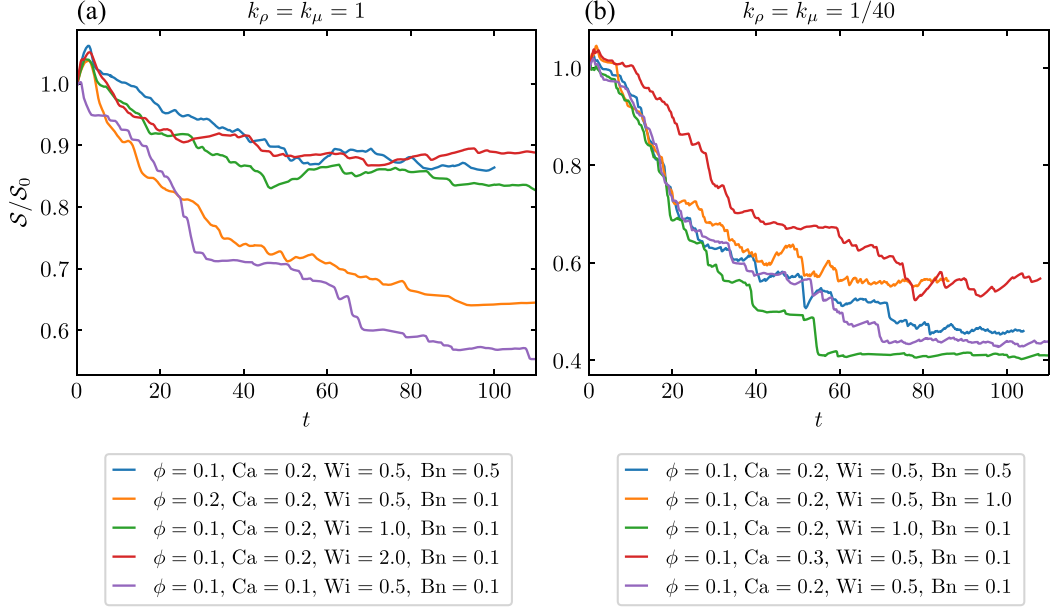


FIG. 20. Temporal evolution of the normalized interfacial area, S/S_0 , for (a) droplets and (b) bubbles for some selected cases of our parametric study.

(a) droplets and (b) bubbles. Generally, the systems reach a statistically stationary state after $t \approx 80$ – 100 time units, after which temporal averaging was performed over an additional 10–20 time units to obtain the statistically stationary results.

Additionally, we conducted simulations to assess the effects of grid resolution and computational domain size. Figure 21 shows results for the case $Wi = 1$, $Bn = 0.1$, $k_\rho = k_\mu = 1$, comparing the

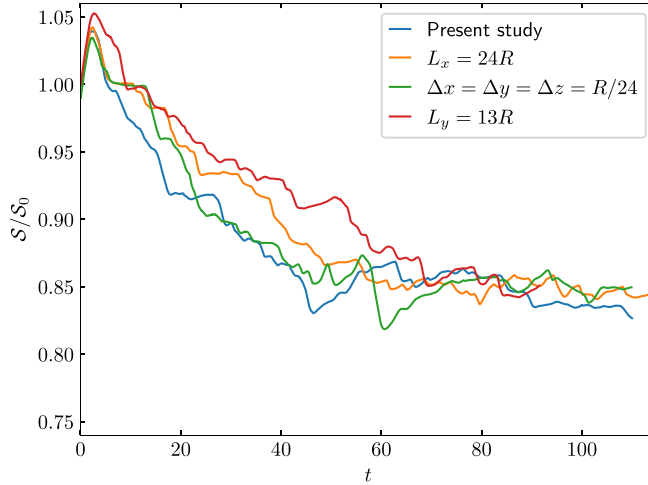


FIG. 21. Temporal evolution of the normalized interfacial area, S/S_0 , for a longer domain in the stream-wise direction ($L_x = 24R$, $L_y = 10R$, $L_z = 16R$, orange line), a higher grid resolution ($L_x = 16R$, $L_y = 10R$, $L_z = 16R$, $\Delta x = R/24$, green line), and a larger domain in the shear-gradient direction ($L_x = 16R$, $L_y = 13R$, $L_z = 16R$, red line). The computational domain used in the present work is shown by the blue line. Here $\phi = 0.1$, $k_\rho = k_\mu = 1$, $Ca = 0.2$, $Wi = 1$, $\beta = 0.09$, $Bn = 0.1$.

present setup (blue line) with a larger domain in the streamwise direction ($L_x = 24R$, orange line), a larger domain in the shear-gradient direction ($L_y = 13R$, red line), and a finer grid resolution ($\Delta x = \Delta y = \Delta z = R/24$, green line). The steady-state values of $\mathcal{S}/\mathcal{S}_0$ differ by less than 2%, indicating that boundary effects from the domain size are negligible and that our chosen resolution ($\Delta x = \Delta y = \Delta z = R/16$) is sufficient to resolve the statistics of the EVP stresses and dispersed-phase morphologies arising in the present study.

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